Polynomials of bounded degree

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Warm-up

- Ways to define the rank of a polynomial?
- How about system of polynomials?
- Special properties of systems of polynomials with high rank (relative to their degrees)?

$\mathbf{C}[x_1, \ldots, x_r]$ is the ring of complex polynomials in r variables,

$f_1, \ldots, f_n$ are *homogeneous* polynomials, assumed linearly independent.

$I = (f_1, \ldots, f_r) = \{\sum_i g_i f_i \mid g_i \text{ polynomial}\}$
is **ideal generated by f's**,

$Z = Z(I) = \{(a_1, \ldots, a_r) \in \mathbf{C}^r \mid f_1(a) = \cdots = f_n(a) = 0\}$
is the **zero set** of I or f's. Use Euclidean or Zariski topology.

Z has a nonempty open subset which is a disjoint union of complex manifolds, define **dimension** as the max dimension.

$f_1, \ldots, f_n \in \mathbf{C}[x_1, \ldots, x_r]$

Codimension is $\operatorname{codim} = r - \dim$. Always have $\operatorname{codim} \leq n$.
If $\operatorname{codim} = n$, $f_1, \ldots, f_n$ is called **regular sequence**.
Regular sequence implies algebraically independent, but is stronger.

If $n = 1$, always true.
If $n = 2$, true if and only if f_1, f_2 have no common factors.

For a random linear space L with $\dim L = \operatorname{codim} Z$, $L \cap Z$ is a finite set of points, **degree** $\deg Z$ is the number.

Bézout bound: $\deg Z \leq \prod_{i=1}^{n} \deg f_i$.

Of note: codim and deg are bounded by n and $\deg f_i$, but independent of r.

$f_1, \ldots, f_n \in \mathbf{C}[x_1, \ldots, x_r]$

Projective dimension pdim is length of minimal free resolution of $\mathbf{C}[x_1, \ldots, x_r]/I$. Higher pdim = more complicated.
Lower bound: pdim $\geq$ codim
Hilbert bound: pdim $\leq r = \#$variables.

(Castelnuovo–Mumford) regularity reg is the "height" of minimal free resolution.
Related to when Hilbert function agrees with Hilbert polynomial
Bound (Galligo, Giusti, Caviglia–Sbarra): reg $\leq (2 \max \deg f_i)^{2^{r-2}}$

Stillman (2000): Is there an upper bound for pdim independent of $r = \#$variables?
If answer is yes, certain Gröbner basis calculations can in principle be replaced by linear algebra calculations

For 1 polynomial: pdim $= 1$
For 2 polynomials: pdim $= 2$
For 3 polynomials: Bruns showed that pdim is unbounded, however his examples use polynomials of higher and higher degree

Refine question: bound should depend on number of polynomials n and their max degree D

Caviglia showed that positive answer also implies bound on regularity independent of r.

Subalgebras generated by regular sequences

Naive improvement to Hilbert bound: if $f_1, \ldots, f_n$ only use s of the variables, then pdim $\leq s$ (others don't matter).
Can try to improve by allowing linear changes of coordinates

Not practical though: $x_1^2 + x_2^2 + \cdots + x_r^2$ cannot be defined using less than r variables (rank of quadric).

Less naive: If there is regular sequence $g_1, \ldots, g_s$ so that f's are in subalgebra generated by g's; then pdim $\leq s$ by flatness argument.

Ananyan–Hochster theorem: can always find $g_1, \ldots, g_s$ where s is bounded by $n = \#$polynomials and $D = \max\deg f_i$.
They call subalgebra generated by g's a *small subalgebra*.

First approximation

First approximation of idea for existence of small subalgebras:

- If $f_1, \ldots, f_n$ is a regular sequence, take $g_i = f_i$, bound is $s = n$.
- Otherwise, decompose one of the polynomials into smaller degree polynomials,

$$f_1 = g_1 h_1 + \cdots + g_e h_e$$

  and consider now $g_1, \ldots, g_e, h_1, \ldots, h_e, f_2, \ldots, f_n$. For a suitable ordering, this is a simpler system of polynomials, and we can continue if we can bound e.

Problem: $\sum x_i^2$ suggests we can't control e. Obvious improvement is not to decompose f_1, but to pick f_i carefully to minimize e. Even better, consider all linear combinations of the f_i to minimize e.

Formalize previous ideas:

- The **strength** ν of a homog. polynomial f is the minimal e such that there exists homog. decomposition

$$f = g_1 h_1 + \cdots + g_e h_e$$

  with $\deg g_i, \deg h_i < \deg f$.
  This always exists if $\deg f > 1$ since can use variables, so strength $\leq$ #variables. Linear forms have ∞ strength.
- The **strength** ν of $f_1, \ldots, f_n$ is the minimal strength of a nonzero homogeneous linear combination.

Ananyan–Hochster theorem: There exists $N = N(n, D)$ such that either $f_1, \ldots, f_n$ is a regular sequence, or strength is $< N$.

Waring rank of f is minimal e such that

$$f = \ell_1^d + \cdots + \ell_e^d$$

where ℓ_i are linear; natural from perspective of secant varieties of Veronese embeddings. For $d = 2$, this is the usual rank of a quadric.

