Curried Lie algebras

Steven Sam

University of California, San Diego

November 24, 2020

Slides available at
`http://math.ucsd.edu/~ssam/icra.pdf`

General linear Lie algebra, revisited

Recall: A representation of $\mathfrak{gl}(V)$ is a space M with a map

$$\varphi\colon \mathfrak{gl}(V) \otimes M \to M$$

satisfying $(*)$ $[\varphi_1(x), \varphi_2(y)] = \varphi([x, y])$ as maps

$$\mathfrak{gl}(V) \otimes \mathfrak{gl}(V) \otimes M \to \mathfrak{gl}(V) \otimes M.$$

Basic idea: $\mathfrak{gl}(V) = V \otimes V^*$ so φ is equivalent to

$$a\colon V \otimes M \to V \otimes M$$

which satisfies some condition $(**)$.

Claim: The condition is $(**)$ $[a_1, a_2] = \tau(a_1 - a_2)$ as endomorphisms of $V \otimes V \otimes M$ where τ is the symmetry on $V \otimes V$.

Proof sketch: $\{e_1, \ldots, e_n\}$ basis for V; $x_{i,j} = e_i \otimes e_j^* \in \mathfrak{gl}(V)$.

$$a(e_i \otimes m) = \sum_{j=1}^{n} e_j \otimes x_{i,j} m$$

$$\begin{aligned}
[a_1, a_2](e_i \otimes e_k \otimes m) &= \sum_{1 \le j, \ell \le n} e_j \otimes e_\ell \otimes [x_{i,j}, x_{k,\ell}] m \\
&= \sum_{1 \le j, \ell \le n} e_j \otimes e_\ell \otimes (\delta_{j,k} x_{i,\ell} - \delta_{i,\ell} x_{k,j}) m \\
&= \sum_{1 \le \ell \le n} e_k \otimes e_\ell \otimes x_{i,\ell} m - \sum_{1 \le j \le n} e_j \otimes e_i \otimes x_{k,j} m \\
&= \tau(a_1 - a_2)(e_i \otimes e_k \otimes m)
\end{aligned}$$

Definition: Let V be an object of a symmetric monoidal abelian category $\mathcal{C}$. A representation of $\underline{\mathfrak{gl}}(V)$ is an object M and a map $a \colon V \otimes M \to V \otimes M$ satisfying $(**)$ $[a_1, a_2] = \tau(a_1 - a_2)$.
Let $\mathrm{Rep}(\underline{\mathfrak{gl}}(V))$ be the category of representations.

$V = \mathbf{k}\langle 1 \rangle$ in $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$

$\mathcal{C}$	Curried Lie algebra
Brauer	Symplectic Lie algebra $\mathfrak{sp}(V \oplus V^*)$
Signed Brauer	Orthogonal Lie algebra $\mathfrak{so}(V \oplus V^*)$
Spin Brauer	Orthosymplectic Lie superalgebra $\mathfrak{osp}(\mathbf{k} \vert V \oplus V^*)$
Signed spin Brauer	Orthogonal Lie algebra $\mathfrak{so}(\mathbf{k} \oplus V \oplus V^*)$
$\mathbf{FI}\sharp(\delta)$	Heisenberg Lie algebra on $V \oplus V^*$
Partition category	Weyl Lie algebra on $V \oplus V^*$
Degenerate partition	Hamiltonian algebra on $V \oplus V^*$
FI	Symmetric algebra on V
$\mathbf{FS}^{\mathrm{op}}$	$(\star)$ Positive Witt Lie algebra on V
$\mathbf{FiniteSet}^{\mathrm{op}}$	$(\star)$ Witt Lie algebra on V
Periplectic Brauer	Periplectic Lie superalgebra on $V \oplus V^*[1]$

Standard actions: V is a vector space, $\mathcal{C} = \mathrm{Rep}(\mathfrak{gl}(V))$. Then any representation M in the usual sense is a $\underline{\mathfrak{gl}}(V)$, and we call $a\colon V \otimes M \to V \otimes M$ the standard action.

Example: Let $M = \mathbf{S}_\lambda(V)$ the Schur module. Then $V \otimes M = \bigoplus_\mu \mathbf{S}_\mu(V)$ sum over $\mu \supset \lambda$ such that $|\mu| = |\lambda| + 1$. a is multiplication by a scalar on each $\mathbf{S}_\mu(V)$, which is the "content" of $\mu \setminus \lambda$ ($= \mu_i - i$ where $\mu_i \neq \lambda_i$).

