Big Polynomial rings and Stillman's conjecture

Note Title 5/6/2020

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M = module over $k[x_1 .. x_n]$

free resolution approximates M by sequence of free modules

Formally, a free resolution of M is a sequence

$$0 \leftarrow M \xleftarrow{d_0} F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} \cdots$$

F_i = free

$d_i d_{i+1} = 0 \ \forall i$

$\text{image}(d_{i+1}) = \ker(d_i)$

Intuitively, pick gens for M

$\rightsquigarrow F_0 \to M$, iterate and pick gens for kernel, $F_1 \to \ker \subset F_0$, etc.

Def. pdim M = length of shortest free resolution of M

= "measure of how non-free M is".

Thm (Hilbert syzygy Thm) pdim $M \le n$

Can we bound pdim M in terms of invariants not depending on n?

Stillman's conjecture:

Thm (Ananyan-Hochster) Fix integers $d_1, \dots d_r$. $\exists$ constant C s.t. pdim $I \le C$ for any ideal I generated by homog. poly of degrees $d_1 \dots d_r$.

Naive idea: bound # vars these polynomials use? (no: even if $r=1$, $d_1 = 2$, consider rk of a quadric)

More flexible idea: find a subalgebra containing the polynomials gen. by a "regular sequence" of bounded size

"small subalgebra"

Def. Let R be graded ring, f homogeneous. f has strength $\le s$ if we can write

$f = g_1 h_1 + \dots + g_s h_s$ where $\deg g_i, \deg h_i < \deg f$ homog.

strength ∞ if no such decomposition.

If W linear space spanned by homog. elts, $\operatorname{str} W = \min \{ \operatorname{str} f \mid f \in W, f \neq 0 \}$

Thm. (Ananyan-Hochster). Fix $d_1 \dots d_r$. $\deg(f_i) = d_i$. If $\operatorname{str}\langle f_1, \dots, f_r\rangle \gg 0$.

then $f_1, \dots, f_r$ is a regular sequence.

$\Longrightarrow$ can find small subalgebras

<u>Ultraproducts</u>: context for proving existence of bounds

I = infinite set. $\mathcal{F}$ non-principal ultrafilter: collection of subsets of I s.t.

1. F has no finite sets
2. $A, B \in F \Rightarrow A \cap B \in F$
3. $A \in F$, $A \subseteq B \Rightarrow B \in F$
4. $\forall A \subseteq I$, either $A \in F$ or $I \setminus A \in F$.

<u>Intuition</u>: sets in F are "neighborhoods of $* \in I$"

Given sets $\{X_i\}_{i \in I}$, ultraproduct $\text{ulim}\, X_i$ is $\prod_{i \in I} X_i / \sim$ where $(x_i) \sim (y_i)$ if $\{i \in I \mid x_i = y_i\} \in \mathcal{F}$.

$\text{ulim}\, X_i$ inherits operations

if all X_i are fields, so is $\text{ulim}\, X_i$

If X_i graded groups define graded ultraproduct $\text{ulim}\, X_i$ by considering only sequences of elements of bounded degree.

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3 facts which imply Ananyan-Hochster:

(1) For $i \geq 1$, let R_i be polynomial rings, $V_i \subseteq R_i$ linear subspaces, $\text{str}(V_i) \to \infty$ as $i \to \infty$. Then $\text{ulim}\, V_i \subseteq \text{ulim}\, R_i$ has infinite strength.

(2) For $i \geq 1$, let $R_i = k_i[x_1, x_2, \ldots]$ w/ $\deg(x_i) = 1$. Let $S = \underset{i}{\text{ulim}}\, R_i$. $m \subset S$ homog. max ideal. Let $\Sigma \subset m$ homog. elts s.t. image of Σ in m/m^2 is a basis (over $\text{ulim}\, k_i$). Then, S is a polynomial ring gen. by Σ.

(3) For $i \geq 1$, let $f_{i,1}, \ldots f_{i,r} \in R_i$ be homog. of degrees $d_1 \ldots d_r$. Then $\text{ulim}\, f_{i,1}, \ldots, \text{ulim}\, f_{i,r} \in \text{ulim}\, R_i$ is a regular sequence $\iff$ $f_{i,1}, \ldots f_{i,r}$ is a regular sequence for i in a neighborhood of $*$.

Using ①, ②, ③, proof: Suppose Thm is false.

i.e., str >> 0 does not imply regular sequence.

$\Rightarrow \exists$ sequences $f_{i,1}, \dots, f_{i,r}$ which are not regular and str $\to \infty$.

Take $\text{ulim}\, f_{i,1}, \dots, \text{ulim}\, f_{i,r} \in \text{ulim}\, R_i$

① strspan$(\text{ulim}\, f_{i,1} \dots \text{ulim}\, f_{i,r}) = \infty$

$\Rightarrow$ images of $\text{ulim}\, f_{i,1}, \dots \text{ulim}\, f_{i,r}$ are lin. ind. in m/m^2

$2 \Rightarrow$ extend to basis of m/m^2

$\Rightarrow \underbrace{\text{lift to homog.}}_{\text{these are variables of a poly ring}}$ elts of S

$\Rightarrow$ regular sequence $\overset{3}{\Rightarrow}$ contradiction

Proof of 2 : Differential criterion for polynomiality.

Thm k = field of char. 0

$R = \bigoplus_{i \geq 0} R_i$, $R_0 = k$.

Assume R has "enough derivations"

$\forall f \in R$, $\deg f > 0$, $\exists$ derivation ∂ s.t. $\deg(\partial) < 0$ and $\partial(f) \neq 0$.

$\Rightarrow R \cong$ polynomial ring $/k$, w/ gen. set any homog lift of basis for m/m^2.

Use this to prove $\varinjlim$ (poly rings) is poly.

Explicit bounds on pdim / size of small subalgebra given $d_1 \ldots d_r$?

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pdim bounds (known results)

n quadrics $\leq 2^{n+1}(n-2)+4$ Ananyan Hochster

4 quadrics ≤ 6 Huneke, Mantero, McCullough, Seceleanu

3 cubics ≤ 5 Mantero, McCullough