Murnaghan-Nakayama rule

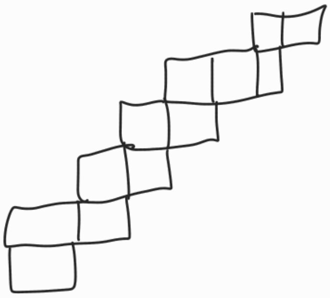
$$\chi^{\lambda} = ch^{-1}(s_\lambda), \quad \chi^{\lambda/\nu} = ch^{-1}(s_{\lambda/\nu})$$

$$\chi^{\lambda} = \sum_{\mu} \chi^{\lambda}(\mu)\, 1_\mu \xrightarrow{ch} s_\lambda = \sum_{\mu} z_\mu^{-1} \chi^{\lambda}(\mu)\, p_\mu .$$

A border strip is a connected skew diagram w/ subdiagram of the form

→ by edges, corners touching don't count

ht = 5



height of B is ht(B) = #rows of B − 1

Thm. r positive integer. Then

$$s_\mu p_r = \sum_{\lambda} (-1)^{ht(\lambda/\mu)} s_\lambda$$

λ → λ s.t. λ/μ borderstrip of size r

Pf. Assume n variables, but n>>0.

Recall, $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1,\dots,n}$ where $\alpha = (\alpha_1, \dots, \alpha_n)$

$\rho = (n-1, n-2, \dots, 1, 0)$, $\varepsilon_j = (0, \dots, \underset{\uparrow j}{1}, \dots 0)$

Know: $s_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$.

$$a_{\mu+\rho} p_r = \left(\sum_{\sigma \in S_n} sgn(\sigma)\, x^{\sigma(\mu+\rho)} \right) \left(\sum_{i=1}^{n} x^{r\varepsilon_i} \right)$$

$$= \sum_{i=1}^{n} \sum_{\sigma \in S_n} sgn(\sigma)\, x^{\sigma(\mu+\rho+r\varepsilon_i)} = \sum_{i=1}^{n} a_{\mu+\rho+r\varepsilon_i} \qquad (*)$$

Note: $\mu + r\varepsilon_i$ might not be a partition!

Observations: if $\beta = (i-1, i)\cdot \alpha$ then $a_\alpha = -a_\beta$

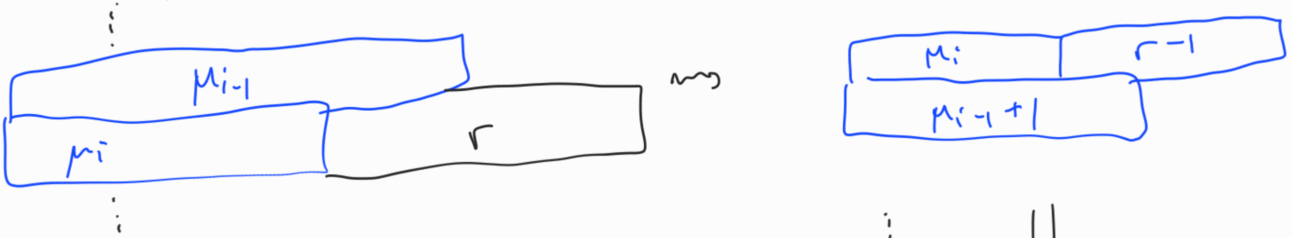
In particular, $a_{\alpha+\rho} = -a_{\alpha'+\rho}$ where $\alpha' = (\dots, \alpha_i - 1, \alpha_{i-1}+1, \dots)$

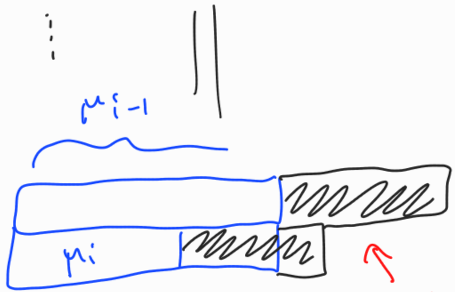
If $\mu + r\varepsilon_i + \rho$ has repeated entry, Then $a_{\mu+\rho+r\varepsilon_i} = 0$

Else, $\exists$ unique reordering so that it is strictly decreasing.