$\mathbf{GL}_\infty$: Let $\mathbf{V} = \bigcup_n \mathbf{C}^n$ and $\mathcal{C} = \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ be category of polynomial representations of $\mathbf{GL}(\mathbf{V})$ ($=$ category of polynomial functors). $a\colon \mathbf{V} \otimes M \to \mathbf{V} \otimes M$ is well-defined (but $\mathbf{V}^*$ is not)

FB: Let **FB** be the category of **F**inite sets and **B**ijections and $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$ the category of functors from **FB** to **k**-vector spaces. This is symmetric monoidal with product

$$(F \otimes G)(S) = \bigoplus_{T \subseteq S} F(T) \otimes G(S \setminus T).$$

Let $\mathbf{k}\langle 1\rangle$ be the point module $\mathbf{k}\langle 1\rangle(S) = \begin{cases} 0 \text{ if } |S| \neq 1 \\ \mathbf{k} \text{ if } |S| = 1 \end{cases}$

Schur–Weyl duality: $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty) \simeq \mathrm{Rep}_{\mathbf{C}}(\mathbf{FB})$ as symmetric monoidal categories such that $\mathbf{V} \leftrightarrow \mathbf{C}\langle 1\rangle$.

Standard action on $M \in \mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$ is endomorphism of

$$(\mathbf{k}\langle 1\rangle \otimes M)(S) = \bigoplus_{x \in S} M(S \setminus x)$$

whose components are $M(S \setminus x) \to M(S \setminus y)$ for $x \neq y$ given by bijection $S \setminus x \to S \setminus y$ swapping x and y.

Symmetric group representations: $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$ is equivalent to $\mathrm{Rep}(\mathfrak{S}_*)$, whose objects are sequences $(M_n)_{n \geq 0}$ where M_n is a $\mathbf{k}$-representation of symmetric group $\mathfrak{S}_n$. Tensor product becomes

$$(M \otimes N)_n = \bigoplus_{i=0}^{n} \mathrm{Ind}_{\mathfrak{S}_i \times \mathfrak{S}_{n-i}}^{\mathfrak{S}_n} (M_i \otimes N_{n-i}).$$

Motivations

This idea came from a few different sources:

FI-modules: Can be interpreted via twisted commutative algebras (= commutative algebras internal to $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$), but variants could not.

Brauer categories: Suggested some connections between classical invariant theory and parabolic category $\mathcal{O}$ of symplectic and orthogonal Lie algebras.

Representations of surjections: We noticed a connection with Witt Lie algebras that we wanted to make precise.

FI is the category of **F**inite sets and **I**njections.
FI-modules M have an intrinsic description: they are **FB**-modules together with a map $\mathbf{k}\langle 1\rangle \otimes M \to M$ so that the action is commutative and associative, i.e., induces a module $\mathrm{Sym}(\mathbf{k}\langle 1\rangle) \otimes M \to M$

They play a central role in representation stability, which I won't recall. There is a variant, $\mathbf{FI}\sharp$ whose morphisms are partially defined injections. [CEF] shows that finitely generated $\mathbf{FI}\sharp$-modules are free as **FI**-modules. Expect analogy with the Weyl algebra.

Usual Weyl algebra: pick vector space V and take tensor algebra on $V \oplus V^*$ modulo: V commutes with itself, V^* commutes with itself, $[v, f] = f(v)$ for $v \in V$, $f \in V^*$.

Curried Weyl algebra: A representation M has maps $b\colon V \otimes M \to M$ and $b'\colon M \to V \otimes M$ such that

(a) b defines commutative action

(b) b' defines cocommutative coaction

(c) $b'b - b_1 b_2' = \mathrm{id}$ as maps $V \otimes M \to V \otimes M$.

There are 2 basic kinds of $\mathbf{FI}\sharp$ morphisms: $S \setminus x \to S$ and $S \to S \setminus x$ that correspond to $\mathbf{k}\langle 1 \rangle \otimes M \to M$ and $M \to \mathbf{k}\langle 1 \rangle \otimes M$.

Theorem: $\mathbf{FI}\sharp$-modules are equivalent to representations of the curried Weyl algebra with $V = \mathbf{k}\langle 1 \rangle$.

