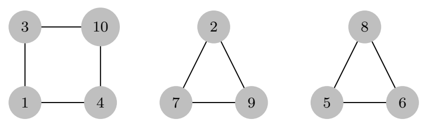
Math 184, Spring 2023
Homework 6
Due: Wednesday, May 31 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions when appropriate. Use complete sentences. I'll put some hints at the very end.

(1) For $n > 0$, let h_n be the number of bijections $f\colon [n] \to [n]$ with the property that $f \circ f \circ f$ is the identity function and set $h_0 = 1$. Give a simple expression for the EGF $H(x) = \sum_{n \ge 0} \frac{h_n}{n!} x^n$.

(2) For $n > 0$, let h_n be the number of set partitions of $[n]$ such that every block has size ≥ 2 (i.e., they have no singleton blocks) and set $h_0 = 1$. Give a simple expression for the EGF $H(x) = \sum_{n \ge 0} \frac{h_n}{n!} x^n$.

(3) In this problem, we will consider labeled graphs such that every vertex has degree 2 (i.e., is contained in exactly 2 edges). Let h_n be the number of such graphs with vertex set $[n]$. First let me point out (you don't need to prove this) that every graph satisfying this condition is a disjoint union of cycles, each one consisting of 3 or more vertices, e.g., if our vertex set is $[10]$, an example might be



(a) Given $n \ge 3$ vertices, how many ways are there to give them the structure of a labeled cycle graph? When $n = 3$ and $n = 4$, the answers are 1 and 3, so make sure your formula matches.

(b) Let $H(x) = \sum_{n \ge 0} h_n \frac{x^n}{n!}$. Using the exponential formula, we have $H(x) = \exp(E_\alpha(x))$ where α is the structure such that

$$\alpha(S) = \begin{cases} \{\text{cycle graphs with vertex set } S\} & \text{if } |S| \ge 3 \\ \emptyset & \text{else} \end{cases}.$$

Using Proposition 7.2.2 and your answer to (a), what recursive formula do you get for h_n?

(4) In Example 7.3.2, we counted the number of labeled trees on 3 and 4 vertices by considering all types of unlabeled trees and counting how many different labelings each one has. Do the same for 5 and 6 vertices.

To make it easier not to go down the wrong path: there are 3 types of unlabeled trees on 5 vertices and 6 types of unlabeled trees on 6 vertices. Also your answers should add up to $5^3 = 125$ and $6^4 = 1296$, respectively.

(5) Use Lagrange inversion to solve these problems:

(a) Let $A(x) = \sum_{n\geq 0} a_n x^n$ be the formal power series satisfying the identity

$$A(x) = \frac{x}{(1 - A(x))^2}.$$

Find a closed formula for a_n.

(b) Let $B(x) = \sum_{n\geq 0} b_n x^n$ be the formal power series satisfying the identity

$$B(x) = 2 + xB(x)^5.$$

Find a closed formula for b_n.

Extra practice problems (don't turn in)

(6) Use Proposition 7.2.2 in #1 above to get a recursive formula for h_n.
(7) Use Proposition 7.2.2 in #2 above to get a recursive formula for h_n.

Hints

1: Thinking of f as a permutation on n letters, the condition we're putting on it is really a restriction on what cycle lengths can appear.

2: You'll want to adapt Example 7.2.7 by first changing the definition of α.

3a: A cycle graph is very similar to a cycle in a permutation, *except* there is no orientation, i.e., there is no difference between clockwise and counterclockwise.