Garnir polynomials

λ = partition of n,

$\mathbb{k}[x_1, \dots, x_n]$ = polynomials in $x_1, \dots, x_n$ w/ coefficents from $\mathbb{k}$.

For non-negative integer d, $\mathbb{k}[x_1, \dots, x_n]_d$ = subspace of homogeneous degree d polynomials

Note: $\mathfrak{S}_n$ acts on $\mathbb{k}[x_1, \dots, x_n]_d$ by permuting variables:

$$\sigma \cdot x_1^{p_1} \cdots x_n^{p_n} = x_{\sigma(1)}^{p_1} \cdots x_{\sigma(n)}^{p_n}$$

Given λ-tableau t, let $m(t) = x_1^{t_1 - 1} \cdots x_n^{t_n - 1}$ where t_i is index of row in which i appears in t.

$$t = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 9 & 7 \\ \hline 5 & 4 & 8 & \\ \cline{1-3} 2 & & & \\ \cline{1-1} \end{array} \qquad m(t) = x_5 x_4 x_8 x_2^2$$

$\deg m(t) = n(\lambda)$ where $n(\lambda) = \sum_i (i-1)\lambda_i$

Note: $m(\sigma \cdot t) = \sigma \cdot m(t)$ & $m(t) = m(t')$ if $t \sim t'$

Prop. $\mathbb{k}$-linear span of $\{m(t) \mid t \ \lambda\text{-tabloid}\}$ is a $\mathfrak{S}_n$-subrep. of $\mathbb{k}[x_1, \dots, x_n]_{n(\lambda)}$ which is isomorphic to M^λ.

Consider new set of variables $y_1, \dots, y_r$.

Lemma. $$\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma)\, y_{\sigma(2)} y_{\sigma(3)}^2 \cdots y_{\sigma(r)}^{r-1} = \prod_{1 \le i < j \le r} (y_j - y_i)$$

Pf. Left side is expansion of $\det\left(y_i^{j-1}\right)_{1 \le i,j \le r}$ which is Vandermonde determinant. $\square$

Define $$\Delta(y_1, \dots, y_r) = \prod_{1 \le i < j \le r} (y_j - y_i)$$

Recall: S^λ spanned by $e_T = \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\{\sigma t\}$

$$e_t \longrightarrow f(t) := \sum_{\sigma \in C_t} (\operatorname{sgn} \sigma)\, \sigma \cdot m(t)$$

$=$ product of Vandermonde determinants, one for each column

Ex. $t = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 9 & 7 \\ \hline 5 & 4 & 8 & \\ \cline{1-3} 2 & & & \\ \cline{1-1} \end{array}$ $\quad f(t) = \Delta(x_1, x_5, x_2)\,\Delta(x_3, x_4)\,\Delta(x_9, x_8)\,\Delta(x_7)$

$= (x_5 - x_1)(x_2 - x_5)(x_2 - x_1)(x_4 - x_3)(x_8 - x_9)$

$f(t)$ are called Garnir polynomials.

Prop. $\Bbbk$-linear span of Garnir polynomials of shape λ is a $\mathfrak{S}_n$-subrep. of $\Bbbk[x_1, \dots, x_n]_{n(\lambda)}$ which is isomorphic to S^λ.

The Garnir polynomials from standard Young tableaux give basis.

Ex. ① If $\lambda = (n)$, then $n(\lambda) = 0$, get trivial rep.

② $\lambda = (n-1, 1)$, $n(\lambda) = 1$.

$t = \begin{array}{|c|c|} \hline i & \cdots \\ \hline j & \\ \cline{1-1} \end{array} \longrightarrow f(t) = x_j - x_i$

Garnir polynomials span $\{c_1 x_1 + \dots + c_n x_n \mid c_1 + \dots + c_n = 0\}$

standard tableaux: $\begin{array}{|c|c|} \hline 1 & \cdots \\ \hline i & \\ \cline{1-1} \end{array}$ $\quad i = 2, \dots, n$

basis: $\{x_2 - x_1,\ x_3 - x_1,\ \dots,\ x_n - x_1\}$

③ $\lambda = (1^n)$, $n(\lambda) = \binom{n}{2} = \dfrac{n(n-1)}{2}$

$t = \begin{array}{|c|} \hline i_1 \\ \hline i_2 \\ \hline \vdots \\ \hline i_n \\ \hline \end{array}$, $f(t) = \Delta(x_{i_1}, \dots, x_{i_n}) = \pm \Delta(x_1, \dots, x_n)$

④ $\lambda = (2,2)$, $n(\lambda) = 2$

2 standard tableaux: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ $\quad$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$

$(x_3 - x_1)(x_4 - x_2) \qquad (x_2 - x_1)(x_4 - x_3)$

$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array}$ not standard, but

$$(x_4 - x_1)(x_3 - x_2) = (x_3 - x_1)(x_4 - x_2) - (x_2 - x_1)(x_4 - x_3)$$