Math 202B, Winter 2022
Homework 4
Due: February 28 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material up to lecture 21.

(1) (a) Consider the permutation $\sigma = 45723618 \in \mathfrak{S}_8$, i.e., $\sigma(1) = 4$, $\sigma(2) = 5$, etc. Interpret it as a matrix by putting a 1 in position $(i, \sigma(i))$ for $i = 1, \dots, 8$ and 0 elsewhere. Apply RSK to this matrix.
(b) Find a permutation such that RSK produces SYT of shape $(4, 2, 1, 1)$. In general, find a permutation such that RSK produces SYT of shape λ for any partition λ.

(2) Show that

$$s_\lambda(x_1, y_1, x_2, y_2, \dots) = \sum_{\mu \subseteq \lambda} s_\mu(x_1, x_2, \dots) s_{\lambda/\mu}(y_1, y_2, \dots).$$

(3) Let $\lambda = (2n-2, 2n-4, \dots, 4, 2, 0)$. Find a simple factorization for $s_\lambda(x_1, \dots, x_n)$.

(4) Simplify the product

$$\left(\sum_{n \ge 0} e_n t^n\right)\left(\sum_\lambda s_{2\lambda} t^{2|\lambda|}\right)$$

where $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ and the second sum is over all partitions.

(5) Explain how to deduce the identity

$$s_\nu h_\mu = \sum_\lambda K_{\lambda/\nu,\mu} s_\lambda$$

using just Pieri's rule (the special case where $\ell(\mu) = 1$).

(6) Let λ, μ, ν be partitions of n. Let χ^λ be the character of the Specht module $\mathbf{S}^\lambda$. The product $\chi^\lambda \chi^\mu$ (naive product, not induction product!) is the character of $\mathbf{S}^\lambda \otimes \mathbf{S}^\mu$, and we can decompose it:

$$\chi^\lambda \chi^\mu = \sum_\nu g^\nu_{\lambda,\mu} \chi^\nu$$

for some non-negative integers g.
(a) Show that g is invariant under permutations of ν, λ, μ, i.e.,

$$g^\nu_{\lambda,\mu} = g^\nu_{\mu,\lambda} = g^\lambda_{\nu,\mu} = g^\lambda_{\mu,\nu} = g^\mu_{\lambda,\nu} = g^\mu_{\nu,\lambda}.$$

For that reason, we write them more symmetrically as $g_{\lambda,\mu,\nu}$.
(b) Using the Frobenius characteristic ch, we can transfer the product of characters to symmetric functions: given $f, g \in \Lambda_{\mathbf{Q}}$, define

$$f * g = \mathrm{ch}(\mathrm{ch}^{-1}(f)\mathrm{ch}^{-1}(g)).$$

For any class function χ, show that $\mathrm{ch}(\chi) * p_\lambda = \chi(\lambda) p_\lambda$.

(c) We showed that $p_1, p_2, \ldots$ are algebraically independent and hence every $f \in \Lambda_{\mathbf{Q}}$ can be written as a polynomial in the p_n with $\mathbf{Q}$-coefficients. Define $\frac{\partial}{\partial p_n} f$ to be partial derivative of this polynomial where we are treating the p_n as variables, so for example, $\frac{\partial}{\partial p_2}(p_2^2 p_5 + p_1 + p_1 p_2^3) = 2p_2 p_5 + 3p_1 p_2^2$. Show that for all $f, g \in \Lambda$, we have

$$\left\langle n \frac{\partial}{\partial p_n} f, g \right\rangle = \langle f, p_n g \rangle.$$

(d) Let Y^1 be the character of the permutation representation of $\mathfrak{S}_n$ on $\mathbf{C}^n$. Show that $\mathrm{ch}(Y^1) * s_\lambda = s_1 s_{\lambda/1}$.
Hint: first show that $\mathrm{ch}(Y^1) * p_\lambda = m_1(\lambda) p_\lambda = p_1 \frac{\partial}{\partial p_1} p_\lambda$ and show that $\frac{\partial}{\partial p_1} s_\lambda = s_{\lambda/1}$ by showing they pair the same way against all symmetric functions.

(e) Use $Y^1 = \chi^{(n-1,1)} + \chi^n$ to deduce a formula for $g_{(n-1,1),\lambda,\mu}$.