Symplectic Lie algebra

Recall: V a vector space, then $V \oplus V^*$ has a symplectic form ω:

$$\omega((v,f),(v',f')) = f(v') - f'(v).$$

The subalgebra of $\mathfrak{gl}(V \oplus V^*)$ preserving ω decomposes as (D=divided power=symmetric power in char. 0)

$$\mathfrak{sp}(V \oplus V^*) = \mathrm{D}^2(V^*) \oplus \mathfrak{gl}(V) \oplus \mathrm{D}^2(V).$$

Hence a linear map $\mathfrak{sp}(V \oplus V^*) \otimes M \to M$ is equivalent to:

$$a\colon V \otimes M \to V \otimes M, \quad b\colon \mathrm{D}^2(V) \otimes M \to M, \quad b'\colon M \to \mathrm{Sym}^2(V) \otimes M.$$

Being a representation is equivalent to list of conditions:

(a) a is $\underline{\mathfrak{gl}(V)}$-representation
(b) b defines action of $\mathrm{Sym}(\mathrm{D}^2(V))$ on M
(c) b' defines coaction of $\mathrm{Sym}(\mathrm{Sym}^2(V))$ on M.
(d) b, b' are $\underline{\mathfrak{gl}(V)}$-equivariant

(e) $b'b - b_1 b_2' = (m \otimes 1)(1 \otimes a)(\Delta \otimes 1)$ as maps (m=mult, Δ=comult)

$$\mathrm{D}^2(V) \otimes M \to \mathsf{Sym}^2(V) \otimes M$$

Definition: Let V be an object of a symmetric monoidal abelian category $\mathcal{C}$. A representation of $\underline{\mathfrak{sp}}(V \oplus V^*)$ is an object M together with maps a, b, b' satisfying (a)–(e).
Let $\mathrm{Rep}(\underline{\mathfrak{sp}}(V \oplus V^*))$ be the category of representations.

Example: Take $\mathcal{C} = \mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$ and $V = \mathbf{k}\langle 1 \rangle$; is there a nice description of $\mathrm{Rep}_{\mathbf{k}}(\underline{\mathfrak{sp}}(V \oplus V^*))$?
$\mathsf{Sym}^2(V) = \mathrm{D}^2(V)$ is the functor that sends 2-element sets to the trivial representation of $\mathfrak{S}_2$ and all other sets to 0.
To describe b, we need maps $M(S \setminus \{x, y\}) \to M(S)$ for all $x \neq y \in S$ and for b', need $M(S) \to M(S \setminus \{x, y\})$.

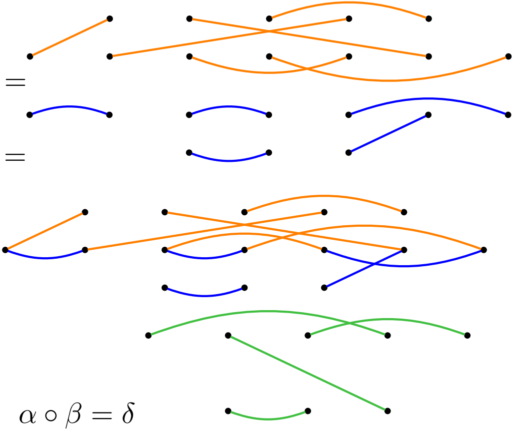
Objects: finite sets S

$\mathrm{Hom}_{\mathfrak{B}(\delta)}(S, T)$**:** **k**-linear span of perfect matchings of $S \amalg T$, i.e., decomposition as disjoint union of 2-element subsets

Composition: merge diagrams, closed loop $\to \delta$

$\alpha =$

$\beta =$



$\alpha \circ \beta = \delta$

Brauer algebra $\mathfrak{B}_n(\delta)$ is the endomorphism algebra of $[n]$. Surjects onto $\mathrm{End}_{\mathbf{O}(V)}(V^{\otimes n})$ for $\delta = \dim V$ and $\mathrm{End}_{\mathbf{Sp}(W)}(W^{\otimes n})$ for $\delta = -\dim W$.

Representations of $\mathfrak{B}(\delta)$ are functors to the category of $\mathbf{k}$-vector spaces.

Horizontal concatenation gives tensor structure to $\mathfrak{B}(\delta)$. Tensor generators for category are the unique diagrams in $\mathrm{Hom}(1, 1)$, $\mathrm{Hom}(0, 2)$, $\mathrm{Hom}(2, 0)$.

