Linear Representations: Definitions

G finite group, 1_G = identity element (or just 1)

gg' is product of g and g'

$\Bbbk$ = field Properties: ① $\Bbbk$ algebraically closed?

↳ every polynomial w/ coefficients in $\Bbbk$ has a root in $\Bbbk$

② characteristic of $\Bbbk$:

char $\Bbbk = 0$ if $1+1+\cdots+1$ never 0 (examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$)

char $\Bbbk = p$ if $\underbrace{1+1+\cdots+1}_{p} = 0$ and p is as small as possible.

$\Rightarrow$ p is prime (example: $\mathbb{Z}/p$)

<u>Defn.</u> V = vector space (finite dimensional) over field $\Bbbk$

$GL(V)$ = set of invertible linear operators on V

group structure is composition of operators

[can pick basis for $V \Rightarrow$ identifies $GL(V)$ w/ matrices

group structure is multiplication]

A (linear) <u>representation</u> of G is a homomorphism

$$\rho_V : G \longrightarrow GL(V).$$

Equivalently, this is a function $G \times V \to V$

$(g, v) \to g \cdot v$

s.t.:

① $g \cdot (v + v') = g \cdot v + g \cdot v'$
② $(gg') \cdot v = g \cdot (g' \cdot v)$
③ $1_G \cdot v = v$
④ $g \cdot (\lambda v) = \lambda (g \cdot v)$

$g, g' \in G$
$v, v' \in V$
$\lambda \in \Bbbk$

$g \cdot v = \rho_V(g)(v)$

If V, V' are representations, a linear map $f: V \to V'$ is <u>G-equivariant</u>

if $\forall g \in G$, $f \circ \rho_V(g) = \rho_{V'}(g) \circ f$

i.e., $f(g \cdot v) = g \cdot f(v) \quad \forall v \in V$

f is an isomorphism of representations if it is invertible (as a linear map). Write $V \cong V'$ if an isomorphism exists.

Examples ① For any vector space V, we can define $\rho_V(g)$ to be identity for all $g \in G$. When $\dim V = 1$, this is called the trivial representation.

When $\dim V = 0$, this is the zero representation.

② Let X be finite set w/ action of G.

Recall: this is a function $G \times X \to X$, $(g, x) \to g \cdot x$ s.t. ① $1_G \cdot x = x$ ② $(gg') \cdot x = g \cdot (g' \cdot x)$

Let $V = \mathbb{k}[X]$ be $\mathbb{k}$-vector space w/ basis $\{ e_x \mid x \in X \}$

Define $g \cdot e_x = e_{g \cdot x}$ ($\&$ $g \cdot \sum_{x \in X} \alpha_x e_x = \sum_{x \in X} \alpha_x e_{g \cdot x}$)

Permutation representation of X

Subexample: $X = G$ w/ $g \cdot x = gx$, $g \in G$, $x \in X = G$

$\mathbb{k}[G]$ = regular representation

Lemma. ① All eigenvalues of $\rho(g)$ are roots of unity.

② If $\mathbb{k}$ algebraically closed, char. 0, then $\rho(g)$ is diagonalizable.

Pf. ① Let λ be eigenvalue of $\rho(g)$, w/ eigenvector v.

Then $\rho(g)^{|G|} = \rho(g^{|G|}) = \mathrm{id}_V$, but

$v = \rho(g)^{|G|} v = \lambda^{|G|} v \implies \lambda^{|G|} = 1$.

② Consider Jordan normal form of $\rho(g)$.

↳ upper-triangular matrix
diagonal entries are eigenvalues
1's and 0's on superdiagonal, 0's elsewhere.

Matrix is diagonalizable $\iff$ no 1's on superdiagonal.

If 1's on superdiagonal $\Rightarrow$ not finite order.

Eg. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ never identity if $n > 0$. $\square$

Rmk. $G = \mathbb{Z}/2 = \{1, z\}$, $\Bbbk$ = field of char. 2

$\rho(z) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives rep. since $\rho(z)^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

but $\rho(z)$ not diagonalizable.

Basic operations. Let V, W be reps of G.

① Direct sum $V \oplus W = \{(v, w) \mid v \in V, w \in W\}$

is rep of G by $g \cdot (v, w) = (g \cdot v, g \cdot w)$

② Dual: V^* = dual space = {linear functionals $V \to \Bbbk$}

Given $g \in G$, $f \in V^*$, $g \cdot f$ is functional

$(g \cdot f)(v) = f(g^{-1} \cdot v)$

③ Tensor product: Recall: $V \otimes W$ is vector space spanned by symbols $\{v \otimes w \mid v \in V, w \in W\}$ satisfying relations

- $(v + v') \otimes w = v \otimes w + v' \otimes w$
- $v \otimes (w + w') = v \otimes w + v \otimes w'$
- For all $\lambda \in \Bbbk$ $(\lambda v) \otimes w = v \otimes (\lambda w) = \lambda (v \otimes w)$

$\dim (V \otimes W) = (\dim V)(\dim W)$

[If $e_1, \dots e_n$ basis for V, $f_1, \dots, f_m$ basis for W,

then $e_i \otimes f_j$ $\begin{smallmatrix} i = 1, \dots, n \\ j = 1, \dots, m \end{smallmatrix}$ basis for $V \otimes W$]

Given $g \in G$, $g \cdot \left(\sum_i v_i \otimes w_i \right) = \sum_i (g \cdot v_i) \otimes (g \cdot w_i)$

$\Rightarrow$ $V \otimes W$ has G-rep structure.

④ Hom spaces $\mathrm{Hom}(V, W)$ = {linear functions $V \to W$}

For $g \in G$, $f : V \to W$, $g \cdot f$ is linear function given by $(g \cdot f)(v) = g \cdot (f(g^{-1} \cdot v))$

$\Rightarrow$ G-rep on $\mathrm{Hom}(V, W)$.

[If $W = \mathbb{k}$ is trivial, this specializes to V^*]

(5) (Invariants) $V^G := \{ v \in V \mid g \cdot v = v \ \forall g \in G \}$

is space of G-invariants

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(1) We have natural isomorphism $V^* \otimes W \longrightarrow \mathrm{Hom}(V, W)$

given by $\sum_i f_i \otimes w_i \mapsto F$ where

$$F(v) = \sum_i f_i(v)\, w_i$$

In fact, this is G-equivariant (exercise)

(2) $\mathrm{Hom}(V, W)^G = \{ f : V \to W \mid f \text{ is } G\text{-equivariant} \}$

If $V \cong W_1 \oplus W_2$ (and $W_1 \neq 0$, $W_2 \neq 0$), think of this as

decomposition of V into simpler building blocks.

Def. V is <u>decomposable</u> if $\exists\ W_1, W_2$ nonzero s.t. $V \cong W_1 \oplus W_2$

V is <u>indecomposable</u> otherwise.