Characters Assume $\mathbb{k} = \mathbb{C}$

Def. The <u>character</u> of $\rho: G \to GL(V)$ is function $\chi_V : G \to \mathbb{C}$ given by $\chi_V(g) = \underbrace{Tr}_{\text{trace}}(\rho_V(g))$

Note: $\chi_V(1_G) = \dim V$

Recall: • Trace of a linear operator is sum of its eigenvalues (w/ multiplicity)

• $\rho_V(g)$ is diagonalizable

Let V, W be representations of G.

complex conjugation

Lemma. $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ and $\chi_{V^*}(g) = \overline{\chi_V(g)}$

Pf. let $\lambda_1, \dots \lambda_n$ be eigenvalues of $\rho_V(g)$

$v_1, \dots, v_n$ be eigenvectors

let $f_1, \dots, f_n$ be dual basis for V^*

Then f_i is an eigenvector w/ eigenvalue $1/\lambda_i$ for $\rho_{V^*}(g)$:

$$(g \cdot f_i)(v_j) = f_i(g^{-1} \cdot v_j) = 1/\lambda_j \, f_i(v_j) = \delta_{ij}/\lambda_j$$

Since λ_i root of unity, $\lambda_i^{-1} = \overline{\lambda_i}$ [$\lambda \overline{\lambda} = |\lambda|$, root of unity has abs. value 1]

$$\Rightarrow \chi_{V^*}(g) = \chi_V(g^{-1}) = 1/\lambda_1 + \dots + 1/\lambda_n = \overline{\lambda_1 + \dots + \lambda_n} = \overline{\rho_V(g)}. \qquad \square$$

Lemma. $\chi_{V \oplus W}(g) = \chi_V(g) + \chi_W(g)$

Pf. Let $\lambda_1, \dots, \lambda_n$ e. values for $\rho_V(g)$
$v_1, \dots, v_n$ e. vectors

$\mu_1, \dots \mu_m$ e. values for $\rho_W(g)$.
$w_1, \dots, w_m$ e. vectors

$\Rightarrow (v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)$ eigenvectors for $\rho_{V \oplus W}(g)$ w/ eigenvalues $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$. $\square$

Lemma. $\chi_{V\otimes W}(g) = \chi_V(g)\chi_W(g)$

Pf. Let $\lambda_1, \dots, \lambda_n$ e. values for $\rho_V(g)$
$v_1, \dots, v_n$ e. vectors

$\mu_1, \dots, \mu_m$ e. values for $\rho_W(g)$.
$w_1, \dots, w_m$ e. vectors

$\Rightarrow \{ v_i \otimes w_j \mid \begin{smallmatrix} i=1,\dots,n \\ j=1,\dots,m \end{smallmatrix} \}$ are eigenvectors for $\rho_{V\otimes W}(g)$

w/ eigenvalues $\{ \lambda_i \mu_j \}$

$$\operatorname{Tr} \rho_{V\otimes W}(g) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j = \left(\sum_{i=1}^{n} \lambda_i\right)\left(\sum_{j=1}^{m} \mu_j\right) = \chi_V(g)\chi_W(g). \quad \square$$

Lemma. Let X be set w/ G-action. Let $V = \mathbb{C}[X]$
$$\chi_V(g) = \#\{ x \in X \mid g \cdot x = x \}$$

Pf. Consider basis $\{ e_x \mid x \in X \}$
In this basis, $\rho_V(g)$ is permutation matrix
Diagonal has 1 for each $x \in X$ s.t. $g \cdot x = x$
$\Rightarrow$ sum of diagonal entries $= \#\{ x \in X \mid g \cdot x = x \}$. $\square$

Lemma (projection formula) Define $\varphi : V \to V$ by
$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$$

Then: φ is a projection w/ image $\varphi = V^G$.
In particular, $\dim V^G = \operatorname{Tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$.

Pf. Pick $v \in V$, $h \in G$:
$$h \cdot \varphi(v) = \frac{1}{|G|} \sum_{g \in G} h \cdot (g \cdot v) = \frac{1}{|G|} \sum_{g \in G} (hg) \cdot v = \frac{1}{|G|} \sum_{g \in G} g \cdot v = \varphi(v)$$

$\Rightarrow$ image $\varphi \subseteq V^G$

If $w \in V^G$, then $\varphi(w) = \frac{1}{|G|} \sum_{g \in G} g \cdot w = \frac{1}{|G|} \sum_{g \in G} w = w$

$\Rightarrow$ image $\varphi = V^G$ & $\varphi^2 = \varphi$

Lastly, eigenvalues of a projection are 0 or 1 since it's a root of $t^2 - t$: rank = multiplicity of 1 as eigenvalue

$\Rightarrow$ Tr = dim of image. $\square$

Def A function $G \to \mathbb{C}$ which is constant on conjugacy classes is a <u>class function</u> and $CF(G)$ = set of class functions

- If $f \in CF(G)$, γ conj. class, $f(\gamma) :=$ value of f on any $g \in \gamma$
- $CF(G)$ is a $\mathbb{C}$-vector space: If $f_1, f_2 \in CF(G)$, $(f_1 + f_2)(g) = f_1(g) + f_2(g)$
  $\lambda \in \mathbb{C}$, $(\lambda f_1)(g) = \lambda f_1(g)$.
- $\chi_V \in CF(G)$ for all representations V:
  $\chi_V(hgh^{-1}) = \operatorname{Trace}(\rho_V(h)\rho_V(g)\rho_V(h)^{-1}) = \operatorname{Trace}(\rho_V(g)) = \chi_V(g)$

For $\varphi, \psi \in CF(G)$, define pairing

$$(\varphi, \psi)_G := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \overline{\psi(g)}$$

Prop. $(,)_G$ is an inner product on $CF(G)$.

Pf. Let c be # of conjugacy classes of G.

$\gamma_1, \dots, \gamma_c$ conjugacy classes of G

Define
$$CF(G) \xrightarrow{\cong} \mathbb{C}^c$$
$$f \longmapsto \frac{1}{\sqrt{|G|}} \left(\sqrt{|\gamma_1|}\, f(\gamma_1), \dots, \sqrt{|\gamma_c|}\, f(\gamma_c) \right)$$

$(,)_G$ becomes usual dot product under this isomorphism. $\square$

Prop. Given reps V, W of G, we have

$$\dim \operatorname{Hom}_G(V,W) = (\chi_V, \chi_W)_G.$$

Pf. $\operatorname{Hom}_G(V,W) = \operatorname{Hom}(V,W)^G \cong (V^* \otimes W)^G$

proj. formula: $\dim (V^* \otimes W)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{V^* \otimes W}(g)$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V^*}(g) \chi_W(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)} \chi_W(g)$$

$$= (\chi_W, \chi_V) = \overline{(\chi_V, \chi_W)} \leftarrow \text{is an integer} = (\chi_V, \chi_W) \qquad \square$$

Prop. Suppose for all $g \in G$, g is conjugate to g^{-1}. Then $\chi_V(g)$ is real forall reps. V.

Pf. $\chi_V(g^{-1}) = \overline{\chi_V(g)}$, but $\chi_V(g^{-1}) = \chi_V(g)$

$\Rightarrow \chi_V(g^{-1}) \in \mathbb{R}$ for all $g \in G$. $\qquad \square$