Formal power series

A **formal power series** (in variable x) is an expression of the form $\sum_{n=0}^{\infty} a_n x^n$ $\to$ scalars (integers or rational numbers)

shorthand ||

$$\sum_{n\geq 0} a_n x^n$$

usually denoted by capital letters, so $A(x) = \sum_{n\geq 0} a_n x^n$

$[x^n]A(x) =$ coefficient of x^n in $A(x) = a_n$

Equality: $A(x) = B(x) \iff a_n = b_n$ for all n

Addition: $A(x) + B(x) = \sum_{n\geq 0} (a_n + b_n) x^n$

Product: $A(x)\,B(x) = \sum_{n\geq 0} c_n x^n$, $c_n = \sum_{i=0}^{n} a_i b_{n-i}$

Special case: if $a_n = 0$ for $n > 0$

Then $A(x)\,B(x) = a_0 \sum_{n\geq 0} b_n x^n = \sum_{n\geq 0} a_0 b_n x^n$

Addition is commutative: $A(x) + B(x) = B(x) + A(x)$

is associative $A(x) + (B(x) + C(x)) = (A(x) + B(x)) + C(x)$

Multiplication is commutative: $A(x)\,B(x) = B(x)\,A(x)$

is associative $A(x)\,(B(x)\,C(x)) = (A(x)\,B(x))\,C(x)$

Ex. $A(x) = \sum_{n\geq 0} x^n$, $B(x) = \sum_{n\geq 0} x^n$

$A(x) + B(x) = \sum_{n\geq 0} 2x^n$, $A(x)B(x) = \sum_{n\geq 0} (n+1)\,x^n$

$c_n = \sum_{i=0}^{n} a_i b_{n-i} = \sum_{i=0}^{n} 1 = n+1$.

Def. $A(x)$ is invertible if there exists formal power series $B(x)$ s.t. $A(x)B(x)=1$

Ex. ① $A(x)=\sum_{n\geq 0} x^n$; $A(x)$ is invertible ✓

Set $B(x)=1-x$

$$A(x)B(x)=\sum_{n\geq 0} c_n x^n \quad \text{where} \quad c_n=\sum_{i=0}^{n} a_i b_{n-i}=\sum_{i=0}^{n} b_{n-i}$$
$$=\sum_{i=0}^{n} b_i$$

$b_0=1,\ b_1=-1,\ b_n=0$ for $n\geq 2$

If $n=0$, $\sum_{i=0}^{0} b_i=1$ | If $n\geq 1$ $\sum_{i=0}^{n} b_i=b_0+b_1=0$

$$\Rightarrow c_n=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\geq 1\end{cases} \Rightarrow A(x)B(x)=1$$

Another way: $\left(\sum_{n\geq 0} x^n\right)(1-x)=\sum_{n\geq 0} x^n - x\sum_{n\geq 0} x^n=1$

$$\begin{aligned} = \quad & 1+x+x^2+x^3+\cdots \\ & \quad -x-x^2-x^3-\cdots \\ \hline & 1 \end{aligned}$$

② $A(x)=x$: not invertible : why?

$$A(x)B(x)=x\sum_{n\geq 0} b_n x^n = b_0x+b_1x^2+b_2x^3+\cdots \neq 1$$

Thm. $A(x)$ is invertible $\Leftrightarrow [x^0]A(x)\neq 0$.

Pf. Want to solve for $B(x)$ (if possible) s.t. $A(x)B(x)=1$

expand this

$$A(x)B(x)=a_0b_0+(a_0b_1+a_1b_0)x+(a_0b_2+a_1b_1+a_2b_0)x^2+\cdots$$

If this is 1, then we get following equalities

$$a_0 b_0 = 1 \qquad (Eq\ 0)$$
$$a_0 b_1 + a_1 b_0 = 0 \qquad (Eq\ 1)$$
$$a_0 b_2 + a_1 b_1 + a_2 b_1 = 0 \qquad (Eq\ 2)$$
$$a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 0 \qquad (Eq\ 3)$$
$$\vdots \qquad \vdots$$

If $a_0 = 0$, first equation is impossible. $\Rightarrow$ $A(x)$ not invertible in that case.

Otherwise assume $a_0 \neq 0$. $(Eq\ 0) \Rightarrow b_0 = \frac{1}{a_0}$.

$(Eq\ 1) \Rightarrow b_1 = \frac{-a_1 b_0}{a_0} = -\frac{a_1}{a_0^2}$

$(Eq\ 2) \Rightarrow b_2 = \frac{-a_1 b_1 - a_2 b_0}{a_0}$ → already determined.

$\vdots$

We can show (by induction) that there is always a choice of $b_0, \ldots b_n$ s.t. $(Eq\ 0), \ldots (Eq\ n)$ are satisfied. (never get stuck) $\Rightarrow$ $B(x)$ exists, so $A(x)$ is invertible □

We take $\boxed{b_n = -\frac{1}{a_0} \sum_{i=1}^{n} a_i b_{n-i}}$

[This proof shows $B(x)$ is unique when $a_0 \neq 0$].

