Math 202B, Winter 2022
Homework 5
Due: March 14 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material up to lecture 29.

All vector spaces in these questions are assumed to be finite-dimensional.

(1) Use both formulas in §5.2 to compute the dimension of the Specht module $\mathbf{S}^{(5,5,2,1)}$.

(2) Compute the Littlewood–Richardson coefficient $c^{(5,4,3,1,1)}_{(3,3,1),(4,2,1)}$ and list the corresponding Littlewood–Richardson tableaux (do it both ways like in Example 5.4.2).

(3) (a) Given partitions λ and μ, enumerate the SSYT of shape μ with entries $\le n$ as $T_1, T_2, \ldots, T_N$. Show that the character of $\mathbf{S}_\lambda(\mathbf{S}_\mu(\mathbf{C}^n))$ is $s_\lambda(x^{T_1}, \ldots, x^{T_N})$ (the variable set is $x_1, \ldots, x_n$ here).

(b) When $n = 2$, by knowing the degree of the polynomial, we lose no information by setting $x_1 = 1$ and thinking of this as a polynomial in a single variable x_2. Use HW3#4(b) to show that there is an isomorphism of $\mathbf{GL}_2(\mathbf{C})$-representations

$$\mathrm{Sym}^d(\mathrm{Sym}^n\,\mathbf{C}^2) \cong \mathrm{Sym}^n(\mathrm{Sym}^d\,\mathbf{C}^2)$$

for all $d, n \ge 0$. What isomorphism does HW3 #4(c) imply?

(4) Let $\mathbf{k}$ either be a field of characteristic 0 or of characteristic p where $p > d$.

(a) For any vector space V over $\mathbf{k}$, construct a $\mathbf{GL}(V)$-equivariant isomorphism between the divided power $\mathrm{D}^d V$ and the symmetric power $\mathrm{Sym}^d V$.

(b) Show that $\mathrm{D}^p V$ and $\mathrm{Sym}^p V$ are not isomorphic when $\mathbf{k}$ has positive characteristic p.

(5) Let V, W be complex vector spaces. Show that, for any partition λ, there is a $\mathbf{GL}(V) \times \mathbf{GL}(W)$-equivariant isomorphism

$$\mathbf{S}_\lambda(V \boxtimes W) \cong \bigoplus_{\mu,\nu} (\mathbf{S}_\mu(V) \boxtimes \mathbf{S}_\nu(W))^{\oplus g_{\lambda,\mu,\nu}}$$

where the sum is over all pairs of partitions and the $g_{\lambda,\mu,\nu}$ are as defined in HW4# 6.

Hint: Apply Schur–Weyl duality to both sides of the isomorphism $(V \otimes W)^{\otimes n} \cong V^{\otimes n} \otimes W^{\otimes n}$.

1. Extra problems (don't turn in)

(6) Show that without any assumptions on $\mathbf{k}$, there is a $\mathbf{GL}(V)$-equivariant isomorphism between $\mathrm{D}^d(V^*)$ and $(\mathrm{Sym}^d V)^*$.