If $\mu + r\varepsilon_i$ not weakly decreasing, Then

$\mu_{i-1} < \mu_i + r$ $\qquad$ $\mu + r\varepsilon_i \to (\mu + r\varepsilon_i)'$





border strip of height 1 of size r.

$\Rightarrow$ $a_{\mu+r\varepsilon_i+\rho} = (-1)^{ht(\lambda/\mu)} a_{\lambda+\rho}$

for some partition λ

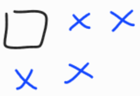
s.t. λ/μ border strip of size r.

$\Rightarrow$ Divide $(*)$ by a_ρ to get

$$s_\mu p_r = \sum_{\lambda \dots} (-1)^{ht(\lambda/\mu)} s_\lambda \qquad \square$$

Ex. $s_1 p_4 = s_5 - s_{32} + s_{221} - s_{15}$

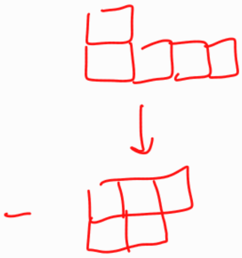
$a_{1+\rho+4\varepsilon_1}$

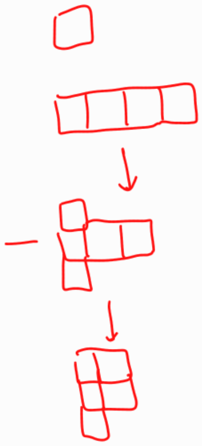


$a_{1+\rho+4\varepsilon_2}$

$a_{1+\rho+4\varepsilon_3}$

$a_{1+\rho+4\varepsilon_5}$





Let $\alpha = (\alpha_1, \dots, \alpha_k)$. A *borderstrip tableau* shape λ/μ, type α is a sequence of partitions $\mu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(k)} = \lambda$ s.t. $\lambda^{(i)}/\lambda^{(i-1)}$ border strip of size α_i

height is $\sum_i ht(\lambda^{(i)}/\lambda^{(i-1)})$ $(ht(\emptyset) = 0)$

Cor. $$s_\mu p_\alpha = \sum_\lambda \sum_T (-1)^{ht(T)} s_\lambda$$

$T \hookrightarrow$ borderstrip tableau of shape λ/μ and type α.

Cor. $$\chi^{\lambda/\nu}(\mu) = \sum_T (-1)^{ht(T)}$$

$T \to$ border strip tableau of shape λ/ν type μ

Pf. $$\chi^{\lambda/\nu}(\mu) = \langle s_{\lambda/\nu}, p_\mu \rangle = \langle p_\mu, s_{\lambda/\nu} \rangle$$
$$= \langle s_\nu p_\mu, s_\lambda \rangle$$
$$= \sum_T (-1)^{ht(T)}$$ □

Cor. Let $n = |\lambda/\mu|$. Then $\chi^{\lambda/\nu}(1^n) = f^{\lambda/\nu} :=$ # SYT of shape λ/ν. In particular, if $\nu = \phi$, $\chi^\lambda(1^n) > 0$, so χ^λ is an irreducible character.

Pf. Border-strip of type 1^n is same as increasing sequence $\nu = \lambda^{(0)} \subseteq \lambda^{(1)} \subseteq \dots \subseteq \lambda^{(n)} = \lambda$ where $\lambda^{(i)}/\lambda^{(i-1)}$ is single box. Label that box w/ i: get SYT. $\square$

Cor. Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$ are non-negative.

Pf. $s_\nu s_\mu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ Apply ch^{-1}:

$$\underbrace{\chi^\nu \circ \chi^\mu}_{\text{character of some rep.}} = \sum_\lambda c^\lambda_{\mu\nu} \chi^\lambda$$

multiplicity of S^λ in that representation

$\Rightarrow c^\lambda_{\mu\nu} \geq 0$. $\square$