Garnir relations

λ = partition, t = λ-tableau, $n = |\lambda|$

X = some subset of values of boxes in ith column of t

Y = ———————————————— $(i+1)$st column of t

$\mathfrak{S}_X$ = permutations of X, $\mathfrak{S}_Y$ = permutations of Y.

$\mathfrak{S}_{X\cup Y}$ = permutations of $X \cup Y$, $\mathfrak{S}_X \times \mathfrak{S}_Y \subseteq \mathfrak{S}_{X\cup Y}$.

Pick coset representatives $\sigma_1, \ldots, \sigma_k$ for $\mathfrak{S}_{X\cup Y} / \mathfrak{S}_X \times \mathfrak{S}_Y$

Define <u>Garnir element</u> $\displaystyle G_{X,Y} = \sum_{j=1}^{k} \operatorname{sgn}(\sigma_j)\, \sigma_j \in \mathbb{k}[\mathfrak{S}_n]$

<u>Ex.</u> $t = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & \\ \cline{1-1} \end{array}$, $X = \{3,5\}$, $Y = \{2,4\}$

write elements of $\mathfrak{S}_{X\cup Y}$ as $\sigma(3)\sigma(5)\sigma(2)\sigma(4)$

choose reps so that $\sigma(3) < \sigma(5)$ & $\sigma(2) < \sigma(4)$

Representatives are: 2345, 2435, 2534, 3425, 3524, 4523

$$G_{XY} e_t = -e_{\begin{smallmatrix} 1 & 4 \\ 2 & 5 \\ 3 \end{smallmatrix}} + e_{\begin{smallmatrix} 1 & 3 \\ 2 & 5 \\ 4 \end{smallmatrix}} - e_{\begin{smallmatrix} 1 & 3 \\ 2 & 4 \\ 5 \end{smallmatrix}} - e_{\begin{smallmatrix} 1 & 2 \\ 3 & 5 \\ 4 \end{smallmatrix}} + e_{\begin{smallmatrix} 1 & 2 \\ 3 & 4 \\ 5 \end{smallmatrix}} - e_{\begin{smallmatrix} 1 & 2 \\ 4 & 3 \\ 5 \end{smallmatrix}}$$

<u>Thm.</u> (Garnir relations) If $|X \cup Y| > \lambda_i^T$ ← size of ith column of $Y(\lambda)$

then $G_{XY} e_t = 0$.

<u>Pf.</u> The left side has integer coefficients in tabloid basis, so to check if it is 0, suffices to assume $\mathbb{k} = \mathbb{Q}$.

Define $\displaystyle \alpha = \sum_{\sigma \in \mathfrak{S}_X \times \mathfrak{S}_Y} \operatorname{sgn}(\sigma)\cdot\sigma$, $\displaystyle \beta = \sum_{\sigma \in \mathfrak{S}_{X\cup Y}} \operatorname{sgn}(\sigma)\cdot\sigma$

Since $|X \cup Y| > \lambda_i^T$, for every $\tau \in C_t$, there are always two values from $X \cup Y$ that are in same row of τt.

$\Rightarrow \beta\{\tau t\} = 0$ for any $\tau \in C_t$.

Why? Let ρ be transposition swapping two elements of $X \cup Y$ in same row of τt. Then for $\sigma \in \mathfrak{S}_{X \cup Y}$, $\sigma\rho \in \mathfrak{S}_{X \cup Y}$ and $\sigma\{\tau t\} = \sigma\rho\{\tau t\}$

but $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\sigma\rho)$ and $\sigma \mapsto \sigma\rho$ gives bijection between one-half of $\mathfrak{S}_{X \cup Y}$ w/ other half.

$\Rightarrow \beta e_t = \beta K_t\{t\} = 0$

Next, α is a factor of K_t. Namely, if we pick coset representatives $\alpha_1, \ldots, \alpha_r$ of $C_t/(\mathfrak{S}_X \cup \mathfrak{S}_Y)$, then

$$K_t = \left(\sum \operatorname{sgn}(\alpha_i)\,\alpha_i\right)\alpha$$

Similarly, $\beta = G_{X,Y}\,\alpha$.

For any $\sigma \in \mathfrak{S}_X \times \mathfrak{S}_Y$, we have $\sigma \cdot K_t = \operatorname{sgn}(\sigma) K_t$.

$$\Rightarrow 0 = \beta e_t = \beta K_t\{t\} = G_{X,Y}\,\alpha K_t\{t\} = G_{X,Y} \sum_{\sigma \in G_X \times G_Y} K_t\{t\}$$

$$= |X|!\,|Y|!\;G_{X,Y}\, e_t$$

$$\Rightarrow G_{X,Y}\, e_t = 0. \qquad \square$$

Def. A tableau t is <u>standard</u> if values increase left to right in each row & top to bottom in each column:

$$t_{ij} < t_{i,j+1} \quad \& \quad t_{ij} < t_{i+1,j}.$$

If t standard, e_t is <u>standard polytabloid</u>, $\{t\}$ <u>standard tabloid</u>.

