Littlewood-Richardson coefficients

$c^{\nu}_{\lambda,\mu}$ $\quad$ λ, μ, ν partitions $\quad$ $|\nu| = m+n$, $|\lambda| = m$, $|\mu| = n$

They appear in these contexts:

- Multiplication of Schur functions: $s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$
- Expansion of skew Schur function: $s_{\nu/\mu} = \sum_\lambda c^\nu_{\lambda\mu} s_\lambda$
- Induction of Specht modules: $\mathrm{Ind}_{\mathfrak{S}_{|\lambda|} \times \mathfrak{S}_{|\mu|}}^{\mathfrak{S}_{|\lambda|+|\mu|}} (S^\lambda \boxtimes S^\mu) \cong \bigoplus_\nu (S^\nu)^{\oplus c^\nu_{\lambda\mu}}$
- Restriction of Specht modules $\mathrm{Res}^{\mathfrak{S}_{m+n}}_{\mathfrak{S}_m \times \mathfrak{S}_n} S^\nu \cong \bigoplus_{\lambda,\mu} (S^\lambda \boxtimes S^\mu)^{\oplus c^\nu_{\lambda\mu}}$

Let $w = w_1 w_2 \cdots w_n$ sequence of positive integers.

let $m_i(w) = \#\{j \mid w_j = i\}$

A prefix of w is sequence $w_1 \cdots w_m$ for some $m \leq n$.

Def. w is a lattice permutation / Yamanouchi word / ballot sequence if, for every prefix v of w, we have $m_i(v) \geq m_{i+1}(v)$ for all i.

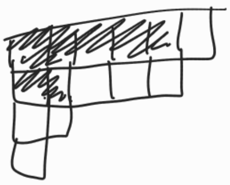
Def. T = tableau, its reverse reading word is sequence of its entries reading right to left starting from first row, and moving down.

Def. A SSYT T is a L-R tableau if its reverse reading word is a lattice permutation.

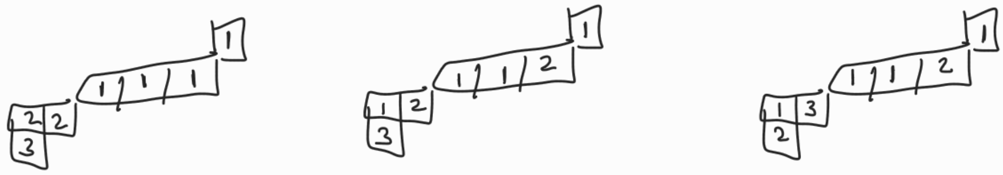
Thm. $c^\nu_{\lambda\mu} = \#$ L-R tableaux of shape ν/μ and type λ

EX. $\lambda = (4,2,1)$, $\mu = (5,2)$, $\nu = (6,5,2,1)$

ν/μ

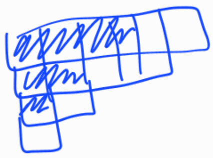


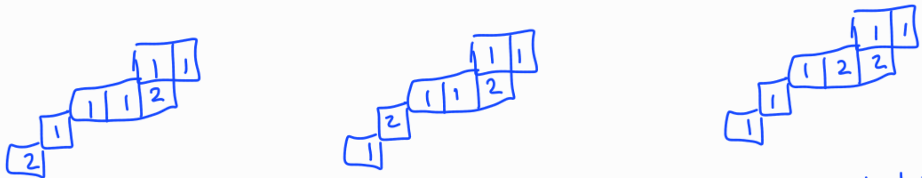
$\boxed{c^\nu_{\lambda\mu} = 3}$



1 1 1 1 2 2 3   1 2 11 21 3   1 2 11 31 2

ν/λ





1 1 2 1 1 1 2   1 1 2 1 1 2 1   1 1 2 2 1 1 1

Rmk. This rule generalizes Pieri rule:

- suppose $\lambda = (d)$. Then $c^{\nu}_{(d)\mu}$ = # L-R tableau of shape ν/μ of type (d), i.e., only using 1's & ν/μ

semistandard $\Rightarrow$ no two boxes in same column

$$\Rightarrow c^{\nu}_{(d),\mu} = \begin{cases} 1 & \text{if } \nu/\mu \text{ is horizontal strip of size } d \\ 0 & \text{else.} \end{cases}$$

- Now suppose $\lambda = (1^d)$. Then $c^{\nu}_{(1^d),\mu}$ = # L-R tableaux of shape ν/μ of type (1^d), i.e, using $1, \dots, d$ each exactly once.

reverse reading word must be $1 2 \cdots d$

$\Rightarrow$ entries in each row decreasing left to right, (since it's also SSYT)

$\Rightarrow$ each row only has ≤ 1 box

$$c^{\nu}_{(1^d)\mu} = \begin{cases} 1 & \text{if } \nu/\mu \text{ is a vertical strip} \\ 0 & \text{else} \end{cases}$$

This rule shows:

$c^{\nu}_{\lambda\mu} > 0 \Longrightarrow$ for any integer $d > 0$, $c^{d\nu}_{d\lambda, d\mu} > 0$.

$c^{\nu}_{\lambda\mu} > 0 \overset{?}{\Longleftarrow}$ There exists $d > 0$ s.t. $c^{d\nu}_{d\lambda, d\mu} > 0$ yes, "saturation property"