Classification of representations

<u>Lemma</u>. Let $\rho: G \to GL(V)$ be rep, $f \in CF(G)$. Define

$$\rho_f = \sum_{g \in G} f(g)\,\rho(g) \quad \text{linear operator on } V.$$

If V is irreducible, then ρ_f is scalar $= \lambda \cdot \mathrm{id}_V$ where

$$\lambda = \frac{|G|}{\dim V} (f, \overline{\chi_V})_G .$$

<u>Pf</u>. Pick $h \in G$. Then

$$\rho(h)\,\rho_f\,\rho(h)^{-1} = \sum_{g \in G} f(g)\,\rho(hgh^{-1}) = \sum_{g \in G} f(hgh^{-1})\rho(hgh^{-1}) = \rho_f .$$

$\Rightarrow \rho(h)\rho_f = \rho_f\rho(h) \quad \forall h \in G \Rightarrow \rho_f$ commutes w/ all elements of G.

Schur's lemma $\Rightarrow \rho_f$ is scalar $= \lambda \cdot \mathrm{id}_V$.

$$\lambda \dim V = \mathrm{Tr}(\rho_f) = \sum_{g \in G} f(g)\,\chi_V(g) = |G|\,(f, \overline{\chi_V})_G$$

$$\Rightarrow \quad \lambda = \frac{|G|}{\dim V}(f, \overline{\chi_V})_G . \qquad \square$$

<u>Thm</u>. ① The characters of irred. reps. form orthonormal basis for $CF(G)$. In particular, # of irred. reps. of G (up to isom.) is # conj. classes of G

② $V \cong W \iff \chi_V = \chi_W$.

<u>Pf</u>. ① Let V, W be irred. reps.

$$(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V,W) \overset{\text{Schur}}{=} \begin{cases} 0 & \text{if } V \not\cong W \\ 1 & \text{if } V \cong W \end{cases}.$$

$\Rightarrow$ If $V_1, V_2, \ldots$ are distinct irred. reps of G (up to isom.)
then $\chi_{V_1}, \chi_{V_2}, \ldots$ are orthonormal $\Rightarrow$ linearly independent.

Need to show: $\chi_{V_1}, \dots$ span $CF(G)$.

Let f be an element in orthogonal complement of span$(\chi_{V_1}, \dots)$

$\Rightarrow \rho_f = 0 \quad \forall$ irred. reps $\rho: G \to GL(V_i)$

$\Rightarrow \rho_f = 0 \quad \forall$ reps. $\rho: G \to GL(V)$ (Maschke)

Consider regular rep. $\rho: G \to GL(\mathbb{C}[G])$

$$0 = \rho_f(1_G) = \sum_{g \in G} f(g) e_g$$

$f(g) = 0 \quad \forall g \in G$

$\Rightarrow f = 0$

$\Rightarrow \chi_{V_1}, \dots$ spans $CF(G)$

$\Rightarrow$ # irred. reps (up to isom) $=$ dim $CF(G) =$ # conj. classes

(2) Let $V_1, \dots, V_c$ be distinct irred. reps of G

For any rep. V, $V \cong V_1^{\oplus m_1} \oplus \dots \oplus V_c^{\oplus m_c}$ (Maschke)

By Schur's lemma, $m_i = (\chi_{V_i}, \chi_V)$

If $\chi_W = \chi_V \Rightarrow W \cong V_1^{\oplus m_1} \oplus \dots \oplus V_c^{\oplus m_c} \cong V$. $\square$

Cor. The multiplicity of an irred. rep V in $\mathbb{C}[G]$ is $\dim V$

Pf. multiplicity $= (\chi_V, \chi_{\mathbb{C}[G]})$

$$\chi_{\mathbb{C}[G]}(g) = \#\{h \in G \mid gh = h\} = \begin{cases} 0 & \text{if } g \neq 1_G \\ |G| & \text{if } g = 1_G \end{cases}.$$

$$(\chi_V, \chi_{\mathbb{C}[G]}) = \frac{1}{|G|} \chi_V(1_G) \cdot |G| = \chi_V(1_G) = \dim V \quad \square$$

Cor. Let $d_1, \dots, d_c$ be dimensions of irred. reps of G. Then $d_1^2 + \dots + d_c^2 = |G|$.

Pf. $\mathbb{C}[G] \cong V_1^{\oplus \dim V_1} \oplus \dots \oplus V_c^{\oplus \dim V_c}$.

$|G| = \dim \mathbb{C}[G] = (\dim V_1)^2 + \cdots + (\dim V_c)^2$. $\square$

Cor. If G is abelian, then every irred. rep is 1-dimensional.

Pf. G abelian $\Rightarrow$ all conj. classes are singletons

$\Rightarrow c = |G|$

dimensions of irred. reps satisfy $d_1^2 + \cdots + d_c^2 = c$

only solution w/ positive integers is $d_1 = \cdots = d_c = 1$. $\square$