Re-interpreting symmetric polynomial identities

Thm. ① Finite-dim polynomial representations of $GL_n(\mathbb{C})$ are semisimple, i.e., always isomorphic to direct sum of irreducible polynomial representations.

② Polynomial reps are isomorphic $\iff$ have same character

③ The Schur functors $S_\lambda(\mathbb{C}^n)$ $(\ell(\lambda) \le n)$ are irreducible polynomial representations. If $\lambda \neq \mu$, then $S_\lambda(\mathbb{C}^n) \not\cong S_\mu(\mathbb{C}^n)$. Every irreducible polynomial rep is isomorphic to $S_\lambda \mathbb{C}^n$ for some λ.

Cauchy identities

Variables $x_1, \dots, x_m$, $y_1, \dots, y_n$

$$G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$$

has representation on $\mathbb{C}^m \boxtimes \mathbb{C}^n$

symmetric powers $Sym^d(\mathbb{C}^m \boxtimes \mathbb{C}^n)$ are polynomial reps.

Its character is $h_d(x_1 y_1, x_1 y_2, \dots, x_m y_n)$

$$= \text{Coeff of } t^d \text{ in } \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (1 - x_i y_j t)^{-1}$$

$$\Longrightarrow \prod_{i,j} (1 - x_i y_j t)^{-1} = \sum_{d \ge 0} h_d(x_i y_j)\, t^d = \sum_\lambda s_\lambda(\underset{x_1,\dots,x_m}{x}) s_\lambda(\underset{y_1,\dots,y_n}{y})\, t^{|\lambda|}.$$

$$h_d(x_i y_j) = \sum_{|\lambda| = d} s_\lambda(x) s_\lambda(y)$$

$$Sym^d(\mathbb{C}^m \boxtimes \mathbb{C}^n) \cong \bigoplus_{|\lambda| = d} S_\lambda(\mathbb{C}^m) \boxtimes S_\lambda(\mathbb{C}^n).$$

Dual version:

character of $\bigwedge^d(\mathbb{C}^m \boxtimes \mathbb{C}^n)$ is $e_d(x_i y_j)$

$$\prod_{i,j}(1+x_i y_j t) = \sum_{d\geq 0} e_d(x_i y_j)\, t^d = \sum_{\lambda} s_\lambda(x)\, s_{\lambda^T}(y)\, t^{|\lambda|}$$

$$e_d(x_i y_j) = \sum_{|\lambda|=d} s_\lambda(x)\, s_{\lambda^T}(y)$$

$$\bigwedge^d(\mathbb{C}^m \boxtimes \mathbb{C}^n) \cong \bigoplus_{|\lambda|=d} S_\lambda(\mathbb{C}^m) \boxtimes S_{\lambda^T}(\mathbb{C}^n)$$

Pieri / Littlewood-Richardson

$$s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda$$

$$\rightsquigarrow \quad S_\mu(\mathbb{C}^n) \otimes S_\nu(\mathbb{C}^n) \cong \bigoplus_\lambda (S_\lambda \mathbb{C}^n)^{\oplus c^\lambda_{\mu\nu}}$$

as reps of $GL_n(\mathbb{C})$

$\Rightarrow \quad c^\lambda_{\mu\nu} \geq 0$ since they are multiplicities.

Pieri is special case when $\nu = (k)$ or $\nu = (1^k)$

$S_{(k)}\mathbb{C}^n = \mathrm{Sym}^k \mathbb{C}^n$, $\quad S_{(1^k)}\mathbb{C}^n = \bigwedge^k \mathbb{C}^n$

i.e., taking tensor product of Schur functor w/ symmetric/exterior power.

Schur-Weyl duality

Consequence of Pieri: $s_1^d = \sum_{|\lambda|=d} f^\lambda s_\lambda$ — # SYT of shape λ.

$$\Rightarrow \quad (\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{|\lambda|=d} (S_\lambda \mathbb{C}^n)^{\oplus f^\lambda}.$$

rep of $\mathfrak{S}_d$ by permuting tensor factors

in fact, rep of $GL_n\mathbb{C} \times \mathfrak{S}_d$

$$(\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{|\lambda|=d} S_\lambda \mathbb{C}^n \boxtimes S^\lambda \quad (\text{as reps of } GL_n\mathbb{C} \times \mathfrak{S}_d)$$