Standard Young Tableaux

$f^\lambda = \#\,$SYT of shape λ

Thm. Pick $k \geq \ell(\lambda)$, $l_i = \lambda_i + k - i$. Then $(n = |\lambda|)$

$$f^\lambda = \frac{n!}{l_1! \cdots l_k!} \prod_{1 \leq i < j \leq k} (l_i - l_j)$$

Pf.

$$s_1^n = \sum_{|\lambda| = n} f^\lambda s_\lambda$$

Work in k variables $x_1, \dots, x_k$. Multiply by a_ρ:

$$a_\rho s_1^n = \sum_{|\lambda|=n} f^\lambda a_{\lambda+\rho} \Rightarrow f^\lambda = \text{coeff. of } x^{\lambda+\rho} \text{ in } a_\rho s_1^n$$

$$a_\rho = \det \left(x_i^{k-j} \right)_{i,j=1,\dots,k} = \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma)\, x_1^{k-\sigma(1)} \cdots x_k^{k-\sigma(k)}$$

$$s_1^n = (x_1 + \cdots + x_k)^n = \sum_{i_1, \dots, i_k} \binom{n}{i_1, \dots, i_k} x_1^{i_1} \cdots x_k^{i_k}$$

$i_1, \dots, i_k \rightarrow i_j \geq 0\ \forall j \ \&\ i_1 + \cdots + i_k = n$

$$\binom{n}{i_1 \cdots i_k} = \frac{n!}{i_1! \cdots i_k!}$$

$l_i = (\lambda + \rho)_i$

Coeff of $x^{\lambda+\rho}$ is:

$$\sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \underbrace{\binom{n}{l_1 - k + \sigma(1), \dots, l_k - k + \sigma(k)}}_{\text{is } 0 \text{ if any } l_i - k + \sigma(i) < 0}$$

$$= \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) \frac{n!}{(l_1 - k + \sigma(1))! \cdots (l_k - k + \sigma(k))!} \qquad (*)$$

Define $(x)_r := x(x-1) \cdots (x - r + 1)$.

Then $$(l_i - k + \sigma(i))! = \frac{l_i!}{(l_i)_{k-\sigma(i)}}$$

$$(*) = \frac{n!}{l_1! \cdots l_k!} \sum_{\sigma \in \mathfrak{S}_k} \operatorname{sgn}(\sigma) (l_1)_{k-\sigma(1)} \cdots (l_k)_{k-\sigma(k)}$$

$$= \frac{n!}{l_1! \cdots l_k!} \det \begin{pmatrix} (l_1)_{k-1} & \cdots & l_1 & 1 \\ \vdots & & & \vdots \\ (l_k)_{k-1} & \cdots & l_k & 1 \end{pmatrix}$$

$$= \frac{n!}{\ell_1! \cdots \ell_k!} \prod_{1 \le i < j \le k} (\ell_i - \ell_j)$$

$\square$

Given box b in $Y(\lambda)$, its <u>hook length</u> is

$h(b) = 1 +$ # boxes to the right in same row

$+$ # boxes below in same column

<u>Ex</u>. hook lengths

$$\begin{array}{|c|c|c|c|c|c|} \hline 8 & 6 & 5 & 3 & 2 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1} \end{array}$$

<u>Thm</u>. (Hook length formula). If $|\lambda| = n$, then

$$f^\lambda = \frac{n!}{\prod_{b \in Y(\lambda)} h(b)}.$$

<u>Pf</u>. Let $g^\lambda = \dfrac{n!}{\prod_{b \in Y(\lambda)} h(b)}$. We show by induction on # columns that

$g^\lambda = f^\lambda$. If one column, then $f^\lambda = 1$

hook lengths are $n, n-1, \dots, 1$, so $g^\lambda = \dfrac{n!}{n!} = 1$ ✓

Suppose λ has ≥ 2 columns. Let μ be obtained by deleting first column. Then $g^\mu = f^\mu$.

Let $k = \ell(\lambda)$. Then hook lengths of first column of λ are

$\lambda_1 + k - 1, \lambda_2 + k - 2, \dots, \lambda_k$ and all hook lengths same in other columns

$$g^\lambda = g^\mu \frac{n!}{(n-k)!} \cdot \frac{1}{(\lambda_1 + k - 1) \cdots \lambda_k} = \frac{n!}{(n-k)!} \frac{1}{\ell_1 \cdots \ell_k} g^\mu$$

Also know that $f^{\lambda} = \frac{n!}{l_1! \cdots l_k!} \prod_{i<j} (l_i - l_j)$

$$f^{\mu} = \frac{(n-k)!}{(l_1-1)! \cdots (l_k-1)!} \prod_{i<j} (l_i - l_j)$$

$$\Rightarrow \quad f^{\lambda} = \frac{n!}{(n-k)!} \frac{1}{l_1 \cdots l_k} f^{\mu} = \frac{n!}{(n-k)!} \frac{1}{l_1 \cdots l_k} g^{\mu} = g^{\lambda} \quad \square$$

Ex. $\lambda = (6,3,1)$: hook length formula:

$$f^{(6,3,1)} = \frac{10 \cdot 9 \cdot \cancel{8} \cdot 7 \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}}{\cancel{8} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{4} \cdot 2} = \frac{630}{2} = 315$$

first formula: $k = 3$, $l_1 = 6+2 = 8$
$l_2 = 3+1 = 4$
$l_3 = 1$

$$f^{(6,3,1)} = \frac{10!}{8!\,4!} (8-4)(8-1)(4-1)$$

$$= \frac{90}{\underset{2}{\cancel{24}}} \cdot \cancel{4} \cdot 7 \cdot \cancel{3} = \frac{630}{2} = 315 \qquad \square$$