Geometric approach to Littlewood inversion formulas

Steven Sam
(joint with Jerzy Weyman)

Massachusetts Institute of Technology

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• We work over the field of complex numbers $\mathbb{C}$.
• Given a partition $\lambda$, its conjugate partition is $\lambda'$.
• $s_{\lambda}, e_{\lambda}, h_{\lambda}$ denotes Schur, elementary, and homogeneous symmetric functions.
• Given a symmetric function $f$, let $s_{\lambda / f}$ be defined by $(s_{\lambda / f}, g) = (s_{\lambda}, fg)$ where $(,)$ is the Hall inner product.
• The polynomial irreps of $GL(V)$ ($\dim V = m$) are indexed by partitions $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$, and are denoted $S_\lambda(V)$—these can be realized as subspaces of $V^{\otimes |\lambda|}$.

• Put a nondegenerate (skew-)symmetric form $\omega$ on $V$.

• Suppose that $\omega$ is skew-symmetric (so $m = 2n$). The irreps of $Sp(V)$ are the traceless tensors in $S_\lambda(V)$: this is the intersection of the kernels of the contractions

$$v_1 \otimes \cdots \otimes v_N \mapsto \omega(v_i, v_j)v_1 \otimes \cdots \hat{v}_i \cdots \hat{v}_j \cdots \otimes v_N.$$ 

Denote this irrep by $S_{[\lambda]}(V)$.

• Ditto for the orthogonal group $O(V)$.
Let $c_{\alpha,\beta}^\gamma$ be the Littlewood–Richardson coefficient, i.e.,

$$s_\alpha s_\beta = \sum_\gamma c_{\alpha,\beta}^\gamma s_\gamma.$$ 

When $\omega$ is symmetric, Littlewood showed:

$$S_\lambda V \cong \bigoplus_{\mu,\nu} (S_\mu V)^{\oplus c_{\mu,2\nu}}.$$ 

When $\omega$ is skew-symmetric, Littlewood showed:

$$S_\lambda V \cong \bigoplus_{\mu,\nu} (S_\mu V)^{\oplus c_{\mu,(2\nu)'}}.$$ 

Note that $S_\lambda V$ appears with multiplicity 1, and if $S_\mu V$ appears, then $\mu = \lambda$ or $|\mu| < |\lambda|$. So the branching matrix can be made upper unitriangular.
We can express the branching as saying how to write a Schur function in terms of irreducible characters of the symplectic / orthogonal group. Littlewood gave the inversion formulas. When $\omega$ is skew-symmetric,

$$s[\lambda] = \sum_{i \geq 0} (-1)^i s_{\lambda / (e_i \circ e_2)}.$$ 

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We want to find a “geometric” interpretation of these formulas in the hope that they may generalize to other groups.
Isotropic maps

For simplicity of notation, assume $\omega$ is skew-symmetric. Consider $Y_\omega = \{\varphi: E \to V \mid \varphi(E) \text{ isotropic}\}$. This is a complete intersection in $\text{Hom}(E, V)$ if and only if $2 \dim E \leq \dim V$, and ideal is generated by $\wedge^2 E$ in degree 2.

So it has the following graded minimal free resolution (set $A = \mathbb{C}[\text{Hom}(E, V)]$):

$$0 \to \bigwedge^{\binom{\dim E}{2}} E \otimes A(\!-\!2 \dim E) \to \cdots \bigwedge^i \bigwedge^{2} E \otimes A(\!-\!2i) \to \cdots \to \bigwedge^2 E \otimes A(\!-\!2) \to A \to \mathbb{C}[Y_\omega] \to 0$$
The coordinate ring of $Y_\omega$ decomposes as (Cauchy identity):

$$C[Y_\omega] = \bigoplus_{\lambda} S_\lambda(E) \otimes S[\lambda](V)$$

Taking isotypic components of the Koszul complex (with respect to $GL(E)$) gives a resolution of irreps for $Sp(V)$ in terms of irreps for $GL(V)$

$$\cdots \rightarrow S_{\lambda/\wedge^i \wedge^2 V} \rightarrow \cdots \rightarrow S_{\lambda/\wedge^2 V} \rightarrow S_\lambda V \rightarrow S[\lambda] V \rightarrow 0$$

The first part is “Weyl’s construction”, and taking the Euler characteristic gives Littlewood’s inversion formula.

