Power sum symmetric functions

$p_k = \sum_{i\geq 1} x_i^k$. For partition $\lambda = (\lambda_1, \dots, \lambda_k)$ define

$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$. power sum symmetric functions

Define $$P(t) = \sum_{n\geq 1} p_n t^{n-1} \in \Lambda[[t]]$$

Recall $E(t) = \sum_{n\geq 0} e_n t^n$, $H(t) = \sum_{n\geq 0} h_n t^n$

<u>Lemma</u>. $$P(t) = \frac{d}{dt} \log H(t) \quad \left(= \frac{H'(t)}{H(t)}\right), \qquad P(-t) = \frac{d}{dt} \log E(t) \quad \left(= \frac{E'(t)}{E(t)}\right)$$

<u>Pf</u>. $$\frac{d}{dt} \log H(t) = \frac{d}{dt} \log \left(\prod_{i\geq 1} (1-x_i t)^{-1} \right)$$

$$= \sum_{i\geq 1} \frac{d}{dt} \log (1-x_i t)^{-1}$$

$$= \sum_{i\geq 1} \frac{x_i}{1-x_i t} = \sum_{i\geq 1} x_i \sum_{n\geq 0} x_i^n t^n = \sum_{i\geq 1} \sum_{n\geq 1} x_i^n \, t^{n-1}$$

$$= P(t).$$ $\square$

λ partition, $m_i(\lambda) = $ # of times i appears in λ.

$$z_\lambda := \prod_{i\geq 1} i^{m_i(\lambda)} m_i(\lambda)!, \qquad \varepsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}$$

<u>Thm</u>. $$E(t) = \sum_\lambda \varepsilon_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|} \qquad H(t) = \sum_\lambda z_\lambda^{-1} p_\lambda t^{|\lambda|}$$

$$\rightsquigarrow e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda \qquad \rightsquigarrow h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda .$$

Pf. Start w/ $p(t) = \frac{d}{dt} \log H(t)$

Integrate: $\sum_{n \geq 1} \frac{p_n}{n} t^n = \log H(t)$

Exponentiate: $\exp\left(\sum_{n \geq 1} \frac{p_n}{n} t^n \right) = H(t)$

$[\exp(F(t)) = \sum_{n \geq 0} F(t)^n / n!$ if $F(0) = 0]$

$[p_\lambda = \prod_{n \geq 1} p_n^{m_n(\lambda)}]$

$$H(t) = \prod_{n \geq 1} \exp\left(\frac{p_n}{n} t^n \right)$$

$$= \prod_{n \geq 1} \sum_{d \geq 0} \frac{p_n^d}{n^d} \cdot \frac{t^{nd}}{d!} = \sum_{\lambda \in \mathrm{Par}} \frac{p_\lambda t^{|\lambda|}}{z_\lambda} \qquad \square$$

Thm. $p_1, p_2, \ldots$ are algebraically independent generators for $\Lambda_{\mathbb{Q}}$. i.e., $\{p_\lambda \mid \lambda \in \mathrm{Par}\}$ is a basis for $\Lambda_{\mathbb{Q}}$.

$p_1(x_1, \ldots x_n), \ldots, p_n(x_1, \ldots, x_n)$ are algebraically independent generators for $\Lambda(n)_{\mathbb{Q}}$.

and $\left\{ p_\lambda(x_1, \ldots, x_n) \,\middle|\, \begin{array}{l} \lambda_1 \leq n \\ |\lambda| = d \end{array} \right\}$ is basis for $\Lambda(n)_{\mathbb{Q},d}$.

Pf. We have $e_n = \sum_{|\lambda| = n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$.

which tells us that p_λ span $\Lambda_{\mathbb{Q}}$ since every e_λ is a product of these sums and product of p_λ's is another p_μ. Note, $p_\lambda \in \Lambda_{|\lambda|}$ and $\dim \Lambda_{\mathbb{Q},d} = p(d)$

$\Rightarrow \{p_\lambda \mid |\lambda| = d\}$ is basis for $\Lambda_{\mathbb{Q}, d}$. $\square$

Rmk. $p_2 = m_2$

$p_{1,1} = p_1^2 = m_2 + 2m_{1,1}$ $\rightsquigarrow$ change of basis matrix has det 2 $\Rightarrow \{p_\lambda\}$ don't span Λ

$\underline{\text{Cor}}$. $\omega(p_\lambda) = \varepsilon_\lambda p_\lambda$

$\underline{\text{pf}}$. Induction on λ_1.

When $\lambda_1 = 1$, we have $p_{1^n} = e_1^n = h_1^n$ and $\omega(e_1^n) = h_1^n$

and $\varepsilon_n = (-1)^{n-n} = 1$ ✓

Now assume known for λ whenever $\lambda_1 < n$.

Start w/ $e_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda$ Apply ω

$$h_n = \sum_{|\lambda|=n} \varepsilon_\lambda z_\lambda^{-1} \omega(p_\lambda) = \varepsilon_{(n)} z_{(n)}^{-1} \omega(p_n) + \sum_{\substack{|\lambda|=n \\ \lambda_1 < n}} z_\lambda^{-1} p_\lambda$$

Also: $$h_n = \sum_{|\lambda|=n} z_\lambda^{-1} p_\lambda = z_{(n)}^{-1} p_n + \sum_{\substack{|\lambda|=n \\ \lambda_1 < n}} z_\lambda^{-1} p_\lambda$$

$$\Rightarrow 0 = z_n^{-1}\left(\varepsilon_{(n)} \omega(p_n) - p_n\right)$$

$\Rightarrow \varepsilon_{(n)} p_n = \omega(p_n) \Rightarrow$ identity holds for $\lambda = (n)$

Suppose λ is general s.t. $\lambda_1 = n$.

$$\omega(p_\lambda) = \omega(p_{\lambda_1}) \cdots \omega(p_{\lambda_k}) = \varepsilon_{\lambda_1} \cdots \varepsilon_{\lambda_k} p_\lambda = \varepsilon_\lambda p_\lambda \quad ✓$$

$$\boxed{\varepsilon_{\lambda_1} \cdots \varepsilon_{\lambda_k} = (-1)^{\lambda_1 - 1} \cdots (-1)^{\lambda_k - 1} = (-1)^{|\lambda| - \ell(\lambda)} = \varepsilon_\lambda}$$

$\square$