# Some recent developments in Boij–Söderberg theory

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- $K$ is a field
- $A = K[x_1, \ldots, x_n]$ with standard grading
- $A(-i)$ denotes $A$ with a grading shift, i.e., $A(-i)_j = A_{j-i}$
- $M$ denotes a graded finitely generated module
- $\mathbf{F}_\bullet$ is the minimal free resolution of $M$
- Write $\mathbf{F}_i = \bigoplus_j A(-j)^{\oplus \beta_{i,j}}$
- $\beta(M) = (\beta_{i,j})$ is the **graded Betti table**

- We will restrict attention to finite length modules.
- A module has a **pure resolution** if each $\mathbf{F}_i$ is generated in a single degree, i.e., for each $i$, $\beta_{i,j} \neq 0$ for at most one value of $j$. In this case, the $j$ for which $\beta_{i,j} \neq 0$ is denoted $d_i$, and $d = (d_0, d_1, \ldots . d_n)$ is the degree sequence.
- The Herzog–Kühl equations state that the Betti numbers for a given degree sequence $d$ are uniquely determined up to scalar multiple.

Theorem (Eisenbud–Fløystad–Weyman, Eisenbud–Schreyer)

*For $d_0 < d_1 < \cdots < d_n$, there exists a finite length module $M$ whose resolution is pure with degree sequence $d$.*

By Herzog–Kühl, each degree sequence defines a ray in the space of all Betti tables. The cone spanned by these rays is the **Boij–Söderberg cone**.

Theorem (Eisenbud–Schreyer)

*Every Betti table of a finite length module $M$ is contained in the Boij–Söderberg cone.*

So for some large positive integer $N$, $N\beta(M)$ is a positive integer linear combination of Betti tables of modules with pure resolution.

Theorem (Erman)

*The semigroup spanned by actual Betti tables is finitely generated (if we bound the degrees that may appear in the Betti table).*

Define $d \leq d'$ if $d_i \leq d_i'$ for $i = 0, \ldots, n$. The Boij–Söderberg cone is a geometric realization of this poset, and hence one gets a simplicial decomposition.

With respect to this triangulation, every Betti table can **uniquely** be expressed as a positive linear combination of Betti tables of pure resolutions (assuming we have chosen a normalized value for each ray).

We can also interpret this poset structure module-theoretically:

Theorem (Berkesch–Erman–Kummini–S.)

$d \leq d'$ if and only if there exist modules $M$ and $M'$ with pure resolutions of type $d$ and $d'$, respectively, such that $\operatorname{Hom}(M', M)_{\leq 0} \neq 0$.

It is natural to ask what the decompositions of Betti tables means in terms of modules.

Naive guess: for any module $M$, some high multiple $M^{\oplus N}$ has a filtration such that the quotient modules have pure resolutions.

### Example

Let $M = K[x, y]/(x, y^2)$. Then

$$\beta(M) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

but any submodule or quotient module of $M^{\oplus 3}$ is annihilated by $y^2$, so must have a degree 2 relation.

Some cases where pure filtrations do exist are given by Eisenbud–Erman–Schreyer.

How to fix the previous example: there is a flat deformation of $M^{\oplus 3}$ to $M' = A/(x, y^2) \oplus A/(x^2, y) \oplus A/(x + y, (x - y)^2)$, and this does possess a pure filtration. [Take the submodule spanned by $(1, 1, 1)$.]

**Conjecture:** The Boij–Söderberg decomposition of $\beta(M)$ corresponds to a pure filtration of a flat deformation (i.e., has the same Hilbert polynomial) of some high multiple of $M$.

Now let $R$ be a regular local ring of dimension $n$.
We can no longer speak about grading, but we can ask about Betti sequences (ranks of the terms in the free resolution).

### Theorem (Berkesch–Erman–Kummini–S.)

*The extremal rays of the closure of the cone of Betti sequences of $R$-modules of finite length are the vectors $\rho_i$ for $i = 0, \ldots, n-1$ where $\rho_0 = (1, 1, 0, \ldots, 0), \rho_1 = (0, 1, 1, 0, \ldots, 0), \ldots, \rho_{n-1} = (0, \ldots, 0, 1, 1)$.*

Surprising consequence: The linear functionals defining the cone above are given by partial Euler characteristics (that they must be nonnegative is easy), so this result essentially says there are no other obstructions to being a Betti sequence of a module (at least up to scalar multiple).

So for example, some multiple of a small perturbation of $(1, 1, 10^{10}, 10^{10}, \ldots, 10^{10}, 1, 1, 10^{100}, 10^{100}, 1, 1)$ is a Betti sequence, and you get even wilder behavior.