Symmetric Functions

$x_1, \ldots x_n$ finite set of variables

$\mathbb{Z}[x_1, \ldots x_n]$ = polynomials w/ $\mathbb{Z}$-coefficients

$\mathfrak{S}_n$ acts by permuting variables.

Ring of symmetric polynomials: $\Lambda(n) := \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$

$$\Lambda(n) = \{ f \in \mathbb{Z}[x_1, \ldots, x_n] \mid \sigma \cdot f = f \ \forall \sigma \in \mathfrak{S}_n \}$$

$\Lambda(n)$ subring of $\mathbb{Z}[x_1, \ldots x_n]$

Consider $x_1, x_2, \ldots$ countable set of variables.

$\mathfrak{S}_\infty$ = permutations of $\{1, 2, \ldots\}$

R = set of power series in $x_1, x_2, \ldots$ of $\boxed{\text{bounded degree}}$ w/ $\mathbb{Z}$-coeff.

$\mathfrak{S}_\infty$ acts on R by permuting variables

$$\Lambda := R^{\mathfrak{S}_\infty} = \{ f \in R \mid \sigma \cdot f = f \ \forall \sigma \in \mathfrak{S}_\infty \}$$

ring of symmetric functions.

R is a ring, Λ is a subring.

For every n, have $\pi_n : \Lambda \to \Lambda(n)$ [set $x_i = 0$ for $i > n$]

$$f \to f(x_1, \ldots, x_n, 0, 0, \ldots)$$

For each integer $d \geq 0$, let

$$\Lambda(n)_d = \{ f \in \Lambda(n) \mid f \text{ homogeneous of degree } d \}$$

$$\Lambda_d = \{ f \in \Lambda \mid \text{———} \}$$

$$\Lambda(n) = \bigoplus_{d \geq 0} \Lambda(n)_d, \qquad \Lambda = \bigoplus_{d \geq 0} \Lambda_d$$

Let $\Lambda(n)_{\mathbb{Q}}$ be symmetric polynomials in $x_1, \ldots x_n$ w/ $\mathbb{Q}$-coeff.

$\Lambda_{\mathbb{Q}}$ be symmetric functions w/ $\mathbb{Q}$-coeff.

Ex. $p_k := \sum_{i \geq 1} x_i^k = x_1^k + x_2^k + \cdots$ (power sum)

$e_k := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ (elementary)

$h_k := \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$ (complete homog.)

sum over all monomials of degree k

$e_1 = h_1$ $\quad e_2 = x_1x_2 + x_1x_3 + \cdots + x_2x_3 + x_2x_4 + \cdots$

$h_2 = e_2 + p_2$

$h_3 = e_3 + p_3 + \sum_{i \neq j} x_i^2 x_j$

Monomial symmetric functions

Given sequence $(\alpha_1, \alpha_2, \ldots)$ finitely many nonzero entries

$$x^\alpha := \prod_{i \geq 1} x_i^{\alpha_i} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

Given partition λ, define monomial symmetric function by

$$m_\lambda := \sum_\alpha x^\alpha$$

$\alpha \rightarrow$ permutations of $(\lambda_1, \lambda_2, \ldots)$ (sum only over distinct values)

Ex. $m_1 = x_1 + x_2 + \cdots = p_1 = e_1 = h_1$

$m_{1,1} = \sum_{i<j} x_i x_j = e_2$

in general, $m_{1^k} = e_k$ & $m_k = p_k$

$$m_{321} = \sum_{\substack{i,j,k \\ i \neq j,\, i \neq k,\, j \neq k}} x_i^3 x_j^2 x_k$$

Thm. $\{m_\lambda \mid \lambda \text{ partition}\}$ is a basis for Λ.

Pf. Linear independence: no 2 m_λ's share common monomial

Span: given $f \in \Lambda$, write $f = \sum_\alpha c_\alpha x^\alpha$, $c_\alpha \in \mathbb{Z}$

& $\exists$ d s.t. $|\alpha| \le d$

$c_\alpha = c_\beta$ if α is permutation of β, so can write $f = \sum_\lambda c_\lambda m_\lambda$

partitions → finite sum $\square$

Cor. Λ_d has basis $\{m_\lambda \mid |\lambda| = d\}$, hence is free abelian group of rank $p(d)$.

$\Lambda(n)_d$ has a basis $\{m_\lambda(x_1, \dots, x_n) \mid |\lambda| = d,\ \ell(\lambda) \le n\}$.

