

Compositions

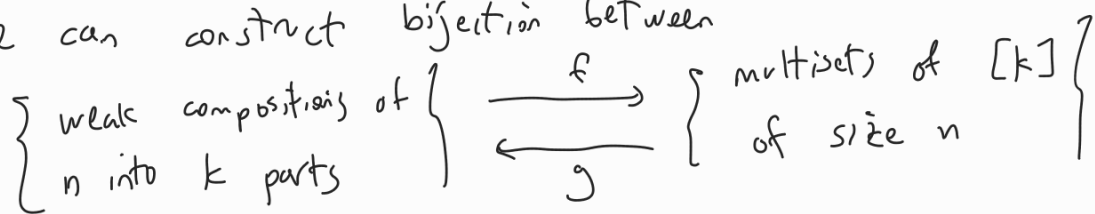
Def. A sequence of non-negative integers (a_1, \dots, a_k) is a weak composition of n if $a_1 + \dots + a_k = n$.

$k = \#$ parts of the weak composition.

If all $a_i > 0$, then this is a composition.

Thm. # weak compositions of n into k parts is $\binom{n+k-1}{n}$
(or $\binom{n+k-1}{k-1}$)

Pf. We can construct bijection between



$f(a_1, \dots, a_k) =$ multiset where i appears exactly a_i many times

Let S be multiset. $g(S) = (\# \text{ times } 1 \text{ appears, } \# \text{ times } 2 \text{ appears, } \dots, \# \text{ times } k \text{ appears})$

\Rightarrow # weak compositions of n into k parts = # multisets of $[k]$ of size $n = \binom{k+n-1}{n} \quad \square$

EX. (weak compositions) of $\overset{n}{3}$ into $\overset{k}{3}$ parts

$(3, 0, 0)$ $(2, 1, 0)$ $(1, 0, 2)$ $(1, 1, 1)$ Only composition of 3 into 3 parts.
 $(0, 3, 0)$ $(2, 0, 1)$ $(0, 1, 2)$
 $(0, 0, 3)$ $(1, 2, 0)$ $(0, 2, 1)$

EX. Distribute 20 pieces of candy (identical) to 4 children.

How many ways? let $a_i = \#$ candy for i th child.

Answer: # weak compositions of 20 into 4 parts

$k=4$
 $n=20$

$$= \binom{20+4-1}{20} = \binom{23}{20}$$

New rule: every child must receive candy.

Answer: # compositions of 20 into 4 parts

First, give each child 1 candy. Then distribute remaining 16 as before.

$$k=4$$

$$n=16$$

weak compositions of 16 into 4 parts.

$$\binom{16+4-1}{16} = \binom{19}{16}$$

Another derivation of # weak compositions:

$$(a_1, \dots, a_k), \quad \sum a_i = n.$$

Consider n 0's, $k-1$ walls

0 0 0 ... 0

place walls to separate groups of 0's:

put wall after first a_1 0's, another after next a_2 , etc.

$$k=4, \quad (2, 0, 3, 1)$$

$$\# \text{ of such configurations} = \frac{(\# \text{ 0's} + \# \text{ walls})!}{\# \text{ 0's!} \# \text{ walls!}} = \frac{(n+k-1)!}{n! (k-1)!} = \binom{n+k-1}{n}$$

Multiset example: $(2, 0, 3, 1) \leftrightarrow \{1, 1, 3, 3, 3, 4\}$
 $n=6$ size=6

Then, # compositions of n into k parts = $\binom{n-1}{k-1}$.

Pf. $\left\{ \begin{array}{l} \text{compositions of } n \\ \text{into } k \text{ parts} \end{array} \right\} \xrightleftharpoons[g]{f} \left\{ \begin{array}{l} \text{weak compositions of} \\ n-k \text{ into } k \text{ parts} \end{array} \right\}$

$$f(a_1, \dots, a_k) = (a_1 - 1, \dots, a_k - 1)$$

$$g(b_1, \dots, b_k) = (b_1 + 1, \dots, b_k + 1)$$

$$\Rightarrow \# \text{ compositions of } n \text{ into } k \text{ parts} = \binom{n-1}{k-1}.$$

□

Cor. # compositions of $n = 2^{n-1}$.

Pf. each composition of n has at least 1 part, at most n parts:
 # compositions of $n = \sum_{k=1}^n \binom{n-1}{k-1} = \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}$. \square

(weak) compositions of n into k parts = assignments of n identical balls to k distinguished boxes

Set partitions.