If $A(x)$ is invertible, write $\frac{1}{A(x)}$ or $A(x)^{-1}$ for its inverse.

Ex. $\left(\sum_{n \geq 0} x^n \right)^{-1} = 1 - x$

Also $\frac{1}{1-x} = (1-x)^{-1} = \sum_{n \geq 0} x^n$ (GEOMETRIC SERIES)

Infinite sums of FPS (formal power series)

Let $A_1(x), A_2(x), A_3(x), \ldots$ be FPS.

Infinite sum $A_1(x) + A_2(x) + A_3(x) + \cdots$ makes sense, if for every $n \geq 0$, $[x^n]\, A_i(x)$ is 0 for all but finitely many i.

intuitively, we allow infinite sums of numbers as long as all but finitely many are 0.

Composition

$B(x)$ FPS

$A(x)$ FPS s.t. $\boxed{a_0 = 0}$

The composition $(B \circ A)(x)$ is FPS $\sum_{n \geq 0} b_n A(x)^n$

$$(B \circ A)(x) = B(A(x))$$

$$A(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$A(x)^2 = a_1^2 x^2 + 2a_1 a_2 x^3 + \cdots$$

$$A(x)^3 = a_1^3 x^3 + \cdots$$

in general, $A(x)^n = a_1^n x^n + \cdots$

i.e., has no terms of degree $< n$

$$\begin{aligned} B(A(x)) = \; & b_0 \\ + \; & b_1 a_1 x + b_1 a_2 x^2 + b_1 a_3 x^3 \\ + \; & b_2 a_1^2 x^2 + 2 b_2 a_1 a_2 x^3 + \cdots \\ + \; & b_3 a_1^3 x^3 + \cdots \\ & \vdots \end{aligned}$$

$$b_0 + b_1 a_1 x + (b_1 a_2 + b_2 a_1^2) x^2 + (b_1 a_3 + 2 b_2 a_1 a_2 + b_3 a_1^3) x^3 + \cdots$$

Ex. d positive integer, $A(x) = x^d$ $\quad B(A(x)) = \sum_{n\geq 0} x^{dn}$

$B(x) = \sum_{n\geq 0} x^n$

From before, $(1-x)B(x) = 1$

sub $A(x)$: $\boxed{(1-x^d)\sum_{n\geq 0} x^{dn} = 1}$

$$\Rightarrow \frac{1}{1-x^d} = \sum_{n\geq 0} x^{dn}$$

Formal derivatives:

$$DA(x) = A'(x) := \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n$$

Basic properties: ① $D(A+B) = DA + DB$

② $D(AB) = (DA)B + A(DB)$

③ $D(B\circ A) = (DA)(DB\circ A)$

④ $D(1/A) = -DA/A^2$

⑤ $D(A^n) = n(DA)A^{n-1}$

Ex. ① From before, $\frac{1}{1-x} = \sum_{n\geq 0} x^n$

Apply derivative: $\frac{1}{(1-x)^2} = \sum_{n\geq 0} n x^{n-1} = \sum_{n\geq 0} (n+1)x^n$

② Simplify $B(x) = \sum_{n\geq 0} n x^n$

option 1: $B(x) = \sum_{n\geq 0} (n+1)x^n - \sum_{n\geq 0} x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x}$

$$= \frac{1-(1-x)}{(1-x)^2} = \frac{x}{(1-x)^2}$$

Option 2: $B(x) = \sum_{n\geq 1} n x^n = x \sum_{n \geq 1} n x^{n-1} = x \sum_{n\geq 0} (n+1) x^n = \frac{x}{(1-x)^2}$

Notation: $\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}$

General Binomial Theorem:

m rational number, k non-negative integer

$$\binom{m}{0} := 1$$

$$\binom{m}{k} := \frac{\overbrace{m(m-1)\cdots(m-k+1)}^{k \text{ terms}}}{k!} \quad \text{if } k>0$$

Thm. m rational number. Then

$$(1+x)^m = \sum_{n\geq 0} \binom{m}{n} x^n \qquad \text{(identity of FPS)}$$

Meaning?

- If $m \geq 0$ integer, this is just multiplying $(1+x)$ m many times,
- If $m < 0$ is integer, this is just multiplying $\frac{1}{1+x}$ $-m$ many times.

$(1+x)^m = \left(\frac{1}{1+x}\right)^{-m}$

- If $m = \frac{1}{b}$, b positive integer, then $(1+x)^m$ is a bth root of $1+x$ i.e., $((1+x)^m)^b = 1+x$

- If $m = \frac{a}{b}$, b positive integer
  a integer

$$(1+x)^m = \left((1+x)^{1/b}\right)^a$$