Splitting rings and cohomology of supergrassmannians

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September 28, 2021

joint work with Andrew Snowden
See also: arXiv:2108.00504

What is it?

The talk will go in a different order, but here is how we came to these topics:

- The syzygies of generic determinantal varieties have an unexpected Lie superalgebra symmetry, as shown in works of Akin, Pragacz, Raicu, Sam, Weyman.
- We realized that this is related to the structure sheaf of the supergrassmannian (more on this later) via Kempf collapsing (the "geometric technique" in Weyman's book).
- This geometric setup naturally constructs finite covers of determinantal varieties; we later realized these are examples of splitting rings (which are fundamental in intersection theory).
- To get a satisfactory picture, we needed to develop basic algebraic properties of these rings.
  The setup is elementary, so we start there.

- Let $f(t)$ be a degree n monic polynomial over A.
- The splitting ring of f, $\mathrm{Split}_A(f)$, is the quotient of $A[\xi_1, \ldots, \xi_n]$ in which the following equality holds:

$$(t - \xi_1) \cdots (t - \xi_n) = f(t).$$

- If $n = 2$, this is the usual construction of adjoining a root of f.
- Compatible with base change: given $\varphi \colon A \to A'$, then

$$\mathrm{Split}_{A'}(\varphi(f)) = A' \otimes_A \mathrm{Split}_A(f).$$

- Universal construction:
  $A^{\mathrm{univ}} = \mathbb{Z}[a_1, \ldots, a_n]$ with $f(t) = t^n + a_1 t^{n-1} + \cdots + a_n$;
  $\mathrm{Split}_{A^{\mathrm{univ}}}(f) \cong \mathbb{Z}[\xi_1, \ldots, \xi_n]$.
  Some things can be deduced from this case and base change.

- The universal case behaves really well; standard argument shows that $a_1 = \cdots = a_n = 0$ is "most degenerate" example. That case is well-known in algebraic combinatorics as the coinvariant ring of the symmetric group.
- It's a complete intersection and $\mathrm{Split}_{A^{\mathrm{univ}}}(f)$ is free of rank $n!$. Monomial basis: $\xi_1^{p_1} \cdots \xi_n^{p_n}$ where $0 \leq p_i \leq n - i$.
(More interesting basis: Schubert polynomials – important objects in intersection theory of flag varieties)
- Thus, $\mathrm{Split}_A(f)$ is a free A-module of rank $n!$ for all A.
($\mathrm{Split}_A(f)$ is isomorphic to the regular representation of $\mathfrak{S}_n$ over A if and only if $n!$ is invertible in A)
- Important example: let $\mathcal{E}$ be a rank n vector bundle over a smooth variety X, A the Chow ring of X, and f the Chern polynomial of $\mathcal{E}$. Then $\mathrm{Split}_A(f)$ is the Chow ring of the relative flag variety of $\mathcal{E}$ (this realizes the "splitting principle" in intersection theory – the ξ_i are the Chern roots).

Let $\psi\colon A \to \mathsf{Split}_A(f)$ be inclusion map.
Let $\Delta = \prod_{i \neq j}(\xi_i - \xi_j)$ be discriminant (it belongs to A).

- ψ is syntomic (i.e., flat of finite presentation and all fibers are local complete intersections).
- (Laksov) ψ admits an A-linear splitting.
- If A is Cohen–Macaulay, then so is $\mathsf{Split}_A(f)$ (more generally, if A satisfies Serre's condition (S_k) so does $\mathsf{Split}_A(f)$).
- If Δ is a unit, then ψ is étale.
- If Δ is NZD and A is reduced, then $\mathsf{Split}_A(f)$ is reduced.

We needed to prove that $\mathrm{Split}_A(f)$ is normal for specific example.

Example

$A = \mathbb{Z}$ and $f(t) = t^2 - 5$.
Then $\mathrm{Split}_A(f) \cong \mathbb{Z}[\sqrt{5}]$ is not normal.

Theorem

Suppose that

- *A is normal,*
- *Δ is a nonzerodivisor,*
- *For any prime $\mathfrak{p} \subset A$, $\Delta \in \mathfrak{p}^2 A_{\mathfrak{p}}$ implies that $\operatorname{codim} \mathfrak{p} \geq 2$.*

Then $\mathrm{Split}_A(f)$ is normal.

In our example, $\Delta = 20$; taking $\mathfrak{p} = (2)$ violates the condition.

