Math 184, Spring 2023, Final exam practice

Here are some final exam problems taken from previous instances of Math 184 that I taught. The first 11 problems make up the complete final from Fall 2019 in case you want to time yourself. I've added some more after from previous offerings. Finally, keep in mind that this does not necessarily represent all possible topics you could see, and this is only provided as a reference point.

Fall 2019 final exam

1. List all of the integer partitions of 5.

2. Evaluate $\displaystyle\sum_{i=0}^{n}(-1)^i\binom{n}{i}2^i6^{n-i}$.

3. Given an alphabet of size k, how many words are there with period 18?

4. If $\displaystyle\sum_{n\geq 0} a_n x^n = \frac{2-3x^3}{(1-4x)^4}$, give a closed formula for a_n.

5. Let $F(x) = \sum_{n\geq 0} f_n x^n$ be a formal power series such that $F(x) = 2 + xF(x)^3$. Find a formula for f_n.

6. Let h_n be the number of set partitions of $[n]$ such that every block has even size ($h_0 = 1$). Find a simple formula for the egf $H(x) = \displaystyle\sum_{n\geq 0}\frac{h_n}{n!}x^n$.

7. How many ways are there to rearrange the letters of the word ACCELERATION so that no two consecutive letters are the same?

8. How many ways can we pick subsets A, B, C of $[n]$ so that $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$?

9. You choose 9 cards from an unusual deck of cards (6 suits, 20 values). How many ways are there to get 3 pairs and a triple? i.e., 2 of the cards have the same value, another 2 cards have the same value, another 2 cards have the same value, and the remaining 3 have the same value (and all 4 of these values are different).

10. Let n be a positive integer. Consider a $2 \times n$ matrix whose entries are the numbers $1, 2, \dots, 2n$, each appearing exactly once. Call this matrix **increasing** if the entries increase in each row going from left to right and, for each column, the bottom entry is larger than the top entry. When $n = 3$, here are the increasing matrices:

$$\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\quad\begin{bmatrix}1&2&4\\3&5&6\end{bmatrix}\quad\begin{bmatrix}1&2&5\\3&4&6\end{bmatrix}\quad\begin{bmatrix}1&3&4\\2&5&6\end{bmatrix}\quad\begin{bmatrix}1&3&5\\2&4&6\end{bmatrix}$$

Show that the number of increasing matrices is the nth Catalan number C_n. By convention, there is exactly one increasing matrix of size 2×0. If you do this with a bijection, then explain the main ideas of why your bijection is correct, but do not worry about explaining minor points.

11. For non-negative integers $m, k \geq 0$, we have integers $t(m,k)$ defined by the following identities:

$$\begin{array}{lll} (1) & t(m,0) = 1 & \text{for all } m, \\ (2) & t(m,k) = 0 & \text{if } m < k, \\ (3) & t(m,k) = 2^k t(m-1,k) + t(m-1,k-1) & \text{if } m > 0 \text{ and } k > 0. \end{array}$$

For $k \geq 0$, define $T_k(x) = \sum_{m \geq 0} t(m,k)x^m$. Express $T_k(x)$ as a rational function of x.

Some more miscellaneous final exam problems

12. How many compositions of 15 into 8 parts are there?

13. What is the coefficient of $w^3x^3y^2z^3$ in $(w+x+y+z)^{11}$?

14. How many paths are there from $(0,0,0)$ to $(3,2,4)$ using only the steps $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$?

15. How many necklaces are there of length 27 using k different colors for the beads?

16. We paint n distinguishable chairs either blue, white, or yellow such that the total number which are blue or white is odd. How many ways can this be done?

17. How many integers $1 \leq x \leq 30000$ are not divisible by 2, 3, or 5?

18. Let a_n be the number of set partitions of $[n]$ such that every block has size ≥ 3. By convention $a_0 = 1$. Give a simple expression for $A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$.

19. Let $A(x) = \sum_{n \geq 0} a_n x^n$ be a formal power series. Express $B(x) = \sum_{n \geq 0} b_n x^n$ in terms of $A(x)$ if $b_n = \begin{cases} a_n + 3a_{n-1} & \text{if } n \geq 1 \\ a_0 & \text{if } n = 0 \end{cases}$.

20. $C(x) = \sum_{n \geq 0} c_n x^n$ is a formal power series satisfying the relation $C(x) = \dfrac{x}{(1-3C(x))^2}$. Find a formula for c_n.

21. Set $a_0 = 1$, and for $n \geq 1$, define $a_n = -a_{n-1} + 2 \cdot 3^n$.

    (a) Express $A(x) = \sum_{n \geq 0} a_n x^n$ as a rational function.

    (b) Give a closed formula for a_n.

22. Let k, n be positive integers such that $n \geq 3$. How many surjective functions $f\colon [n] \to [k]$ are there such that $f(1), f(2), f(3)$ are all distinct?

    Hint: You may express your answer in terms of Stirling numbers.

23. For $n \geq 1$, let a_n be the number of sequences $(w_1, \ldots, w_n)$ of length n with $w_i \in \{1, 2\}$ such that

$$w_1 \leq w_2 \geq w_3 \leq w_4 \geq \cdots$$

(i.e., if i is even, then $w_i \geq w_{i+1}$ and if i is odd, then $w_i \leq w_{i+1}$). Find an order 2 homogeneous linear recurrence relation that a_n satisfies and prove that it is correct.

24. For $n > 0$, let a_n be the number of partitions of n such that every part appears at most 3 times, and let b_n be the number of partitions of n such that no part is divisible by 4. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n.

25. A word in the alphabet of size 4 $\{(,), x, y\}$ is **grammatically correct (gc)** if, when the symbols x and y are all deleted, the remainder is a balanced set of parentheses (an empty word is balanced). Let a_n be the number of grammatically correct words of length n.

    (a) Prove that $a_n = 2a_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-2-i}$ for $n \geq 2$.

    (b) What are the initial conditions?