Compositions

Def. A sequence of non-negative integers $(a_1, \ldots, a_k)$ is a <u>weak composition</u> of n if $a_1 + \cdots + a_k = n$.

k = # <u>parts</u> of the weak composition.

If all $a_i > 0$, then this is a <u>composition</u>.

Thm. # weak compositions of n into k parts is $\binom{n+k-1}{n}$ (or $\binom{n+k-1}{k-1}$)

Pf. We can construct bijection between

$$\left\{\begin{matrix}\text{weak compositions of} \\ n \text{ into } k \text{ parts}\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\text{multisets of } [k] \\ \text{of size } n\end{matrix}\right\}$$

$f(a_1, \ldots, a_k)$ = multiset where i appears exactly a_i many times

Let S be multiset. $g(S)$ = (# times 1 appears, # times 2 appears, ... # times k appears)

$\Longrightarrow$ # weak compositions of n into k parts = # multisets of $[k]$ of size n = $\binom{k+n-1}{n}$ $\square$

EX. (weak compositions) of $\overset{n}{\overset{\shortparallel}{3}}$ into $\overset{k}{\overset{\shortparallel}{3}}$ parts

(3,0,0)	(2,1,0)	(1,0,2)	$\boxed{(1,1,1)}$
(0,3,0)	(2,0,1)	(0,1,2)	
(0,0,3)	(1,2,0)	(0,2,1)	

only composition of 3 into 3 parts.

EX. Distribute 20 pieces of candy (identical) to 4 children. How many ways? let a_i = # candy for ith child.

Answer: # weak compositions of 20 into 4 parts

$$= \binom{20+4-1}{20} = \binom{23}{20}$$

$k = 4$
$n = 20$

New rule: every child must receive candy.

Answer: # compositions of 20 into 4 parts

First, give each child 1 candy. Then distribute remaining 16 as before.

weak compositions of 16 into 4 parts.

$k=4$

$n=16$

$$\binom{16+4-1}{16} = \binom{19}{16}$$

Another derivation of # weak compositions:

$(a_1, \dots, a_k)$, $\sum a_i = n$.

Consider n 0's, $k-1$ walls

0 0 0 0

place walls to separate groups of 0's:

put wall after first a_1 0's, another after next a_2, etc.

$k=4$, $(2, 0, 3, 1)$

0 0 | | 0 0 0 | 0

$$\text{\# of such configurations} = \frac{(\text{\# 0's} + \text{\# walls})!}{\text{\# 0's}!\ \text{\# walls}!} = \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{n}$$

Multiset example: $(2,0,3,1) \leftrightarrow \{1,1,3,3,3,4\}$

$n=6$ size $=6$

__Thm.__ # compositions of n into k parts $= \binom{n-1}{k-1}$.

__Pf.__

$$\left\{\begin{matrix}\text{compositions of } n\\ \text{into } k \text{ parts}\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\text{weak compositions of}\\ n-k \text{ into } k \text{ parts}\end{matrix}\right\}$$

$$f(a_1, \dots, a_k) = (a_1 - 1, \dots, a_k - 1)$$

$$g(b_1, \dots b_k) = (b_1 + 1, \dots, b_k + 1)$$

$\Rightarrow$ # compositions of n into k parts $= \binom{n-1}{k-1}$. $\square$

Cor. # compositions of n = 2^{n-1}.

Pf. each composition of n has at least 1 part, at most n parts:

$$\#\text{ compositions of } n = \sum_{k=1}^{n} \binom{n-1}{k-1} \underset{j=k-1}{=} \sum_{j=0}^{n-1} \binom{n-1}{j} = 2^{n-1}. \qquad \square$$

(weak) compositions of n into k parts = assignments of n identical balls to k distinguished boxes

Set partitions.

Let X be a set. A partition of X is an unordered collection of nonempty subsets $S_1, \ldots, S_k$ of X s.t. every element of X belongs to exactly one S_i.

An ordered partition is same, but $S_1, \ldots, S_k$ considered w/ order.

S_i are blocks of partition.

k = # blocks.

Ex. $X = \{1,2,3\}$.

$k=1$	$k=2$			$k=3$
123	12|3	13|2	23|1	1|2|3

ordered version:

123	12|3	13|2	23|1	1|2|3	2|3|1
	3|12	2|13	1|23	1|3|2	3|1|2
				2|1|3	3|2|1

5 partitions of X, 13 ordered partitions of X.

Ex. 20 different candy $\longrightarrow$ 4 children s.t. everyone gets candy

$\longleftrightarrow$ ordered partitions of candy into 4 blocks

20 different candy $\longrightarrow$ 4 clones s.t. everyone gets candy

$\longleftrightarrow$ partitions of candy into 4 blocks.

Def. $S(n,k) :=$ # partitions of set of size n into k blocks

"Stirling number of second kind"

$S(0,0) = 1$ (by convention)

[If $k > n$, then $S(n,k) = 0$

Rmk. # ordered partitions of set of size n into k blocks

$= k!\, S(n,k)$.

Ex. $n \geq 1$: $S(n,1) = 1$

$S(n,n) = 1$

$\boxed{n \geq 2}$ $S(n,2) = \dfrac{2^n - 2}{2} = 2^{n-1} - 1$

$S(n, n-1) = \binom{n}{2}$

Thm. If $n \geq k$, then

$$S(n,k) = S(n-1,k-1) + k\, S(n-1,k).$$

Pf. Two types of partitions of $[n]$ into k blocks:

Type I: n is by itself.

Type II: n shares its block.

I. $\{\text{Type I}\} \underset{g}{\overset{f}{\rightleftarrows}} \{\text{set partitions of } [n-1] \text{ into } k-1 \text{ blocks}\}$

f takes set partition and deletes block containing n.

g takes set partition and adds new block $\{n\}$

type I $= S(n-1, k-1)$.

II. $\{\text{Type II}\} \underset{g}{\overset{f}{\rightleftarrows}} \{(S_0, T) \mid S_0 \text{ is a set partition of } [n-1] \text{ into } k \text{ blocks},\ T \text{ is a block in } S_0\}$

f deletes n from partition.

Set S_0 = result, T = block which contained n originally

g adds n back to T to get

new partition of $[n]$

$\square$

$\#$type II $= k\,S(n-1,k)$

$1|2|34 \longrightarrow 1|2|3$

$14|2|3 \longrightarrow 1|2|3$

$1|24|3 \longrightarrow 1|2|3$