Polynomial Representations

$GL_n(\mathbb{C})$ = group of $n\times n$ invertible complex matrices.

A polynomial representation of $GL_n\mathbb{C}$ is a group homomorphism

V = complex f. dim vector space

$$\rho: GL_n(\mathbb{C}) \longrightarrow GL(V)$$

s.t. ρ can be expressed using polynomial functions in the entries of $g \in GL_n\mathbb{C}$.

EX. • $\rho = id: GL_n\mathbb{C} \to GL_n\mathbb{C}$ is polynomial

• $n=2$, $\{x,y\}$ is basis for $\mathbb{C}^2$, let $V = S^2\mathbb{C}^2 = \text{Span}\{x^2, xy, y^2\}$

$$\rho(g)(ax^2+bxy+cy^2) = a(gx)^2 + b(gx)(gy) + c(gy)^2$$

If $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, $\rho(g) = \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ 2\alpha\gamma & \beta\gamma+\alpha\delta & 2\beta\delta \\ \gamma^2 & \gamma\delta & \delta^2 \end{pmatrix}$

$$\rho(g)(x^2) = (\alpha x+\gamma y)^2 = \alpha^2x^2 + 2\alpha\gamma xy + \gamma^2y^2$$
$$\rho(g)(xy) = (\alpha x+\gamma y)(\beta x+\delta y) = \alpha\beta x^2 + (\beta\gamma+\alpha\delta)xy + \gamma\delta y^2$$
$$\rho(g)(y^2) = (\beta x+\delta y)^2 = \beta^2x^2 + 2\beta\delta xy + \delta^2y^2$$

• As above, general n, w/ $V = S^d\mathbb{C}^n$ = degree d homog. polynomials in $x_1, \dots, x_n$.

• Also, can do exterior powers $V = \bigwedge^d \mathbb{C}^n$

Polynomial representations closed under:

- Direct sum
- Subrepresentations
- Quotient representations
- Tensor products

$\Rightarrow S_\lambda\mathbb{C}^n$ gives polynomial representation of $GL_n\mathbb{C}$.

Dual of polynomial representation not generally polynomial.

Rmk. Dual of polynomial does give rational representation.

[main difference is we can divide by det polynomial]

If ρ is polynomial rep. of $GL_n\mathbb{C}$, define its character by

$$\mathrm{char}(\rho)(x_1,\dots,x_n) = \mathrm{Tr}\,\rho\begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ & & \ddots \\ 0 & & & x_n \end{pmatrix}$$

Lemma. $\mathrm{char}\,\rho \in \Lambda(n)$

Pf. $\mathrm{char}\,\rho$ is polynomial because ρ is polynomial rep.

Pick $\sigma \in \mathfrak{S}_n$. Define $M(\sigma)$ permutation matrix as 0-1 matrix w/ 1 in row $\sigma(i)$ and column i for all $i=1,\dots,n$, 0's elsewhere.

$$M(\sigma)^{-1}\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} M(\sigma) = \begin{pmatrix} x_{\sigma(1)} & & 0 \\ & \ddots & \\ 0 & & x_{\sigma(n)} \end{pmatrix}$$

$$\begin{aligned}
\mathrm{char}\,\rho(x_1,\dots,x_n) &= \mathrm{Tr}\,\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \\
&= \mathrm{Tr}\left(\rho(M(\sigma))^{-1}\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}\rho(M(\sigma))\right) \\
&= \mathrm{Tr}\left(\rho\left(M(\sigma)^{-1}\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}M(\sigma)\right)\right) \\
&= \mathrm{Tr}\,\rho\begin{pmatrix} x_{\sigma(1)} & & \\ & \ddots & \\ & & x_{\sigma(n)} \end{pmatrix} = \mathrm{char}\,\rho(x_{\sigma(1)},\dots,x_{\sigma(n)}) \quad \square
\end{aligned}$$

Ex. ① $\rho = \mathrm{id}$. $\mathrm{char} = x_1 + \dots + x_n$

② (a) $n=2$, $V = S^2\mathbb{C}^2$, $\mathrm{char} = x_1^2 + x_1x_2 + x_2^2 = h_2(x_1,x_2)$