Elementary symmetric functions

For partition $\lambda = (\lambda_1, \dots, \lambda_k)$, define

$$e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$$

elementary symmetric functions.

$$e_\lambda \in \Lambda_{|\lambda|}.$$

$\exists$ coeff. $M_{\lambda\mu} \in \mathbb{Z}$ s.t.

$$e_\lambda = \sum_\mu M_{\lambda\mu} m_\mu.$$

Given infinite matrix A w/ finitely many nonzero entries,

let $$\operatorname{row}(A) = \Big(\sum_{i \ge 1} A_{1,i},\ \sum_{i \ge 1} A_{2,i},\ \dots \Big)$$

$$\operatorname{col}(A) = \Big(\sum_{j \ge 1} A_{j,1},\ \sum_{j \ge 2} A_{j,2},\ \dots \Big)$$

A is $(0,1)$-matrix if every entry is 0 or 1.

Lemma. $M_{\lambda,\mu}$ is # of (0,1)-matrices w/ $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.

Pf. $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} = \left(\sum x_{i_1} x_{i_2} \cdots x_{i_{\lambda_1}} \right) \left(\sum x_{j_1} \cdots x_{j_{\lambda_2}} \right) \cdots$

$M_{\lambda\mu}$ = coeff. of x^μ in e_λ

= sum over all choices of monomials whose product is x^μ

Given monomial x^α, encode as sequence $(\alpha_1, \alpha_2, \ldots)$

where α_i = exponent of x_i

gives (0,1)-matrix A by letting row i be sequence for monomial chosen from e_{λ_i} s.t. $\mathrm{col}(A) = \mu$ & $\mathrm{row}(A) = \lambda$. $\square$

Cor. $M_{\lambda,\mu} = M_{\mu,\lambda}$

Pf. Get bijection between matrices by taking transpose. $\square$

Thm. If $M_{\lambda\mu} \neq 0$, then $\mu \leq \lambda^T$. Also, $M_{\lambda,\lambda^T} = 1$.

In particular, $\{e_\lambda \mid \lambda \text{ partition}\}$ is basis for Λ.

Pf. Suppose $M_{\lambda\mu} \neq 0$. Then $\exists A$ (0,1)-matrix w/ $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.

Let A' be result of left-justifying all 1's in each row

$\mathrm{row}(A') = \lambda$, $\mathrm{col}(A') = \lambda^T$.

For each i, number of 1's in first i columns of A'

$\geq$ number of 1's in first i columns of A

$\Rightarrow \lambda_1^T + \cdots + \lambda_i^T \geq \mu_1 + \cdots + \mu_i \quad \forall i \Rightarrow \mu \leq \lambda^T$.

Also note if $\mu = \lambda^T$, then 1's must already be left-justified

$\Rightarrow$ only one way, so $M_{\lambda,\lambda^T} = 1$.

Lemma. Let $a_{\lambda,\mu}$ be integers indexed by partitions of size n.

Assume that: $\cdot\ a_{\lambda\lambda} = 1 \quad \forall \lambda$

$\cdot\ a_{\lambda\mu} \neq 0 \Rightarrow \mu \leq \lambda$

For any ordering of $\mathrm{Par}(n)$, $(a_{\lambda\mu})_{\lambda,\mu}$ is invertible (over $\mathbb{Z}$)

i.e. has $\det = \pm 1$.

Same conclusion if instead we have $\cdot\ a_{\lambda\lambda^T} = 1$, $\cdot\ a_{\lambda\mu} \neq 0 \Rightarrow \mu \leq \lambda^T$

$\Rightarrow \det(M_{\lambda\mu}) = \pm 1$ $\square$

Thm. $\left\{ e_\lambda(x_1,\dots,x_n) \,\middle|\, \begin{matrix} \lambda_1 \leq n \\ |\lambda| = d \end{matrix} \right\}$ is basis for $\Lambda(n)_d$

Rmk. $\{e_\lambda\}$ basis for $\Lambda \Rightarrow e_1, e_2, \dots$ are algebraically independent

all nontrivial polynomial expressions in $e_1, e_2,$ are nonzero

$\{e_\lambda(x_1,\dots,x_n) \mid \lambda_1 \leq n\}$ basis for $\Lambda(n)$

$\Rightarrow$ $e_1(x_1,\dots,x_n), \dots e_n(x_1,\dots,x_n)$ are algebraically independent.

fundamental theorem of symmetric polynomials/functions