Math 184, Spring 2023
Homework 1
Due: Friday, Apr. 14 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. I'll put some hints at the very end.

(1) How many ways are there to list the letters of the following words:
   (a) BRIGHTLY
   (b) ENGAGEMENT

(2) Let's change Example 1.5.1 a little bit: we want to show that every polynomial in x can be written as a linear combination of powers of $2x-1$:
$$1, 2x-1, (2x-1)^2, (2x-1)^3, (2x-1)^4, \dots.$$
The same proof *almost* works; explain what needs to be changed (you don't need to rewrite the whole proof).

(3) Let $n \geq 2$ be an integer and define $[n] = \{1, 2, \dots, n\}$. Define the following sets:
$$A = \{S \subseteq [n] \mid 1 \in S \text{ and } 2 \in S\},$$
$$B = \{S \subseteq [n] \mid |\{1,2\} \cap S| = 1\}.$$
In words: A is the collection of subsets that contain *both* 1 and 2, while B is the collection of subsets that contain *exactly one* of 1 and 2.
   Find formulas for the sizes of A and B (with explanation).

(4) Consider the polynomial $f(x) = 3x^3 - x^2 + 1$. Write $f(x)$ as a linear combination of falling factorials and use that to get a formula for $\sum_{i=0}^{n} f(i)$.
   (You can leave the answer in terms of falling factorials.)

(5) Let n be a positive integer.
   Define A_n to be the set of sequences (of any length) whose entries are either 1 or 2 and such that the sum of the entries is $n+1$. For example, $|A_3| = 5$ and consists of the following sequences:
$$(1,1,1,1), (2,2), (2,1,1), (1,2,1), (1,1,2).$$
   Let B_n be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i+1 \notin S$. For example, $|B_3| = 5$ and consists of the following subsets:
$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}.$$
   (a) Describe a bijection $f\colon A_n \to B_n$ (along with an inverse $g\colon B_n \to A_n$). You don't need to explain why they are inverses, but your description of f and g should be clear and detailed enough so that it is obvious.
   (b) Explain how the elements of A_3 and B_3 are matched up by the bijection you gave in (a). I recommend simply drawing a diagram.
   (c) What does your bijection do to the sequence $(1,1,1,2,1,1,2,2,1,2)$?

Extra practice problems (don't turn in)

(6) Let $n \geq 2$ be an integer and define $[n] = \{1, 2, \dots, n\}$. Define the following sets:

$$C = \{S \subseteq [n] \mid \{1,2\} \cap S \neq \emptyset\},$$
$$D = \{S \subseteq [n] \mid \{1,2\} \cap S = \emptyset\}.$$

Find formulas for the sizes of C and D.

(7) Let S be a nonempty set. Define S_{odd} to be the collection of subsets of S with an odd number of elements. Similarly, define S_{even} to be the collection of subsets of S with an even number of elements. Carefully describe a bijection f (and inverse g) between S_{odd} and S_{even}.

(8) Let n be a positive integer.

Define A_n to be the set of (finite) sequences whose entries are either 1 or 2 and such that the sum of the entries is $n-1$ (the length of the sequence is not predetermined)

Define B_n to be the set of sequences whose entries are odd positive integers and such that the sum of the entries is n.

Describe a bijection between A_n and B_n.

Hints

(3): The proof of Theorem 1.4.2 contains a couple tricks on how to think about subsets in different sets. Try to adapt these tricks to this setting.

(5): Think of 2's in sequences of A_n as "skips". The subsets in B_n also have something to do with "skips". Try to match up the intuition for both and this should lead you to a bijection.