Upwards Brauer category is subcategory generated by $\mathrm{Hom}(1, 1)$ and $\mathrm{Hom}(0, 2)$ (δ is irrelevant). We had observed earlier that its representations are modules over the tca $\mathrm{Sym}(\mathrm{Sym}^2(\mathbf{k}\langle 1\rangle))$.

Recall: We are trying to describe $\mathrm{Rep}(\underline{\mathfrak{sp}}(V \oplus V^*))$ for $V = \mathbf{k}\langle 1\rangle \in \mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$.

Let $M \in \mathrm{Rep}_{\mathbf{k}}(\mathfrak{B}(\delta))$. Define maps a, b, b':

a**:** Since $\mathbf{FB} \subset \mathfrak{B}(\delta)$, M has a standard action a'. Let $a = a' + \frac{\delta}{2}\mathrm{id}$.

b**:** For $x \neq y \in S$, there is a unique Brauer morphism $S \setminus \{x, y\} \to S$ that is the identity on $S \setminus \{x, y\}$. These define $b\colon \mathrm{Sym}^2(V) \otimes M \to M$.

b'**:** Similar to the construction of b.

Theorem: The above defines an equivalence

$$\mathrm{Rep}_{\mathbf{k}}(\mathfrak{B}(\delta)) \to \mathrm{Rep}_{\delta/2}(\underline{\mathfrak{sp}}(V \oplus V^*)).$$

The subscript $\delta/2$ denotes the subcategory of $\underline{\mathfrak{sp}}$ representations whose restriction to $\underline{\mathfrak{gl}}$ is standard after subtracting $\frac{\delta}{2}\mathrm{id}$.

Motivation: The right hand side, when transferred to $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$, is an infinite-rank analogue of parabolic category $\mathcal{O}$ for the symplectic Lie algebra. The left side gives combinatorial model which we use to connect to orthosymplectic Lie supergroups and Deligne's category for orthogonal group via Serre quotients.

Similar examples: Orthogonal Lie algebras $\mathfrak{so}(V \oplus V^*)$ and $\mathfrak{so}(V \oplus V^* \oplus \mathbf{k})$, and some other classical Lie (super)algebras.

FS is the category of **F**inite sets and **S**urjections.
$\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}})$ also plays an important role in representation stability; attempts to find a tca structure proved futile.
Goal was to describe $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}})$ ($\mathrm{Rep}_{\mathbf{k}}(\mathbf{FI})$ understood)

Projective generators are $P_d = \mathbf{k}[\mathrm{Hom}_{\mathbf{FS}^{\mathrm{op}}}([d], -)]$
Can identify $\bigoplus_n P_d(n)/\mathfrak{S}_d = x_1 \cdots x_d \mathbf{k}[x_1, \ldots, x_d]$

For $x \leq a$, define
$L_{r,a,x} \colon [a + r] \to [a]$ by $i \mapsto i$ for $i \leq a$ and $j \mapsto x$ for $j > a$ and $L_{r,a} = \sum_{x=1}^{a} L_{r,a,x}$.

Then $L_{r,n} \colon P_d(n)/\mathfrak{S}_d \to P_d(d + n)/\mathfrak{S}_d$, let L_r be sum of $L_{r,n}$.
Have $L_r = \sum_{i=1}^{d} x_i^{r+1}\partial_i$. Note that $[L_r, L_s] = (s - r)L_{r+s}$, so get action of positive half of Witt Lie algebra.

Witt Lie algebra

Given a vector space V with basis $\{x_i\}$, define $W_{\geq 0}(V)$ to be the Lie algebra spanned by $f\partial_i$ where f is a polynomial in x_i.
Let $W_{>0}(V)$ be subalgebra where $f(0) = 0$.

$$W_{\geq 0}(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V) \otimes V^*, \qquad W_{>0}(V) = \bigoplus_{n>0} \mathrm{Sym}^n(V) \otimes V^*,$$

Hence a representation M of the curried Lie algebra $W_{\geq 0}(V)$ is a map $a\colon \mathrm{Sym}\, V \otimes M \to V \otimes M$.
The condition is $[a_1, a_2] = a' - \tau a' \tau$ as maps $\mathrm{Sym}(V) \otimes \mathrm{Sym}(V) \otimes M \to V \otimes V \otimes M$ where a' is

$$\begin{aligned}
\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V \otimes M &\xrightarrow{\mathrm{id}\otimes\Delta\otimes\mathrm{id}} \mathrm{Sym}\, V \otimes V \otimes \mathrm{Sym}\, V \otimes M \\
&\xrightarrow{\tau\otimes\mathrm{id}\otimes\mathrm{id}} V \otimes \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V \otimes M \\
&\xrightarrow{\mathrm{id}\otimes m\otimes\mathrm{id}} V \otimes \mathrm{Sym}\, V \otimes M \\
&\xrightarrow{\mathrm{id}\otimes a} V \otimes V \otimes M
\end{aligned}$$

Let $a^{(n)}\colon \mathrm{Sym}^n V \otimes M \to V \otimes M$ be the component maps.