$f(t) \in A[t]$ is a degree n monic polynomial.
Pick $p+q=n$ and let $\mathrm{Fact}_A^{p,q}(f)$ be the quotient of $A[b_1,\ldots,b_p,c_1,\ldots,c_q]$ by relations $g(t)h(t)=f(t)$ where

$$g(t) = t^p + b_1 t^{p-1} + \cdots + b_p,$$
$$h(t) = t^q + c_1 t^{q-1} + \cdots + c_q.$$

Let $A' = \mathrm{Fact}_A^{p,q}(f)$; we have a natural isomorphism

$$\mathrm{Split}_A(f) = \mathrm{Split}_{A'}(g) \otimes_A \mathrm{Split}_{A'}(h).$$

- A' is free of rank $\binom{n}{p}$ over A
- $A \to A'$ is syntomic (flat, finite presentation, and lci)
- If A is Cohen–Macaulay then so is A'.
- If $\mathrm{Split}_A(f)$ is reduced or normal, then so is A'.
- Gives Chow ring of Grassmannian bundles

Motivation comes from superalgebra structure on Tor groups of determinantal variety.

Let E, F be complex vector spaces and consider $X = \mathrm{Hom}(E, F)$ and for each $r \geq 0$, X_r is subvariety of rank $\leq r$ linear maps.

$\mathfrak{gl}(E) \times \mathfrak{gl}(F)$ acts on $\mathrm{Tor}_\bullet(\mathbb{C}[X_r], \mathbb{C})$ by linear substitution
$\mathrm{Hom}(E, F)^* = \mathrm{Hom}(F, E)$ acts (skew-commutatively) via Tor algebra multiplication

$$\mathrm{Tor}_\bullet(\mathbb{C}[X_r], \mathbb{C}) \to \mathrm{Tor}_{\bullet+1}(\mathbb{C}[X_r], \mathbb{C}).$$

This combines to give action of Lie superalgebra

$$\begin{bmatrix} \mathfrak{gl}(E) & \mathrm{Hom}(F, E) \\ 0 & \mathfrak{gl}(F) \end{bmatrix}$$

Surprisingly, there exists an action of Hom(E, F)

$$\mathrm{Tor}_\bullet(\mathbb{C}[X_r], \mathbb{C}) \to \mathrm{Tor}_{\bullet-1}(\mathbb{C}[X_r], \mathbb{C}).$$

that fills in the whole matrix, i.e., gives an action of the entire Lie superalgebra

$$\mathfrak{gl}(E|F) = \begin{bmatrix} \mathfrak{gl}(E) & \mathrm{Hom}(F, E) \\ \mathrm{Hom}(E, F) & \mathfrak{gl}(F) \end{bmatrix}$$

(There is also a commutative version of this involving $\mathfrak{gl}(E \oplus F)$ acting on $\mathbb{C}[X_r]$ – not sure if there's any connection)
This rigidifies the Tor groups quite a bit; result was proven in several different ways (Pragacz–Weyman, Akin–Weyman, Sam, Raicu–Weyman)

We wanted to understand a "natural" reason for this action.

Kempf collapsing: example

Consider case $\dim E = r + 1 \leq \dim F$, so that X_r consists of linear maps with a nonzero kernel.

If $L \subset E$ is 1-dimensional, we "linearize" X_r by considering all maps $E \to F$ which factor through E/L (this is just $\mathrm{Hom}(E/L, F)$) and varying choice of L.

More precisely: choice of L is a point in projective space $\mathbb{P}(E)$. $\mathcal{O}(-1)$ embeds into $E \times \mathbb{P}(E)$ and fiber over $[L] \in \mathbb{P}(E)$ is just $L \times [L]$. The "linearization" above is the vector bundle $\mathrm{Hom}(E/\mathcal{O}(-1), F)$ over $\mathbb{P}(E)$.

This maps to $\mathrm{Hom}(E, F)$ and the image is precisely X_r.

We can find a locally free resolution of $\mathrm{Hom}(E/\mathcal{O}(-1), F)$ inside the trivial bundle $\mathrm{Hom}(E, F) \times \mathbb{P}(E)$ easily since it's a subbundle.

Namely, it is locally cut out by linear equations – the dual of the quotient bundle $\mathcal{O}(-1) \otimes F^*$, so we get the Koszul complex on this bundle.

