The involution ω

Since $e_1, e_2, \ldots$ alg. independent generators for Λ

we can define ring homomorphism $f: \Lambda \longrightarrow R$ (R = comm ring)

by picking $f(e_1), f(e_2), \ldots$ arbitrarily:

Every $r \in \Lambda$ is uniquely of the form $\sum_\lambda c_\lambda e_{\lambda_1} \cdots e_{\lambda_k}$

$\rightsquigarrow f(r) = \sum_\lambda c_\lambda f(e_{\lambda_1}) \cdots f(e_{\lambda_k})$

Define $\omega: \Lambda \to \Lambda$ by $\omega(e_i) = h_i$ for $i \geq 1$

where $h_n = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n}$ (sum of all monomials of degree n)

Thm. $\omega^2 = id$, equivalently, $\omega(h_n) = e_n$ for all $n \geq 1$.

Pf. Consider $\Lambda[[t]]$ = power series w/ coefficients in Λ.

Define $E(t) = \sum_{n \geq 0} e_n t^n$, $H(t) = \sum_{n \geq 0} h_n t^n$

We have $E(t) = \prod_{i \geq 1} (1 + x_i t)$

$$H(t) = \prod_{i \geq 1} (1 + x_i t + x_i^2 t^2 + \ldots) = \prod_{i \geq 1} (1 - x_i t)^{-1}$$

$\Rightarrow$ $E(t) H(-t) = 1$

$\uparrow$ nth coefficient is 0 for $n > 0$

$$0 = (-1)^n \sum_{i=0}^{n} e_i (-1)^i h_{n-i} \xrightarrow{\text{Apply } \omega} 0 = \sum_{i=0}^{n} (-1)^i h_i \omega(h_{n-i})$$

$\Rightarrow$ $\sum_{n \geq 0} \omega(h_n) t^n$ is inverse of $H(-t)$

$\Rightarrow$ $\sum_{n \geq 0} \omega(h_n) t^n = E(t)$ $\Rightarrow$ $\omega(h_n) = e_n$ for $n \geq 1$. $\square$

$\Lambda(n)$ is polynomial ring w/ generators $e_1(x_1,\ldots x_n), \ldots e_n(x_1,\ldots,x_n)$

Define $\omega_n : \Lambda(n) \to \Lambda(n)$ by $\omega_n(e_i) = h_i$

Thm. $\omega_n^2 = id$, equivalently, $\omega_n(h_i) = e_i$ for $i=1,\ldots,n$.

Pf. Define $E_n(t) = \sum_{i=0}^{n} e_i(x_1,\ldots,x_n)\, t^i$, $H_n(t) = \sum_{i\geq 0} h_i(x_1,\ldots,x_n)\, t^i$.

$$\Rightarrow \quad E_n(t) = \prod_{i=1}^{n} (1+x_i t), \quad H_n(t) = \prod_{i=1}^{n} (1-x_i t)^{-1}$$

$$\Rightarrow \quad E_n(t)\, H_n(-t) = 1$$

$$\Rightarrow \sum_{i=0}^{\min(k,n)} e_i(x_1,\ldots,x_n)(-1)^{k-i}\, h_{k-i}(x_1,\ldots x_n) = 0 \quad \text{for } k>0.$$

$$\xRightarrow{\omega_n} \sum_{i=0}^{\min(k,n)} h_i(x_1,\ldots x_n)(-1)^{k-i}\, \omega_n(h_{k-i}(x_1,\ldots x_n)) = 0$$

In particular, $\sum_{i\geq 0} \omega_n(h_i(x_1,\ldots,x_n))\, t^i$ agrees w/ inverse of $H(-t)$ up to degree n. $\Rightarrow \omega_n(h_i) = e_i$ for $i=1,\ldots,n$. $\square$

Complete homogeneous symmetric functions

For partition $\lambda = (\lambda_1,\ldots,\lambda_k)$ define

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$$

$\Rightarrow \omega(e_\lambda) = h_\lambda$.

Thm. $\{h_\lambda \mid \lambda \in Par\}$ is basis for Λ.

Pf. ω is isom, and $\{e_\lambda \mid \lambda \in Par\}$ is a basis. $\square$

Note: $\exists$ coeffs $N_{\lambda,\mu}$ s.t.

$$h_\lambda = \sum_\mu N_{\lambda\mu} \, m_\mu .$$

$N_{\lambda,\mu}$ = # of matrices A w/ non-negative integer entries s.t. $\mathrm{row}(A) = \lambda$, $\mathrm{col}(A) = \mu$.

Thm. $h_1(x_1, \dots, x_n), \dots, h_n(x_1, \dots x_n)$ are alg. independent generators for $\Lambda(n)$ and $\{ h_\lambda(x_1, \dots x_n) \mid \lambda_1 \leq n, |\lambda| = d \}$ is a basis for $\Lambda(n)_d$.