Bi-graded Koszul modules, K3 carpets, and Green's conjecture

Steven V Sam

October 5, 2021

(joint work with Claudiu Raicu)

Let C be a smooth curve of genus g over an algebraically closed field $\mathbf{k}$ of characteristic 0 and define its canonical ring via

$$\Gamma_C = \bigoplus_{d \geq 0} \mathrm{H}^0(C; \omega_C^{\otimes d}).$$

This is a finitely generated module over $A = \mathrm{Sym}(\mathrm{H}^0(C; \omega_C)) \cong \mathbf{k}[x_1, \dots, x_g]$. We are concerned with the vanishing of the graded Betti numbers

$$\beta_{i,j}(C) = \dim_{\mathbf{k}} \mathrm{Tor}_i^A(\Gamma_C, \mathbf{k})_j.$$

Green's conjecture states that $\beta_{i,i+2}(C) = 0$ for $i < \mathrm{Cliff}(C)$, the Clifford index of C. This governs for how many steps the equations of C have only linear syzygies. Rather than define the Clifford index, we just remark that for most curves (in a sense which can be made precise using the moduli space of curves), the Clifford index of C is $\mathrm{gon}(C) - 2$ where $\mathrm{gon}(C)$ (the gonality of C) is the minimum degree of a non-constant map from C to the projective line.

Voisin [4, 5] showed that Green's conjecture holds generically. That is, there is a nonempty Zariski dense set in the moduli space of curves such that Green's conjecture holds for the curves in this set. The Clifford index of a curve is bounded from above by $(g-1)/2$ and a finer version of the result shows that this set contains curves of every gonality. Various strengthenings or refinements were introduced since then, but of interest to us is a recent proof by Aprodu, Farkas, Papadima, Raicu, and Weyman [1] which reproves Voisin's result using ideas from representation theory. The method of proof is in some ways simpler and extends the result to fields of characteristic $p \geq (g+2)/2$. (Of relevance is work of Schreyer [3] which shows that Green's conjecture does fail in small characteristic.)

The idea behind [1] is to consider rational cuspidal curves. These are smoothable, and due to the upper semicontinuity of Betti numbers in flat families, it suffices to prove that a version of Green's conjecture holds for them. Rational cuspidal curves (in their canonical embedding) with g cusps can be realized as hyperplane sections of the tangential surface T_g of the degree g rational normal curve, and hence the rational cuspidal curve and T_g have the same graded Betti numbers. There is a short exact sequence of graded modules

$$0 \to \mathbf{k}[T_g] \to \mathbf{k}[\widetilde{T_g}] \to M(-1) \to 0$$

where the first term is the homogeneous coordinate ring of T_g, the middle term is its normalization, and M is the canonical module of the homogeneous coordinate ring of the degree g rational normal curve (the (-1) denotes a grading shift). The Tor groups of M can be computed explicitly using the Eagon–Northcott complex, and the Tor groups of the middle term can be computing using the method of Kempf collapsing of vector bundles. Hence one is left to analyze the corresponding long exact sequence on Tor, and the goal is to prove that the following map is surjective for $i \geq (g-1)/2$:

$$\mathrm{Tor}_i^A(\mathbf{k}[\widetilde{T_g}], \mathbf{k})_{i+1} \to \mathrm{Tor}_i^A(M, \mathbf{k})_i.$$

Using careful analysis of the representation theory of $\mathbf{SL}_2(\mathbf{k})$, the authors show that the cokernel of this map is identified with the middle homology of the following complex

$$\mathrm{Sym}^{g-2-i}(\mathrm{D}^{i+1}\mathbf{k}^2) \otimes \mathrm{D}^{2i}(\mathbf{k}^2) \to \mathrm{Sym}^{g-1-i}(\mathrm{D}^{i+1}\mathbf{k}^2) \otimes \mathrm{D}^{i+1}(\mathbf{k}^2) \to \mathrm{Sym}^{g-i}(\mathrm{D}^{i+1}\mathbf{k}^2)$$

where D denotes the divided power functor. The upshot is that by summing over all g, this gives a Koszul module, in the following sense. Given a vector space V and a linear subspace $K \subset \bigwedge^2 V$, the Koszul module $W(V, K)$ is the middle homology of the modified Koszul complex

$$\mathrm{Sym}(V) \otimes K \to \mathrm{Sym}(V) \otimes V(1) \to \mathrm{Sym}(V)(2).$$

In the above setting, $V = \mathrm{D}^{i+1}(\mathbf{k}^2)$ and $K = \mathrm{D}^{2i}(\mathbf{k}^2)$. One of the main results of [1] is that the following are equivalent:

- $K^\perp \subset \bigwedge^2(V^*)$ contains no nonzero rank 2 matrix,
- $W(V, K)$ is finite-dimensional,
- $W(V, K)_d = 0$ for all $d \geq \dim V - 3$.

