

Math 202B, Winter 2022  
 Homework 2  
 Due: January 31 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material up to lecture 9.

- (1) Let  $G_1, G_2$  be finite groups and consider representations over  $\mathbf{C}$  (the results below extend to other situations, but this is simplest using what we've learned).
  - (a) If  $V, W$  are irreducible representations of  $G_1, G_2$ , respectively, show that  $V \boxtimes W$  is an irreducible representation of  $G_1 \times G_2$ .
  - (b) Let  $V_1, \dots, V_n$  and  $W_1, \dots, W_m$  be complete lists of irreducible representations (up to isomorphism) of  $G_1$  and  $G_2$ , respectively. Show that

$$\{V_i \boxtimes W_j \mid i = 1, \dots, n, j = 1, \dots, m\}$$

is a complete list of irreducible representations (up to isomorphism) of  $G_1 \times G_2$ .

- (2) Compute the character table for the quaternion group  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ . The product is determined by  $i^2 = j^2 = k^2 = ijk = -1$  and requiring  $-1$  to commute with everything.
- (3) Let  $p$  be a prime and let  $G$  be the group (under multiplication) of upper-triangular  $3 \times 3$  matrices whose entries are in  $\mathbf{Z}/p$  and whose diagonal entries are all 1:

$$G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbf{Z}/p \right\}.$$

- (a) Determine the size of  $G/[G, G]$ .
- (b) Let  $V$  be the set of functions  $f: \mathbf{Z}/p \rightarrow \mathbf{C}$ . This is a complex vector space under addition of functions and scalar multiplication by complex numbers. Pick  $\omega \in \mathbf{C}$  such that  $\omega^p = 1$ . Show that there is a unique homomorphism  $\rho_\omega: G \rightarrow \mathbf{GL}(V)$  that satisfies
  - (c) Show that  $\rho_\omega$  is irreducible if  $\omega \neq 1$  and that  $\rho_\omega \not\cong \rho_{\omega'}$  if  $\omega \neq \omega'$ .
  - (d) Classify all of the irreducible complex representations of  $G$ .
- (4) Let  $G$  be a finite group and let  $H \subseteq G$  be a subgroup of index 2.

- (a) Show that  $H$  is a normal subgroup and hence is a union of conjugacy classes of  $G$ .
- (b) Let  $\gamma$  be a conjugacy class of  $G$  which is contained in  $H$ . Show that either  $\gamma$  is still a single conjugacy class of  $H$ , or that it is a union of two conjugacy classes of  $H$ , which both have the same size.

(Recall that the size of the conjugacy class of  $x \in G$  is  $|G/C_G(x)|$  where  $C_G(x) = \{h \in G \mid hx = xh\}$  and consider  $C_G(x) \cap H$ .)

- (c) When  $G = \mathfrak{S}_n$  and  $H = \mathfrak{A}_n$  is the alternating group, determine what happens for each conjugacy class of  $\mathfrak{S}_n$  when restricted to  $\mathfrak{A}_n$ .
- (d) Let  $V$  be an irreducible complex representation of  $G$ . Show that  $(\chi_V, \chi_V)_H \leq 2$ . Deduce that the restriction of  $G$  to  $H$  is either irreducible, or a direct sum of two non-isomorphic irreducible representations.
- (e) In the second case of (c), suppose that  $V|_H \cong V_1 \oplus V_2$  for irreducible representations  $V_1, V_2$  of  $H$ . Pick  $g \in G \setminus H$ . Show that

$$g \cdot V_1 := \{g \cdot v \mid v \in V_1\}$$

is an  $H$ -subrepresentation of  $V$  and that in fact  $g \cdot V_1 = V_2$ . Conclude that  $\dim V_1 = \dim V_2$ .

- (5) (a) Show that the sign representation of  $\mathfrak{S}_n$  is isomorphic to the Specht module  $\mathbf{S}^{(1^n)}$ .
- (b) Prove that for any finite group  $G$ , given an irreducible representation  $V$  and a 1-dimensional representation  $W$ , the tensor product  $V \otimes W$  is also irreducible.
- (c) Now assume our field has characteristic 0. Combining the above, we see that, for any partition  $\lambda$ ,  $\mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}$  is irreducible, so must be isomorphic to a Specht module. We will now describe which one.