Def. Total ordering on λ-tabloids: $\{t_1\} < \{t_2\}$ if $\exists i$ s.t.:

1. For all $j > i$, j is in same row of $\{t_1\}$ & $\{t_2\}$
2. i is in higher row of $\{t_1\}$ than in $\{t_2\}$.

EX. $\lambda = (3,2)$

$$\begin{matrix}345\\12\end{matrix} \overset{3}{<} \begin{matrix}245\\13\end{matrix} \overset{2}{<} \begin{matrix}145\\23\end{matrix} \overset{4}{<} \begin{matrix}235\\14\end{matrix} \overset{2}{<} \begin{matrix}135\\24\end{matrix} \overset{3}{<} \begin{matrix}125\\34\end{matrix} \overset{5}{<} \begin{matrix}234\\15\end{matrix} \overset{2}{<} \begin{matrix}134\\25\end{matrix} \overset{3}{<} \begin{matrix}124\\35\end{matrix} \overset{4}{<} \begin{matrix}123\\45\end{matrix}$$

Def. Given tableau t, let $[t]$ be "column version" of tabloid, namely equivalence class under considering two tableaux same if they have same entries in each column.

$[t_1] < [t_2]$ same definition if we replace "row" w/ "column", "higher" w/ "more to right".

Thm. $\{e_t \mid t \text{ standard}\}$ is a basis for S^λ.

Pf. Linearly independent: If t standard tableau and $\sigma \in C_t$, $\sigma \neq id$, then $\{t\} > \{\sigma t\}$

Suppose $t_1, \dots, t_r$ are distinct standard tableaux and have

$$c_1 e_{t_1} + \dots + c_r e_{t_r} = 0.$$

Assume $\{t_1\} < \dots < \{t_r\}$.

$\Rightarrow$ $\{t_r\}$ appears w/ coeff. c_r.

$\Rightarrow$ $c_r = 0$ since tabloids linearly ind. in M^λ.

$\Rightarrow$ repeat to see that $c_1 = \dots = c_{r-1} = 0$.

Span: Need to show for every tableau t that

$$e_t \in \text{span}\{e_s \mid s \text{ standard tableau}\}.$$

Note: if $[t] = [t']$, then $e_t = \pm e_{t'}$

So we may assume entries in each column of t increase top to bottom.

We will prove that $e_t \in \text{Span} \dots$ by descending induction on column equivalence classes.

<u>Base case</u>: largest equivalence class: put $1, \dots \lambda_1^T$ in first column, $\lambda_1^T+1, \dots, \lambda_1^T+\lambda_2^T$ in second column, etc.

This is standard, so nothing to show.

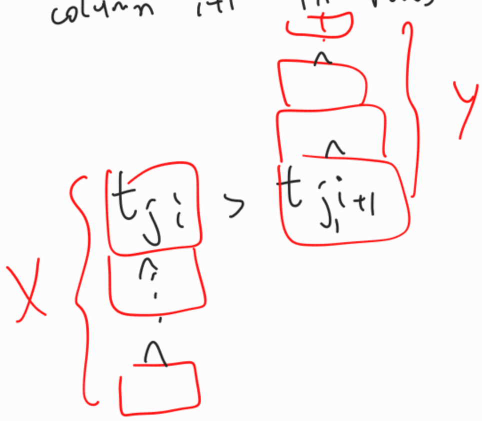
<u>Induction step</u>: let t non-standard tableau.

$\exists\, i,j$ s.t. $t_{j,i} > t_{j,i+1}$ (row j, columns $i, i+1$)

let X = values in column i in rows j and below

Y = values in column $i+1$ in rows j and above.

$|X \cup Y| = \lambda_i^T + 1$



Pick coset reps $\sigma_1, \dots \sigma_k$ for $G_{X\cup Y} / G_X \cup G_Y$.

Assume $\sigma_1 = id$. We have $G_{X,Y}\, e_t = 0$

$$e_t = -\sum_{r=2}^{k} \operatorname{sgn}(\sigma_r)\, e_{\sigma_r t}$$

<u>Claim</u>: $[\sigma_r t] > [t]$ for $r = 2, \dots K$.

<u>Note</u>: $t_{1,i+1} < t_{2,i+1} < \dots < t_{j,i+1} < t_{j,i} < t_{j+1,i} < \dots < t_{\lambda_i^T, i}$

For each σ_r $(r \geq 2)$, some element in X gets moved one column to the right. Consider the biggest one. This will be tiebreaker when comparing $[\sigma_r t] > [t]$. $\square$

<u>Cor.</u> $\dim S^\lambda$ = # standard tableaux of shape λ.

Does not depend on field!