Via standard homological algebra, we then get

$$\mathrm{Tor}_i(\mathbb{C}[X_r], \mathbb{C})_{i+j} = \mathrm{H}^j(\mathbb{P}(E); \bigwedge^{i+j}(\mathcal{O}(-1) \otimes F^*))$$

and the latter can be computed explicitly if you know the cohomology of line bundles on projective space.
This is Kempf's construction of the Eagon–Northcott complex.
This relies on knowing that $\mathbb{C}[X_r]$ is the *derived* pushforward of the structure sheaf of $\mathrm{Hom}(E/\mathcal{O}(-1), F)$ which follows from $\mathrm{Tor}_0 = \mathbb{C}$ and $\mathrm{Tor}_i = 0$ for $i < 0$.

This generalizes quite a bit. Consider the following data:

- V is a projective variety
- E is a vector space
- $\xi \subset E \times V$ is a subbundle with quotient bundle η.

Cohomology of $\bigwedge^\bullet(\xi)$ computes Tor of the derived pushforward of Sym(η) under projection $E^* \times V \to E^*$.

Best situation is if higher direct images vanish, map is birational, and image is normal, so we get Tor groups of a subvariety of E^*.

Can do general determinantal varieties: $\mathbb{P}(E)$ is replaced by a Grassmannian and $\mathcal{O}(-1)$ is replaced with the tautological subbundle $\mathcal{R}$.
Hence: is there a more obvious structure on $\bigwedge^\bullet(\xi)$ which gives superalgebra symmetry?

Given a graded-commutative superalgebra A, can define prime spectrum $\mathrm{Spec}(A)$ as a locally ringed space by mimicking usual definition (now have a sheaf of superalgebras). Superschemes are locally ringed spaces which are locally of the form $\mathrm{Spec}(A)$.

These are generally non-reduced, so we can't describe them using geometric points; better to use the functor of points perspective.

Given a super vector space V and integers $0 \leq d \leq \dim V_0$ and $0 \leq e \leq \dim V_1$, there is a supergrassmannian $\mathrm{Gr}(d|e, V)$.

For a superalgebra T, the T-valued points are submodules of $T \otimes V$ which are locally free of rank $d|e$ (i.e., locally isomorphic to $T^{\oplus d} \oplus T[1]^{\oplus e}$).

The underlying topological space of $\mathrm{Gr}(d|e, V)$ is $\mathrm{Gr}(d, V_0) \times \mathrm{Gr}(e, V_1)$; the structure sheaf $\mathcal{O}$ is not easy to describe.

However, let $\mathcal{J}$ be the ideal sheaf generated by odd elements.

The associated graded of $\mathcal{O}$ with respect to $\mathcal{J}$ is the exterior algebra on vector bundle $\mathcal{J}/\mathcal{J}^2$ (this is true for any smooth superscheme) and

$$\mathcal{J}/\mathcal{J}^2 \cong \mathrm{Hom}(\mathcal{Q}_1, \mathcal{R}_0) \oplus \mathrm{Hom}(\mathcal{Q}_0, \mathcal{R}_1)$$

where $\mathcal{R}_0$ is tautological subbundle on $\mathrm{Gr}(d, V_0)$, $\mathcal{Q}_0 = V_0/\mathcal{R}_0$, and similarly for $\mathcal{R}_1, \mathcal{Q}_1$.

Connection to determinantal varieties

$\xi = \mathrm{Hom}(\mathcal{Q}_1, \mathcal{R}_0) \oplus \mathrm{Hom}(\mathcal{Q}_0, \mathcal{R}_1)$ is a subbundle of the trivial bundle $\mathrm{Hom}(V_1, V_0) \oplus \mathrm{Hom}(V_0, V_1)$. Let η be the quotient.

The cohomology of $\bigwedge^\bullet(\xi)$ computes two things:

- Tor groups of derived pushforward of $\mathrm{Sym}(\eta)$.
- Input to spectral sequence

$$\mathrm{H}^{p+q}(\mathrm{Gr}(d, V_0) \times \mathrm{Gr}(e, V_1); \bigwedge^p \xi) \Longrightarrow \mathrm{H}^{p+q}(\mathrm{Gr}(d|e, V); \mathcal{O}_{\mathrm{Gr}(d|e,V)}).$$

We show:

- Higher direct images of $\mathrm{Sym}(\eta)$ vanish (not straightforward), but map is not birational!
- The spectral sequence degenerates.

Structure sheaf of $\mathrm{Gr}(d|e, V)$ has $\mathfrak{gl}(V)$-action, and so does its cohomology. Degeneracy implies Tor groups also have this structure.

Consider $e = 0$, so $\mathcal{R}_1 = 0$ and $\mathcal{Q}_1 = V_1$. Then

$$\xi = \mathrm{Hom}(V_1, \mathcal{R}_0).$$

This is exactly the bundle used in Kempf collapsing approach to determinantal variety.
There is an extra $\mathrm{Hom}(V_1, V_0)$ factor which doesn't affect Tor calculations.