Let  $\mu = \lambda^\dagger$  be the transpose partition. Pick a  $\lambda$ -tableau  $t$  and let  $t^\dagger$  be the  $\lambda^\dagger$ -tableau obtained by flipping the values across the diagonal. Let  $R_{t^\dagger}$  denote the row-stabilizer of  $t^\dagger$ , i.e., the set of  $\sigma \in \mathfrak{S}_n$  such that  $\{\sigma t^\dagger\} = \{t^\dagger\}$  and let  $\rho_{t^\dagger} = \sum_{\sigma \in R_{t^\dagger}} \sigma$ .

Let  $u \in \mathbf{S}^{(1^n)}$  be a nonzero element and recall that every  $\lambda^\dagger$ -tabloid is of the form  $\{\sigma t^\dagger\}$  for some  $\sigma \in \mathfrak{S}_n$  (though the choice is not unique). Show that there is a well-defined  $\mathfrak{S}_n$ -equivariant linear map

$$\varphi: \mathbf{M}^{\lambda^\dagger} \rightarrow \mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}, \quad \varphi(\{\sigma t^\dagger\}) = \sigma \rho_{t^\dagger}(\{t\} \otimes u).$$

- (d) Show that  $\varphi(e_{t^\dagger}) \neq 0$ . Conclude that  $\mathbf{S}^{\lambda^\dagger} \cong \mathbf{S}^\lambda \otimes \mathbf{S}^{(1^n)}$ .  
(Hint: First show that  $\varphi(e_{t^\dagger}) = (\rho_{t^\dagger} \kappa_t \{t\}) \otimes u$ . Then show that  $\langle \rho_{t^\dagger} \kappa_t \{t\}, \{t\} \rangle \neq 0$  where  $\langle \cdot, \cdot \rangle$  is the pairing on  $\mathbf{M}^{\lambda^\dagger}$ .)
- (6) Let  $p$  be a prime dividing  $n$  and let  $\mathbf{k}$  be a field of characteristic  $p$ . Let  $U = \{(x_1, \dots, x_n) \in \mathbf{k}^n \mid x_1 + \dots + x_n = 0\}$  and let  $L$  be the line spanned by  $(1, 1, \dots, 1)$ . Show that  $U/L$  is an irreducible  $\mathfrak{S}_n$ -representation.

## 1. EXTRA PROBLEMS (DON'T TURN IN)

- (7) Compute the character table of the alternating group  $\mathfrak{A}_4 \subset \mathfrak{S}_4$ .
- (8) Let  $G$  be a finite group and  $H \subseteq G$  be a subgroup. Let  $\mathbf{k}$  be a field. Let  $\rho: H \rightarrow \mathbf{GL}(W)$  be a representation. Define

$$F_H^G(W) = \{f: G \rightarrow W \mid f(xh) = \rho(h^{-1})(f(x)) \text{ for all } x \in G, h \in H\}.$$

- (a) Show that  $F_H^G(W)$  is a  $G$ -representation with the action  $(g \cdot f)(x) = f(g^{-1}x)$ .
- (b) Pick coset representatives  $g_1, \dots, g_r$  for  $G/H$ . Define

$$\Phi: F_H^G(W) \rightarrow \text{Ind}_H^G(W) \quad \Phi(f) = \sum_{i=1}^r e_{g_i} \otimes f(g_i).$$

Show that  $\Phi$  is a  $G$ -equivariant isomorphism.

- (9) Given any finite-dimensional vector space  $V$  with a basis  $v_1, \dots, v_m$  and a symmetric bilinear form  $\langle, \rangle$ , prove that the dimension of  $V/V^\perp$  is the rank of the Gram matrix  $(\langle v_i, v_j \rangle)_{i,j=1,\dots,m}$ .