Compare: In non-commutative setting, $v_1 \otimes \cdots \otimes v_d \in V_1 \otimes \cdots V_d$ are rank 1 tensors (rank r means sum of r rank 1)
Slice rank 1 tensors are of the form $v_i \otimes \omega$ where $v_i \in V$ and $\omega \in \bigotimes_{j \neq i} V_j$ (introduced in study of "cap-set problem")

A-H theorem implies small subalgebras

- Order polynomials by their degree list $\deg f_1 \geq \cdots \geq \deg f_n$ lexicographically. When decomposing a polynomial, the degree list gets smaller. The outlined process terminates by well-ordering property of lexicographic order.
- At all stages, if we don't have a regular sequence, the list of possibilities for new degree sequences is finite by A-H theorem.
- So the whole process is a tree where each node has finitely many children and each path is finite. So the whole tree is finite, which gives bound for s and hence existence of small subalgebras.

Other implications

- Existence of small subalgebras gives unifying perspective on finding bounds for invariants depending only on #polynomials and $\max \deg$.
- Having large enough strength implies more than just regular sequence. Also implies:
  - $Z(f)$ is connected and irreducible,
  - $f_1, \ldots, f_n$ generates a prime ideal,
  - $Z(f)$ is smooth away from 0.
  - More generally, singular locus of $Z(f)$ has codimension $\geq c$ for some fixed c.
  - $Z(f)$ is unirational (Harris–Mazur–Pandharipande)

Rephrasing: if $\nu(f_1, \ldots, f_n) \gg n \max \deg(f_i)$, then $f_1, \ldots, f_n$ is regular sequence.
Convenient to replace $\gg$ with some type of limit.
With Erman and Snowden, we consider two types of limits to give new proofs of A-H.

Limit 1 Let $R(d)$ be the ultrapower of complex polynomials of degree d in a set of variables $x_1, x_2, \ldots$. Then $R = \bigoplus_{d \geq 0} R(d)$ is a graded algebra.

Limit 2 Let $S(d)$ be the set of formal linear combinations of all monomials of degree d in $x_1, x_2, \ldots$. Again $S = \bigoplus_{d \geq 0} S(d)$ is a graded algebra.

For any graded algebra we can define strength. Now there can be non-decomposable elements of degree > 1, they have ∞ strength.

Erman–Sam–Snowden: Both R and S are isomorphic to polynomial rings. We have $\nu(f_1, \dots, f_n) = \infty$ if and only if $\{f_1, \dots, f_n\}$ can be extended to a list of algebraically independent generators.

Proof of A-H: If A-H were false, then we can find collections of non-regular sequences $f_1^{(i)}, \dots, f_n^{(i)}$ of degree $\leq D$ such that $\lim_i \nu \to \infty$.
Take ultralimit $f_j = \mathrm{ulim}_i f_j^{(i)}$ to get elements of R.
Theorem says that $f_1, \dots, f_n$ are a regular sequence. A technical argument implies that this must be true for infinitely many of the original sequences.

Start with non-principal ultrafilter, a collection $\mathcal{F}$ of infinite subsets of $\mathbf{N}$ such that

- Closed under intersection and taking supersets
- For all $S \subset \mathbf{N}$, either $S \in \mathcal{F}$ or $\mathbf{N} \setminus S \in \mathcal{F}$.

For sequences (x_i) and (y_i), $x \sim y$ if $\{i \mid x_i = y_i\} \in \mathcal{F}$.

Ultraproduct of $X_0, X_1, \ldots$ is $\mathbf{X} = \prod_i X_i / \sim$.
Inherits structure, for example X_i are rings implies $\mathbf{X}$ is a ring
More subtle: X_i are fields implies $\mathbf{X}$ is a field

Very flexible: any sequence has a well-defined limit (so proposed counterexamples don't need to be checked for "convergence")

Derivational criterion for polynomiality

The proof of our theorem uses:
Theorem (ESS): Let $A = \bigoplus_{d \geq 0} A_d$ be a commutative algebra with A_0 a field of characteristic 0. Suppose for all $f \in A_d$ with $d > 0$, there exists a negative degree derivation ∂ such that $\partial(f) \neq 0$. Then any minimal set of generators of A is algebraically independent.

Verification in our examples is easy: let ∂_i be partial derivative with respect to x_i; then for all f, there is some i such that $\partial_i(f) \neq 0$.

A topological space X is **noetherian** if every descending chain of closed subsets $Z_1 \supseteq Z_2 \supseteq \cdots$ satisfies $Z_i = Z_{i+1}$ for $i \gg 0$.

Holds for algebraic varieties (e.g., f.dim. vector spaces) with Zariski topology
A general method for proving boundedness of f:
$Z_i = \{x \mid f(x) \geq i\}$ (if closed)

Big if: a lot of functions require further refinement of X (flattening stratification)

Naive strategy: for fixed $d_1, \ldots, d_n$, tuple of homogeneous polynomials with those degrees in variables $x_1, x_2, \ldots$ has Zariski topology. Take $f = \text{pdim}$. Try to find flattening stratification. Not noetherian though.

GL-noetherianity

Space of polynomials not noetherian, but has GL group action (change of basis). We only care about subsets invariant under GL. Refine noetherian to **GL-noetherian**: only consider chains of GL-invariant closed subsets.

Draisma proved that space of tuples of polynomials is GL-noetherian. Can use this idea to give different proof of Stillman conjecture.
In fact, Draisma shows any polynomial functor is GL-noetherian.