Semistandard Young Tableaux

λ = partition, a <u>semistandard Young tableau</u> (SSYT) of shape λ is a filling T of $Y(\lambda)$ w/ positive integers s.t.:

- entries weakly increase in each row left to right
- entries strictly increase in each column top to bottom.

<u>Ex</u>. $\lambda = (4,3,1)$

$$\begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & \\ \cline{1-3} h & & & \\ \cline{1-1}\end{array}$$

$a \leq b \leq c \leq d$

$e \leq f \leq g$

$a < e < h$

$b < f$

$c < g$

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 3 & 5 \\ \hline 2 & 3 & 4 & \\ \cline{1-3} 7 & & & \\ \cline{1-1}\end{array}$$

$\text{type}(T) = (2, 1, 2, 1, 1, 0, 1, 0, 0, \ldots)$

$x^T = x_1^2 x_2 x_3^2 x_4 x_5 x_7$

The <u>type</u> of SSYT T is $\text{type}(T) = (\alpha_1, \alpha_2, \ldots)$ where α_i = # times i appears in T.

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

The <u>Schur function</u> is $\displaystyle s_\lambda = \sum_{T} x^T$

$T \to$ all SSYT T of shape λ

Given $\mu \subseteq \lambda$, we define $Y(\lambda/\mu)$ to be $Y(\lambda) \setminus Y(\mu)$.

Define SSYT of λ/μ to be filling of $Y(\lambda/\mu)$ w/ same conditions.

<u>Ex</u>. $\lambda = (5,3,1)$, $\mu = (2,1)$

$$\begin{array}{ccccc} & & \boxed{a} & \boxed{b} & \boxed{c} \\ & \boxed{d} & \boxed{e} & & \\ \boxed{f} & & & & \end{array}$$

$a \leq b \leq c$

$d \leq e$

$a < e$

Define type(T), x^T same way.

The <u>skew Schur function</u> $s_{\lambda/\mu} = \sum_T x^T$

$T \to$ all SSYT T of shape $Y(\lambda/\mu)$

If $\mu = \emptyset$, then $s_{\lambda/\emptyset} = s_\lambda$.

Can make same definitions if we use finitely many values $1, \dots k$ to get polynomials $s_{\lambda/\mu}(x_1, \dots, x_k)$

<u>Ex</u>. $\lambda = (1)$ $\boxed{a}$ $s_1 = \sum_{i \geq 1} x_i = h_1 = e_1 = m_1 = p_1$

$\lambda = (1,1)$ $\begin{array}{|c|}\hline a \\ \hline b \\ \hline\end{array}$ $a < b$ $s_{11} = \sum_{i<j} x_i x_j = e_2$

More generally, $s_{1^k} = e_k$

$\lambda = (k)$: $\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \dots & a_k \\ \hline\end{array}$ $a_1 \leq \dots \leq a_k$

$s_k = h_k$

$\lambda = (2,1)$ in 3 variables:

$\begin{array}{c}1\,1 \\ 2\end{array}$ $\begin{array}{c}1\,1 \\ 3\end{array}$ $\begin{array}{c}1\,2 \\ 2\end{array}$ $\begin{array}{c}1\,2 \\ 3\end{array}$ $\begin{array}{c}1\,3 \\ 2\end{array}$ $\begin{array}{c}1\,3 \\ 3\end{array}$ $\begin{array}{c}2\,2 \\ 3\end{array}$ $\begin{array}{c}2\,3 \\ 3\end{array}$

$s_{21}(x_1, x_2, x_3) = m_{21}(x_1, x_2, x_3) + \underline{2 m_{111}(x_1, x_2, x_3)}$

<u>Thm</u>. For all $\mu \subseteq \lambda$, $s_{\lambda/\mu}$ is symmetric.

<u>pf</u>. <u>Need to show</u>: for any α and permutation σ,

SSYT of type α and shape λ/μ = # SSYT of type $\sigma(\alpha)$ and shape λ/μ.

Suffices to consider σ that are identity on all but finitely many indices.

By transitivity, enough to consider $\sigma = (i, i+1)$.

Define function

$$\left\{ \begin{array}{c} \text{SSYT of shape } \lambda/\mu \\ \text{and type } \alpha \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{SSYT of shape } \lambda/\mu \\ \text{and type } (i,i+1)\cdot\alpha \end{array} \right\}$$

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Given T, consider columns of λ/μ, separate into 3 cases:

1. Both i and $i+1$ appear in that column
2. Neither appear in that column.

} ignore these.

3. Exactly one of $i, i+1$ appears in that column.

focus on these entries:

set disjoint copies of

no i above here

$$\underbrace{i \; i \; i \cdots i}_{a} \; \underbrace{i+1 \; i+1 \cdots i+1}_{b} \qquad a, b \geq 0$$

no $i+1$ below

swap a, b:

$$\underbrace{i \cdots i}_{b} \; \underbrace{i+1 \cdots i+1}_{a}$$

Doing all these swaps gives SSYT of shape λ/μ and type $(i,i+1)\cdot\alpha$. This function is bijective. $\square$

Let $K_{\lambda,\alpha}$ = # of SSYT of shape λ and type α. (Kostka numbers)

From above, $K_{\lambda,\alpha} = K_{\lambda,\sigma(\alpha)}$ for any permutation σ.

$$\Longrightarrow \quad s_\lambda = \sum_{\mu} K_{\lambda,\mu} m_\mu$$

μ s.t. $|\mu| = |\lambda|$.

If $|\lambda| = n$, then $K_{\lambda,1^n}$ = # standard Young tableaux of shape λ $=: f^\lambda$.

<u>Thm.</u> If $K_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$. Also $K_{\lambda\lambda} = 1$.

In particular, $\{ s_\lambda \mid |\lambda| = d \}$ is a basis for Λ_d

$\{ s_\lambda \mid \lambda \in \mathrm{Par} \}$ is a basis for Λ.

Pf. Suppose $K_{\lambda\mu} \neq 0$. Pick SSYT T of shape λ, type μ.

Note: for each k, k must appear in first k rows of $Y(\lambda)$

$\Rightarrow \mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$. $\Rightarrow \mu \leq \lambda$.

A SSYT of shape λ and type λ must have all values k in k^{th} row. $\Rightarrow K_{\lambda\lambda} = 1$. $\square$

Cor. $\left\{ s_\lambda(x_1, \ldots x_n) \;\middle|\; \begin{array}{l} |\lambda| = d \\ \ell(\lambda) \leq n \end{array} \right\}$ is a basis for $\Lambda(n)_d$.

Pf. If $\ell(\lambda) > n$, no SSYT of shape λ using only $1, \ldots, n$.

Now $\#\left\{ \lambda \;\middle|\; \begin{array}{l} |\lambda| = d \\ \ell(\lambda) \leq n \end{array} \right\} = \operatorname{rank} \Lambda(n)_d$, so is a basis since it spans. $\square$