Let X be a set. A partition of X is an unordered collection of \neq ^{nonempty} subsets S_1, \dots, S_k of X s.t. every element of X belongs to exactly one S_i .

An ordered partition is same, but S_1, \dots, S_k considered w/ order.
 S_i are blocks of partition.
 $k = \#$ blocks.

Ex. $X = \{1, 2, 3\}$.
 $k=1$: 123
 $k=2$: 12|3, 13|2, 23|1
 $k=3$: 1|2|3

ordered version:

123 12|3 13|2 23|1 1|2|3 2|3|1
 3|12 2|13 1|23 1|3|2 3|1|2
 2|1|3 3|2|1

5 partitions of X , 13 ordered partitions of X .

Ex. 20 different candy \rightarrow 4 children s.t. everyone gets candy
 \leftrightarrow ordered partitions of candy into 4 blocks
 20 different candy \rightarrow 4 clones s.t. everyone gets candy
 \leftrightarrow partitions of candy into 4 blocks.

Def. $S(n, k) := \#$ partitions of set of size n into k blocks
 "Stirling number of second kind"

$S(0, 0) = 1$ (by convention)

[IF $k > n$, then $S(n, k) = 0$]

Rmk. $\#$ ordered partitions of set of size n into k blocks
 $= k! S(n, k)$.

Ex. $n \geq 1$: $S(n, 1) = 1$
 $S(n, n) = 1$

$n \geq 2$ $S(n, 2) = \frac{2^n - 2}{2} = 2^{n-1} - 1$

$S(n, n-1) = \binom{n}{2}$

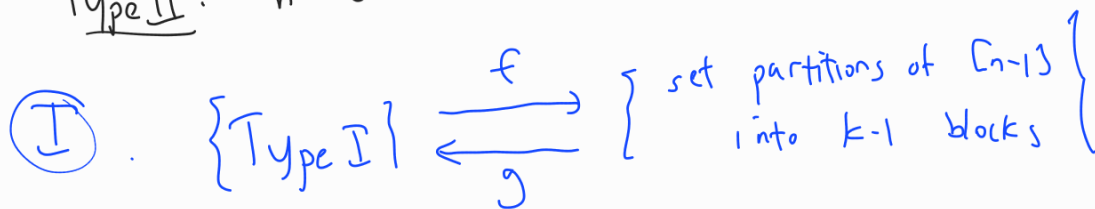
Thm. If $n \geq k$, then

$S(n, k) = S(n-1, k-1) + k S(n-1, k)$.

Pf. Two types of partitions of $[n]$ into k blocks:

Type I: n is by itself.

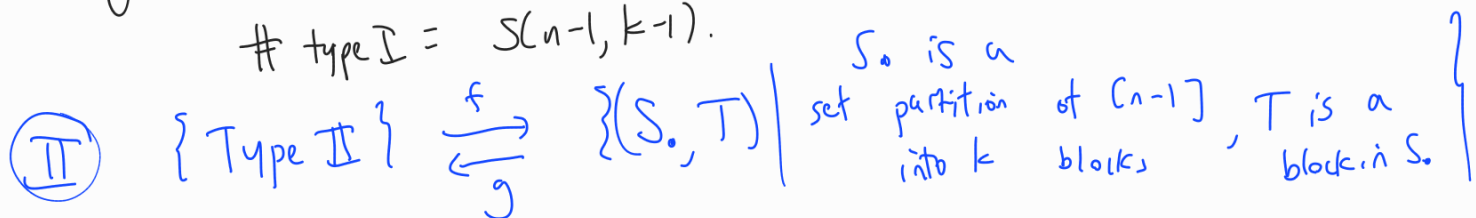
Type II: n shares its block.



f takes set partition and deletes block containing n .

g takes set partition and adds new block $\{n\}$

$\# \text{ type I} = S(n-1, k-1)$.



f deletes n from partition.

Set S = result, T = block which contained n originally

g adds n back to T to get
new partition of $[n]$

$$\# \text{type II} = k S(n-1, k)$$

$$1|2|34 \longrightarrow 1|2|\underline{3}$$

$$14|2|3 \longrightarrow \underline{1}|2|3$$

$$1|24|3 \longrightarrow 1|\underline{2}|3$$

□