(b) n general, $V = S^d\mathbb{C}^n$, pick basis $v_1,\dots,v_n$ for $\mathbb{C}^n$

monomials of deg d form basis for V.

$$\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} v_1^{d_1}\cdots v_n^{d_n} = x_1^{d_1}\cdots x_n^{d_n}\, v_1^{d_1}\cdots v_n^{d_n}$$

$$\Rightarrow \mathrm{char} = \sum_{\substack{\text{monomials}\\ \text{of deg } d \text{ in } x_1,\dots,x_n}} = h_d(x_1,\dots,x_n)$$

(3) In general, $V = \bigwedge^d \mathbb{C}^n$, pick basis $v_1, \dots v_n$ for $\mathbb{C}^n$

for every $1 \le i_1 < i_2 < \dots < i_d \le n$, get

$v_{i_1} \wedge \dots \wedge v_{i_d} \in \bigwedge^d \mathbb{C}^n$. These form basis which are

simultaneous eigenvectors for $\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, eigenvalue is $x_{i_1} \cdots x_{i_d}$

$$\Rightarrow \text{char} = \sum_{1 \le i_1 < \dots < i_d \le n} x_{i_1} \cdots x_{i_d} = e_d(x_1, \dots x_n)$$

(4) $V = S_\lambda \mathbb{C}^n$. pick basis $v_1, \dots v_n$ for $\mathbb{C}^n$.

Have basis for $S_\lambda \mathbb{C}^n$ indexed by SSYT w/ values $1, \dots, n$.

These are simultaneous eigenvectors for $\rho\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, eigenvalue is

$$x^T = x_1^{\#1\text{'s in } T} \cdots x_n^{\#n\text{'s in } T}$$

$$\Rightarrow \text{char} = \sum_{\text{SSYT } T} x^T = s_\lambda(x_1, \dots, x_n)$$

Let V_1, V_2 be polynomial reps of $GL_n \mathbb{C}$.

Pick basis $v_1 \dots v_m$ for V_1, $w_1, \dots w_n$ for V_2,

(1) Then $(v_1, 0), \dots, (v_m, 0), (0, w_1) \dots (0, w_n)$ is basis for $V_1 \oplus V_2$ and

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

$$\Rightarrow \text{char}(\rho_1 \oplus \rho_2) = \text{char}\, \rho_1 + \text{char}\, \rho_2 .$$

(2) Then $v_1 \otimes w_1, v_1 \otimes w_2, \dots v_1 \otimes w_n, v_2 \otimes w_1, \dots, v_m \otimes w_n$

is basis for $V_1 \otimes V_2$.

$$(\rho_1 \otimes \rho_2)(g) = \begin{pmatrix} \rho_1(g)_{1,1}\, \rho_2(g) & \rho_1(g)_{1,2}\, \rho_2(g) & \cdots & \rho_1(g)_{1,m}\, \rho_2(g) \\ \rho_1(g)_{2,1}\, \rho_2(g) & \vdots & & \vdots \\ \vdots & & & \\ \rho_1(g)_{m,1}\, \rho_2(g) & \rho_1(g)_{m,2}\, \rho_2(g) & & \rho_1(g)_{m,m}\, \rho_2(g) \end{pmatrix}$$

$\Rightarrow \operatorname{char}(\rho_1 \otimes \rho_2) = (\operatorname{char} \rho_1)(\operatorname{char} \rho_2)$.

$\Rightarrow$ For any partition $\lambda = (\lambda_1, \dots \lambda_r)$:

$$\operatorname{char}(S^{\lambda_1}\mathbb{C}^n \otimes \cdots \otimes S^{\lambda_r}\mathbb{C}^n) = h_\lambda(x_1, \dots x_n)$$

$$\operatorname{char}(\Lambda^{\lambda_1}\mathbb{C}^n \otimes \cdots \otimes \Lambda^{\lambda_r}\mathbb{C}^n) = e_\lambda(x_1, \dots x_n)$$