Examples of representations

Direct products G_1, G_2 groups, $\rho_i : G_i \to GL(V_i)$ reps $i=1,2$

$G_1 \times G_2$ has rep. on $V_1 \otimes V_2$ by

$$(g_1, g_2) \cdot \sum_{i \in I} v_i \otimes w_i = \sum_{i \in I} (g_1 \cdot v_i) \otimes (g_2 \cdot w_i)$$

$g_1 \in G_1$, $g_2 \in G_2$ $\quad v_i \in V_1$, $w_i \in V_2$

Notation: $V_1 \boxtimes V_2$ is this rep of $G_1 \times G_2$ external tensor product

$$\chi_{V_1 \boxtimes V_2}(g_1, g_2) = \chi_{V_1}(g_1)\, \chi_{V_2}(g_2)$$

Facts: ① If V, W irred., then $V \boxtimes W$ is irred.

② If $V_1, \ldots, V_n$ are all irred. reps of G_1,
— $W_1, \ldots, W_m$ ——— G_2,
then $\left\{ V_i \boxtimes W_j \;\middle|\; \begin{matrix} i=1,\ldots,n \\ j=1,\ldots,m \end{matrix} \right\}$ are all irred. reps of $G_1 \times G_2$.

Abelian groups Every finite abelian group is isomorphic to direct product of cyclic groups, so suffices to understand cyclic groups.

$G = \mathbb{Z}/m$, let ω be a primitive mth root of unity.
$[\omega = e^{2\pi i/m}]$

For $i = 0, \ldots, m-1$ define $\rho_i : \mathbb{Z}/m \to GL(\mathbb{C}^1)$
$$j \mapsto \omega^{ij}$$

These define non-isomorphic reps.
$\Rightarrow$ There are m of them, so we have them all.

Dihedral groups For $n \geq 3$, let D_n be symmetry group of regular n-gon $(|D_n| = 2n)$

If n odd, $\Rightarrow \frac{n+3}{2}$ conj. classes

If n even, $\Rightarrow \frac{n+6}{2}$ conj. classes

Center regular n-gon at origin $\Rightarrow$ D_n acts by linear transformations, so get rep. on $\mathbb{R}^2$ [reflection representation]

Can extend scalars to $\mathbb{C}$, get rep. on $\mathbb{C}^2$

exercise: show this is irreducible.

sign representation: $g \longrightarrow \det \rho(g)$ ρ = reflection rep.

D_5: 4 conj. classes, size 10,

let $d_1 \le d_2 \le d_3 \le d_4$ be dimensions of irred. reps.

$d_1^2 + d_2^2 + d_3^2 + d_4^2 = 10 \Rightarrow$ $d_1 = d_2 = 1 \rightarrow$ trivial, sign

$d_3 = d_4 = 2 \rightarrow$ reflection V

Need 1 more 2-dim rep.

guesses: $\cancel{V^*}$, $\cancel{V \otimes \mathrm{Sgn}}$

Recall: $\chi_{V^*} = \overline{\chi_V}$, but χ_V real-valued, $\Rightarrow$ $V \cong V^*$

can show $\chi_V \chi_{\mathrm{sgn}} = \chi_V$, so $V \otimes \mathrm{sgn} \cong V$

how to build a new 2-dim rep?

Symmetric groups $\mathfrak{S}_n$ = permutations of size n

$|\mathfrak{S}_n| = n!$

Lemma. Two permutations are conjugate $\Leftrightarrow$ have same cycle type.

In particular, # conj. classes of $\mathfrak{S}_n$ = $\underbrace{\text{\# partitions of } n}_{p(n)}$

Pf. If $(i_1, i_2, \ldots, i_k)$ denotes cycle $i_1 \to i_2 \to i_3 \to \cdots \to i_k \to i_1$

then we have $\tau (i_1, \ldots, i_k) \tau^{-1} = (\tau(i_1), \tau(i_2), \ldots, \tau(i_k))$ $\square$

$\mathfrak{S}_1$ = trivial ✓

$\mathfrak{S}_2 = \mathbb{Z}/2$ ✓

Assume $n \ge 3$

Every $\mathfrak{S}_n$ acts on $\mathbb{C}^n$ by permuting coordinates

$\mathbb{C}^n = \mathbb{C}[\{1,\dots,n\}]$: has basis $e_1,\dots,e_n$

$\sigma \cdot e_i = e_{\sigma(i)}$, $\sigma \in \mathfrak{S}_n$.

permutation rep of $\mathfrak{S}_n$.

If $n \geq 2$, $\mathbb{C}^n$ is reducible: $e_1 + \cdots + e_n$ spans 1-dim subrep.

complement: $\{(x_1,\dots,x_n) \mid x_1 + \dots + x_n = 0\}$ standard rep.

__exercise:__ standard rep is irreducible.

__Sign rep:__ $\sigma \mapsto \mathrm{sgn}(\sigma)$ $\left[\begin{array}{l} \mathrm{sgn}(\text{odd cycle}) = 1 \\ \mathrm{sgn}(\text{even cycle}) = -1 \end{array}\right]$

For $n=2$: sgn = standard, {trivial, sgn} give all irred. reps.

$\mathfrak{S}_3$: $|\mathfrak{S}_3| = 6$, 3 conj. classes $\left[\begin{array}{l} 3 \\ 2+1 \\ 1+1+1 \end{array}\right]$

$d_1 \leq d_2 \leq d_3$ dimensions of irred. reps

$d_1^2 + d_2^2 + d_3^2 = 6 \Rightarrow d_1 = d_2 = 1$ trivial, sgn

$d_3 = 2$ standard.

__charactertable__

	(1,1,1)	(2,1)	(3)
trivial	1	1	1
sgn	1	−1	1
standard	2	0	−1

observations
- all characters are real-valued
- standard ⊗ sgn = standard

standard ⊕ trivial ≅ permutation

permutation	3	1	0

$\mathfrak{S}_4$: $|\mathfrak{S}_4| = 24$, 5 conj. classes $\left[\begin{array}{ll} 4 & 2+2 \\ 3+1 & 1+1+1+1 \\ 2+1+1 & \end{array}\right]$

$d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5$ be dim's of irred. reps.

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 = 24$$

	$(1,1,1,1)$	$(2,1,1)$	$(3,1)$	$(2,2)$	(4)
trivial	1	1	1	1	1
sgn	1	-1	1	1	-1
standard	3	1	0	-1	-1
standard $\otimes$ sgn	3	-1	0	-1	1
V	2	0	-1	2	0

Heuristic: find set of size 3 w/ action of $\mathfrak{S}_4$:

$X=$ set of ways to decompose $\{1,2,3,4\}$ into 2 halves

$=\{12|34,\ 13|24,\ 14|23\}$ $\mathfrak{S}_4$ acts by relabeling

$\chi_{\mathbb{C}[X]}$ 3 1 0 3 1

$\sigma=(1,2,3,4)$:
$12|34 \to 23|41 = 14|23$
$13|24 \to 24|31 = 13|24$ ✓
$14|23 \to 21|34 = 12|34$

Is V irred? yes, can show that $(\chi_V, \chi_V)=1$
or observe χ_V not sum of 1-dim characters.