Theorem: $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FS}^{\mathrm{op}}) \simeq \mathrm{Rep}(\underline{W}_{>0}(\mathbf{k}\langle 1\rangle))^{\star}$.

$a^{(n)}$ comes from surjections $[n] \amalg S \to * \amalg S$ that collapse $[n]$ to $*$.

Theorem: $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FiniteSet}^{\mathrm{op}}) \simeq \mathrm{Rep}(\underline{W}_{\geq 0}(\mathbf{k}\langle 1\rangle))^{\star}$.

If we allowed additional generating morphisms we can build the

partition category. We tried to guess the curried Lie algebra this comes from, as we get maps

$$\mathrm{Sym}^n(V) \otimes M \to \mathrm{D}^m(V) \otimes M,$$

so the Lie algebra would have underlying space $\mathrm{Sym}(V \oplus V^*)$.

Sym($V \oplus V^*$) supports the **Hamiltonian Lie algebra** $H(V \oplus V^*)$ with bracket (pick basis $\{x_i\}$ for V and dual basis $\{y_i\}$ for V^*)

$$[f, g] = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}.$$

This is not the right match though, in fact, $H(V \oplus V^*)$ matches up with a "degenerate" partition category (in composition, if two blocks meet in 2 or more vertices, result is 0).

The right thing is to take the Lie algebra underlying the Weyl algebra on $V \oplus V^*$, the latter is the tensor algebra on $V \otimes V^*$ modulo the 2-sided ideal $fx - xf = f(x)$ for $x \in V$, $f \in V^*$.

$V = \mathbf{k}\langle 1\rangle$ in $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$

$\mathcal{C}$	Curried Lie algebra
Brauer	Symplectic Lie algebra $\mathfrak{sp}(V \oplus V^*)$
Signed Brauer	Orthogonal Lie algebra $\mathfrak{so}(V \oplus V^*)$
Spin Brauer	Orthosymplectic Lie superalgebra $\mathfrak{osp}(\mathbf{k}\vert V \oplus V^*)$
Signed spin Brauer	Orthogonal Lie algebra $\mathfrak{so}(\mathbf{k} \oplus V \oplus V^*)$
$\mathbf{FI}\sharp(\delta)$	Heisenberg Lie algebra on $V \oplus V^*$
Partition category	Weyl Lie algebra on $V \oplus V^*$
Degenerate partition	Hamiltonian algebra on $V \oplus V^*$
FI	Symmetric algebra on V
$\mathbf{FS}^{\mathrm{op}}$	$(\star)$ Positive Witt Lie algebra on V
$\mathbf{FiniteSet}^{\mathrm{op}}$	$(\star)$ Witt Lie algebra on V
Periplectic Brauer	Periplectic Lie superalgebra on $V \oplus V^*[1]$

Other categories besides $\mathrm{Rep}_{\mathbf{k}}(\mathbf{FB})$? One source is $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VB}(\mathbf{F}_q))$ where $\mathbf{VB}(\mathbf{F}_q)$ is $\mathbf{F}_q$-vector spaces and invertible operators with product $(F \otimes G)(V) = \bigoplus_{V=W\oplus W'} F(W) \otimes F(W')$. Also, category of graded vector spaces with shuffle product

Braided monoidal categories? Should we use τ or τ^{-1}? One interesting example is parabolic product on $\mathrm{Rep}_{\mathbf{k}}(\mathbf{VB}(\mathbf{F}_q))$ (with $q^{-1} \in \mathbf{k}$) given by $(F \otimes G)(V) = \bigoplus_{W\subseteq V} F(W) \otimes G(V/W)$

Lie theory for Brauer algebras Cox–de Visscher–Martin found descriptions using Weyl group for blocks of Brauer algebra $\mathfrak{B}_n(\delta)$ which match up with combinatorics in parabolic category $\mathcal{O}$ for the symplectic Lie algebra. Can curried description of Brauer category find a more direct link?

Other curried Lie algebras?