This translates to the desired vanishing result for the rational cuspidal curve.

In [2] we follow a similar strategy using ribbons, a different degeneration of canonical curves. These are non-reduced double structures on the projective line, which can be realized as hyperplane sections of non-reduced double structures on rational normal scrolls known as K3 carpets. For the scroll, we choose two parameters a and g and consider the join of the degree a rational normal curve with the degree $g-1-a$ rational normal curve (we assume that $a \leq g-1-a$). The corresponding ribbon is a degeneration of a genus g curve with Clifford index a and gonality $a+2$. In this case, we get an extension

$$0 \to \omega_B \to B' \to B \to 0$$

where B is the homogeneous coordinate ring of the rational normal scroll, ω_B is its canonical module, and B' is the homogeneous coordinate ring of the K3 carpet. As before, the Eagon–Northcott complex can be used to explicitly compute the Tor groups of B and ω_B, so one needs to analyze the corresponding long exact sequence and show that the following map is surjective for $i < a$:

$$\mathrm{Tor}_{i+1}^A(B, \mathbf{k})_{i+2} \to \mathrm{Tor}_i^A(\omega_B, \mathbf{k})_{i+2}.$$

Everything in sight is bi-graded, and so can be further refined. We omit this detail but it is possible to identify the cokernel with bi-graded components of a bi-graded analogue of a Koszul module, in the following sense. Let V_1, V_2 be vector spaces and let K be a linear subspace of $V_1 \otimes V_2 \subset \bigwedge^2(V_1 \oplus V_2)$. Then the bi-graded Koszul module $W(V,K)$ is the middle homology of the modified Koszul complex (which is now bi-graded):

$$\mathrm{Sym}(V_1 \oplus V_2) \otimes K \to \begin{matrix} \mathrm{Sym}(V_1 \oplus V_2) \otimes V_1(0,1) \\ \mathrm{Sym}(V_1 \oplus V_2) \otimes V_2(1,0) \end{matrix} \to \mathrm{Sym}(V_1 \oplus V_2)(1,1)$$

We prove in [2] that the following are equivalent for bi-graded Koszul modules:

- $K^\perp \subset V_1^* \otimes V_2^*$ contains no nonzero rank ≤ 2 matrix,
- $W(V,K)_{d,e} = 0$ for $d, e \gg 0$,
- $W(V,K)_{d,e} = 0$ for $d \geq \dim V_2 - 2$ and $e \geq \dim V_1 - 2$.

Modulo the missing explanation above, this proves that an analogue of Green's conjecture holds for ribbons of Clifford index a and genus g. In fact, this result holds as long as the characteristic of $\mathbf{k}$ is at least a, so that we get a bound for where the refined generic Green conjecture holds in positive characteristic. Furthermore, this also improves the bound in [1] since the Clifford index of a genus g curve is at most $(g-1)/2$.

References

[1] Marian Aprodu, Gavril Farkas, Ştefan Papadima, Claudiu Raicu, Jerzy Weyman, Koszul modules and Green's conjecture, *Invent. Math.* **218** (2019), no. 3, 657–720.

[2] Claudiu Raicu, Steven V Sam, Bi-graded Koszul modules, K3 carpets, and Green's conjecture, *Compos. Math.*, to appear.

[3] Frank-Olaf Schreyer, Syzygies of canonical curves and special linear series, *Math. Ann.* **275** (1986), no. 1, 105–137.

[4] Claire Voisin, Green's generic syzygy conjecture for curves of even genus lying on a K3 surface, *J. Eur. Math. Soc. (JEMS)* **4** (2002), no. 4, 363–404.

[5] Claire Voisin, Green's canonical syzygy conjecture for generic curves of odd genus, *Compos. Math.* **141** (2005), no. 5, 1163–1190.