Bi-graded Koszul modules, K3 carpets, and Green's conjecture

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joint work with Claudiu Raicu
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- Fix alg. closed field k of characteristic 0.
- C is a (smooth) genus $g \geq 2$ curve with canonical bundle ω_C.
- The canonical ring $\Gamma_C = \bigoplus_{d \geq 0} \mathrm{H}^0(C; \omega_C^{\otimes d})$ is finitely generated over $A = \mathrm{Sym}\, \mathrm{H}^0(C; \omega_C) \cong \mathrm{k}[x_1, \ldots, x_g]$.
- We're concerned with vanishing of Betti numbers

$$\beta_{i,j}(C) = \dim_{\mathrm{k}} \mathrm{Tor}_i^A(\Gamma_C, \mathrm{k})_j.$$

- **Green's conjecture** states that $\beta_{i,i+2}(C) = 0$ for $i < \mathrm{Cliff}(C)$, the Clifford index of C. This governs for how many steps the equations of C have only linear syzygies.
- For "most" curves, $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 2$ where $\mathrm{gon}(C)$ is the minimum degree of a non-constant map $C \to \mathrm{P}^1$.

What's known

- Voisin (2002, 2005): There is a nonempty Zariski dense set of curves in the moduli space of curves for which Green's conjecture holds. Geometric proof involving K3 surfaces.
  Note: $\mathrm{Cliff}(C) \leq (g-1)/2$ for all C
  Refinement: In fact, this set contains curves of each gonality.
- Aprodu–Farkas (2011): Green's conjecture holds for any curve that lies on a K3 surface.
- Many other variations...
- Aprodu–Farkas–Papadima–Raicu–Weyman (2019): Reproved Voisin's result using representation theory ideas (next slide). Method of proof is simpler and extends to positive characteristic $p \geq (g+2)/2$
- Schreyer (1986): Green's conjecture fails in low characteristic

- Betti numbers are upper semicontinuous, i.e., the Betti numbers of a flat degeneration can only go up. Hence, to prove (generic) vanishing, it suffices to prove it for a single example, and we can consider singular (smoothable) curves.
- A rational curve with g cusps has genus g and can be realized as a hyperplane section of the tangential surface T_g of the g-uple rational normal curve (= the union of its tangent lines).
- There is a short exact sequence of graded modules

$$0 \to \mathsf{k}[T_g] \to \widetilde{\mathsf{k}[T_g]} \to \omega_{\mathsf{k}[\mathsf{P}^1,\mathcal{O}(g)]}(-1) \to 0,$$

  consisting of the homog. coordinate ring of T_g, its normalization, and the canonical module of the homog. coordinate ring of the g-uple RNC.
- The latter two can be understood, so it amounts to understanding a long exact sequence on Tor.

$$0 \to \mathsf{k}[T_g] \to \widetilde{\mathsf{k}[T_g]} \to \omega_{\mathsf{k}[\mathrm{P}^1,\mathcal{O}(g)]}(-1) \to 0$$

Recall that rational normal curve is cut out by maximal minors of

$$\begin{bmatrix} x_0 & x_1 & \cdots & x_{g-1} \\ x_1 & x_2 & \cdots & x_g \end{bmatrix}$$

thought of as multiplication map $\mathrm{Sym}^{g-1}\mathsf{k}^2 \to \mathsf{k}^2(g)$.
The last module is dual of resulting Eagon–Northcott complex.

$$\mathrm{Tor}_i^A(\omega_{\mathsf{k}[\mathrm{P}^1,\mathcal{O}(g)]}, \mathsf{k})_i = \bigwedge^i(\mathrm{Sym}^{g-1}\mathsf{k}^2) \otimes \mathrm{Sym}^{g-2-i}(\mathsf{k}^2).$$

for $i = 0, \ldots, g-2$
(dim $\mathrm{Tor}_{g-1} = 1$ but unimportant for discussion)

$$0 \to \mathsf{k}[T_g] \to \widetilde{\mathsf{k}[T_g]} \to \omega_{\mathsf{k}[\mathsf{P}^1, \mathcal{O}(g)]}(-1) \to 0$$

T_g has vector bundle desingularization:
it is projection of total space of $\mathcal{J}^*$ in $\mathrm{Sym}^g \mathsf{k}^2 \times \mathsf{P}^1$ where

$$\mathcal{J} = \mathrm{coker}(\mathrm{Sym}^{g-2} \mathsf{k}^2(-2) \to \mathrm{Sym}^g \mathsf{k}^2)$$

Pushing forward Koszul complex on $\mathrm{Sym}^{g-2} \mathsf{k}^2(-2)$ gives

$$\mathrm{Tor}_i^A(\widetilde{\mathsf{k}[T_g]}, \mathsf{k})_{i+1} = \bigwedge^{i+1}(\mathrm{Sym}^{g-2} \mathsf{k}^2) \otimes \mathrm{D}^{2i}(\mathsf{k}^2)$$