So up to shifting incides, the Tor groups of determinantal variety give cohomology of structure sheaf of $\mathrm{Gr}(d|0, V)$.

Degeneracy of spectral sequence comes from the fact that the Tor groups are multiplicity-free as representation of $\mathfrak{gl}(V_0) \times \mathfrak{gl}(V_1)$.

- Let $n = \dim V_0$, $m = \dim V_1$, assume WLOG $d \geq e \geq 0$. Set
$$\delta = \begin{cases} m - n + d - e & \text{if } d - e > n - m \\ 0 & \text{else} \end{cases}.$$
- $W = \mathrm{Hom}(V_0, V_1) \times \mathrm{Hom}(V_1, V_0)$ with coordinate ring $\mathcal{O}_W$.
- Let $Z \subset W$ consist of pairs (f, g) such that $\mathrm{rank}\, f \leq m - \delta$.
- $\chi(u) \in \mathcal{O}_Z[u]$ is characteristic polynomial of fg; $\overline{\chi} = \chi/u^\delta$.

Zero locus of ξ is subbundle of $W \times \mathrm{Gr}(d, V_0) \times \mathrm{Gr}(e, V_1)$ of tuples (f, g, R_0, R_1) such that $f(R_0) \subseteq R_1$, $g(R_1) \subseteq R_0$.

Theorem

- *Higher direct images of* $\mathrm{Sym}(\eta)$ *vanish*
- *Pushforward of* $\mathrm{Sym}(\eta)$ *is* $Fact^{\mathcal{O}_Z}_{e,m-\delta-e}(\overline{\chi})$. *It is normal, C-M and has rational singularities.*
- $\mathrm{Tor}_0^{\mathcal{O}_W}(Fact^{\mathcal{O}_Z}_{e,m-\delta-e}(\overline{\chi}), \mathbb{C}) \cong \mathrm{H}^*_{\mathrm{sing}}(\mathrm{Gr}(e, \mathbb{C}^{m-\delta}), \mathbb{C})$.
- $\mathrm{H}^*(\mathrm{Gr}(d|e, V), \mathcal{O})$ *is this tensored with Tor algebra of* Z.

Normality of $\mathrm{Fact}^{\mathcal{O}_Z}_{e,m-\delta-e}(\overline{\chi})$

Need: (1) $\mathcal{O}_Z$ normal; (2) Δ NZD; (3) $\Delta \in \mathfrak{p}^2\mathcal{O}_{Z,\mathfrak{p}} \implies \operatorname{codim}\mathfrak{p} \geq 2$

(1) Well-known; can be deduced from Kempf approach

(2) $\Delta \neq 0$: write down example (diagonal matrices) where char. poly. of fg has no repeated roots

In fact, we show Δ is irreducible using classification of pairs (f, g).

(3) Since Δ is irreducible, just need one closed point $\mathfrak{m}$ where $\mathfrak{m} \in V(\Delta)$ but $\Delta \notin \mathfrak{m}^2\mathcal{O}_{Z,\mathfrak{m}}$:

$$f = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}, \qquad A = \begin{bmatrix} \lambda & 1 & 0 \\ \varepsilon & \lambda & 0 \\ 0 & 0 & diag \end{bmatrix}, \qquad g = \begin{bmatrix} 0 & 0 \\ 0 & id \end{bmatrix}$$

diag has distinct eigenvalues from λ and $\varepsilon^2 = 0$; Δ is unit times ε

Have explicit presentation of splitting ring, so can prove normality using Jacobian criterion (calculation somewhat similar but more bookkeeping)

Determinantal varieties in symmetric or skew-symmetric matrices?
$\mathfrak{gl}(n|m)$ replaced by periplectic Lie superalgebra $\mathfrak{pe}(n) \subset \mathfrak{gl}(n|n)$

$$\begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}, \qquad B = B^T, \quad C = -C^T.$$

Supergeometry approach leads to determinantal varieties in $\mathrm{Sym}^2(V) \oplus \bigwedge^2(V^*)$ (unifies both cases strangely – hint of that: representations in Tor are the same). [work in progress]

Splitting rings replaced by signed splitting rings (for monic polynomial f in u^2, consider factorizations into $(u^2 - \xi_i^2)$) and similar analogue for factorization rings.
These are related to Chow rings of isotropic flag varieties of symplectic vector bundles.

Instead of supergrassmannians, makes sense to consider general super flag varieties. [complicated, some partial progress]