This works for the orthogonal group with the appropriate changes. (Caveat: if $m = 2n$ and $\dim E = n$, then $Y_\omega$ has 2 irreducible components.)
For $G = G_2(\mathbb{C})$, there is something similar: $G$ has an irrep $V$ of dimension 7 which has an alternating trilinear form $\gamma$. Call a subspace $R$ of dimension 2 isotropic if $\gamma(u, v, w) = 0$ for all $u, v \in R$ and $w \in V$. Can define $Y_\omega$ as before, taking $\dim E \leq 2$. We get the following resolutions:

$$0 \to S_{\mu/(4,4)} V \to (S_{\mu/(3,3)} V \otimes V) \oplus S_{\mu/(4,2)} V \to$$

$$S_{\mu/(3,2)} V \otimes (\mathbb{C} \oplus V) \to S_{\mu/(2,1)} V \otimes (\mathbb{C} \oplus V) \to$$

$$(S_{\mu/(1,1)} V \otimes V) \oplus S_{\mu/(2)} V \to S_{\mu} V \to V_{(\mu_1 - \mu_2, \mu_2)} \to 0$$

The first part encodes “Weyl’s construction” due to (Huang–Zhu 1999), and the Euler characteristic gives an inversion formula.
Let $\Sigma_m$ be the symmetric group and let $V$ be the $m$-dimensional standard representation. The right analogue of $S_{[\lambda]}(V)$ is the irrep indexed by $(m - |\lambda|, \lambda_1, \ldots)$ if it’s a partition, and 0 otherwise. Let $\mathfrak{h}$ be the subspace of traceless diagonal matrices in $\text{End}(V)$.

Given $n$, let

\[
Y = \{ \varphi : \mathbb{C}^n \to \text{End}(V) \mid \varphi(x)^2 = 0 \text{ for all } x \in \mathbb{C}^n \},
\]
\[
D = \{ \varphi : \mathbb{C}^n \to \text{End}(V) \mid \varphi(x) \in \mathfrak{h} \text{ for all } x \in \mathbb{C}^n \}.
\]

Let $Y_D = Y \cap D$ (scheme-theoretic). Then

\[
\mathbb{C}[Y_D] = \bigoplus_{\lambda} S_{\lambda}(\mathbb{C}^n) \otimes S_{[\lambda]}(V)
\]

Free resolution of $Y_D$ over $\mathbb{C}[\text{Hom}(E, \mathfrak{h})]$ should yield inversion formula ($V = \mathfrak{h} \oplus \mathbb{C}$), but seems difficult to obtain.
• The other exceptional groups are not so nice, i.e., one probably cannot resolve irreps in terms of Schur functors (at least not with the geometric methods I was using). This roughly corresponds to the fact that the map $G/B \to G/P_\alpha$ is a “twisted” orthogonal / symplectic flag variety where $P_\alpha$ is a certain maximal parabolic subgroup.

• However, we can still define varieties whose coordinate rings give the right analogue of the Cauchy identity if we only pay attention to a certain codimension 1 lattice of the weight lattice.
Recall that for classical groups, $Y_\omega = \{\varphi : E \to V \mid \omega|_{\varphi(E)} = 0\}$ is a complete intersection if and only if $2 \dim E \leq \dim V$. So when $2 \dim E > \dim V$, the complex (for $\omega$ skew-symmetric)

$$
\cdots \to S_{\lambda/\wedge^i} \wedge^2 V \to \cdots \to S_{\lambda/\wedge^2} V \to S_{\lambda} V.
$$

can have higher homology. Results of Koike–Terada imply that the Euler characteristic of this complex is $\pm s_{[\mu]}$ for some $\mu$ or 0.

Wenzl shows that $\mu$ is obtained from $\lambda$ via a dotted Weyl group (of type $D_\infty$ for orthogonal case and type $B_\infty = C_\infty$ for symplectic case) action (i.e., $\mu = (-1)^{\ell(w)} w(\lambda + \rho_m) - \rho_m$ for some $\rho_m$).

This is formally analogous to the Borel–Weil–Bott theorem, so we conjecture that the complex above has at most 1 nonzero homology group.