Scalar product

Define $\langle\, ,\rangle : \Lambda \otimes \Lambda \to \mathbb{Z}$ by setting $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$

If $f, g \in \Lambda$, $f = \sum_\lambda a_\lambda m_\lambda$, $g = \sum_\mu b_\mu h_\mu$,

then $\langle f, g\rangle = \sum_\lambda a_\lambda b_\lambda$

Prop. $\sum_\lambda m_\lambda(x) h_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$

Pf. $\prod_i \prod_j (1 - x_i y_j)^{-1} = \prod_i \sum_{n \geq 0} h_n(y) x_i^n = \sum_\alpha h_\alpha(y) x^\alpha \quad (*)$

sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$

$h_\alpha(y) = h_{\alpha_1}(y) h_{\alpha_2}(y) \cdots, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$

$(*) = \sum_\lambda h_\lambda(y) m_\lambda(x)$ $\square$

Prop. Let $\{u_\lambda\}_\lambda$, $\{v_\mu\}_\mu$ be homogeneous bases for Λ (or $\Lambda_\mathbb{Q}$)

s.t. $\deg(u_\lambda) = |\lambda|$. Then
$\deg(v_\mu) = |\mu|$

$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \iff \sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$

Pf. Write $u_\lambda = \sum_\alpha a_{\lambda,\alpha} m_\alpha$, $v_\mu = \sum_\beta b_{\mu,\beta} h_\beta$

Then $\langle u_\lambda, v_\mu \rangle = \sum_\gamma a_{\lambda\gamma} b_{\mu,\gamma}$

$\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu} \iff \sum_\gamma a_{\lambda\gamma} b_{\mu\gamma} = \delta_{\lambda\mu} \quad (*)$

Fix d pick ordering on $\mathrm{Par}(d)$

Set $A = (a_{\lambda,\alpha})$, $B = (b_{\mu,\beta})$ matrices w.r.t. this order.

$$(*) \iff AB^T = I \iff B^T = A^{-1} \iff B^T A = I$$

$$\iff \sum_\gamma a_{\gamma\lambda} b_{\gamma\mu} = \delta_{\lambda\mu}$$

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\lambda \sum_{\alpha,\beta} a_{\lambda\alpha} b_{\lambda\beta} m_\alpha(x) h_\beta(y) = \sum_{\alpha,\beta} \Big(\sum_\lambda a_{\lambda\alpha} b_{\lambda\beta} \Big) m_\alpha(x) h_\beta(y)$$

By linear independence, $\sum_\lambda u_\lambda(x) v_\lambda(y) = \sum_\lambda m_\lambda(x) h_\lambda(y)$ agree

$$\iff \sum_\lambda a_{\lambda\alpha} b_{\lambda\beta} = \delta_{\alpha\beta}$$ $\square$

Cor. $\langle f, g \rangle = \langle g, f \rangle$ for all f, g.

Pf. $\prod_{i,j} (1 - x_i y_j)^{-1}$ is invariant under swapping x, y.

So: $$\sum_\lambda m_\lambda(y) h_\lambda(x) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

$$\Rightarrow \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu} = \langle m_\mu, h_\lambda \rangle$$ $\square$

Prop. $$\sum_\lambda z_\lambda^{-1} p_\lambda(x) p_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

So: $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$, $\{p_\lambda\}$ orthogonal basis for $\Lambda_{\mathbb{Q}}$.

Pf. $$\prod_j \prod_i (1 - x_i y_j)^{-1} = \prod_j \sum_{n \geq 0} h_n(x) y_j^n$$

$$= \prod_j \exp\Big(\sum_{n \geq 1} \frac{p_n(x) y_j^n}{n} \Big)$$

$$= \exp\Big(\sum_j \sum_{n \geq 1} \frac{p_n(x) y_j^n}{n} \Big)$$

$$= \exp\left(\sum_{n \geq 1} \frac{p_n(x)\, p_n(y)}{n} \right)$$

$$= \prod_{n \geq 1} \exp\left(\frac{p_n(x)\, p_n(y)}{n} \right) = \prod_{n \geq 1} \sum_{d \geq 0} \frac{p_n(x)^d\, p_n(y)^d}{n^d\, d!}$$

$$= \sum_{\lambda} \frac{p_\lambda(x)\, p_\lambda(y)}{z_\lambda} \qquad \square$$

<u>Cor.</u> $\forall f, g \in \Lambda_{\mathbb{Q}}$, $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$

i.e., ω is an isometry.

<u>Pf.</u> $\langle \omega(p_\lambda), \omega(p_\mu) \rangle = \varepsilon_\lambda \varepsilon_\mu \langle p_\lambda, p_\mu \rangle = \varepsilon_\lambda \varepsilon_\mu z_\lambda \delta_{\lambda\mu}$

$$= z_\lambda \delta_{\lambda\mu} = \langle p_\lambda, p_\mu \rangle.$$

Write $f = \sum_\lambda a_\lambda p_\lambda$, $g = \sum_\mu b_\mu p_\mu$.

Then $\langle \omega(f), \omega(g) \rangle = \langle \sum_\lambda a_\lambda \omega(p_\lambda), \sum_\mu b_\mu \omega(p_\mu) \rangle$

$$= \sum_{\lambda,\mu} a_\lambda b_\mu \langle \omega(p_\lambda), \omega(p_\mu) \rangle$$

$$= \sum_{\lambda,\mu} a_\lambda b_\mu \langle p_\lambda, p_\mu \rangle = \langle \sum_\lambda a_\lambda p_\lambda, \sum_\mu b_\mu p_\mu \rangle = \langle f, g \rangle. \qquad \square$$

<u>Cor.</u> $\langle \,,\, \rangle$ is positive definite: if $f \in \Lambda_{\mathbb{R}}$ and $f \neq 0$ then $\langle f, f \rangle > 0$.

<u>Pf.</u> Assume $f \neq 0$. Write $f = \sum_\lambda a_\lambda p_\lambda$. Then

$$\langle f, f \rangle = \langle \sum_\lambda a_\lambda p_\lambda, \sum_\mu a_\mu p_\mu \rangle = \sum_\lambda a_\lambda^2 z_\lambda.$$

since $z_\lambda > 0$ and $a_\lambda^2 \geq 0$ and at least one $a_\lambda^2 > 0$ get last sum is positive. $\square$