(also Tor_0 has degree 0 piece)

- The problem reduces to showing that the following map is surjective for $i \leq (g-1)/2$:

$$\begin{array}{ccc} \operatorname{Tor}_i^A(\widetilde{\mathrm{k}[T_g]}, \mathrm{k})_{i+1} & \longrightarrow & \operatorname{Tor}_i^A(\omega_{\mathrm{k}[\mathbf{P}^1, \mathcal{O}(g)]}, \mathrm{k})_i \\ \| & & \| \\ \bigwedge^{i+1}(\operatorname{Sym}^{g-2}\mathrm{k}^2) \otimes \mathrm{D}^{2i}(\mathrm{k}^2) & \overset{?}{\dashrightarrow} & \bigwedge^{i}(\operatorname{Sym}^{g-1}\mathrm{k}^2) \otimes \operatorname{Sym}^{g-2-i}(\mathrm{k}^2) \end{array}$$

- The group $SL_2(\mathrm{k})$ acts on everything in sight, so the map is equivariant. However, it is difficult to guess a formula.
- The actual way forward is technical but insight comes from Hermite reciprocity isomorphism:

$$\operatorname{Sym}^n(\mathrm{D}^m \mathrm{k}^2) \cong \bigwedge^m(\operatorname{Sym}^{m+n-1}\mathrm{k}^2).$$

$$\mathrm{Sym}^n(\mathrm{D}^m \mathrm{k}^2) \cong \bigwedge^m (\mathrm{Sym}^{m+n-1}\,\mathrm{k}^2).$$

Doesn't yet help; but also have map such that

$$\bigwedge^i (\mathrm{Sym}^{g-1}\,\mathrm{k}^2) \otimes \mathrm{Sym}^{g-2-i}(\mathrm{k}^2) = \ker\left(\bigwedge^{i+1}(\mathrm{Sym}^{g-1}(\mathrm{k}^2)) \otimes \mathrm{D}^{i+1}(\mathrm{k}^2) \to \bigwedge^{i+1}(\mathrm{Sym}^{g}(\mathrm{k}^2))\right)$$

So cokernel of the last slide is the middle homology of a sequence:

$$\begin{array}{ccccc}
\bigwedge^{i+1}(\mathrm{Sym}^{g-2}\,\mathrm{k}^2) \otimes \mathrm{D}^{2i}(\mathrm{k}^2) & \longrightarrow & \bigwedge^{i+1}(\mathrm{Sym}^{g-1}\,\mathrm{k}^2) \otimes \mathrm{D}^{i+1}(\mathrm{k}^2) & \longrightarrow & \bigwedge^{i+1}(\mathrm{Sym}^{g}\,\mathrm{k}^2) \\
\| & & \| & & \| \\
\mathrm{Sym}^{g-2-i}(\mathrm{D}^{i+1}\mathrm{k}^2) \otimes \mathrm{D}^{2i}(\mathrm{k}^2) & \longrightarrow & \mathrm{Sym}^{g-i-1}(\mathrm{D}^{i+1}\mathrm{k}^2) \otimes \mathrm{D}^{i+1}(\mathrm{k}^2) & \longrightarrow & \mathrm{Sym}^{g-i}(\mathrm{D}^{i+1}\mathrm{k}^2)
\end{array}$$

Suggestion: fix i and sum over all g. The bottom row looks like beginning of Koszul complex for $\mathrm{Sym}(\mathrm{D}^{i+1}\mathrm{k}^2)$ but not quite.

- The key to using the module structure on the sum is that the cokernel can be recast as a **Koszul module**.
  Warning: This "looks" like the case in previous slide, but actual identification is subtle
- Given a subspace $K \subset \bigwedge^2 V$, the Koszul module $W(V, K)$ is the middle homology of the modified Koszul complex

$$\mathrm{Sym}\, V \otimes K \to \mathrm{Sym}\, V \otimes V(1) \to \mathrm{Sym}\, V(2)$$

- In previous setting, $V = \mathrm{D}^{i+1}\mathrm{k}^2$ and $K = \mathrm{D}^{2i}\mathrm{k}^2$.
- AFPRW proved the following are equivalent:
  - $K^\perp \subset \bigwedge^2 V^*$ contains no nonzero rank 2 matrix
  - $W(V, K)$ is finite length
  - $W(V, K)_d = 0$ for all $d \geq \dim V - 3$

  This is enough to prove Green's conjecture for rational cuspidal curves, and hence for a nonempty dense subset of curves in the moduli space of curves.

- Double structures on $\mathbb{P}^1$ (ribbons) give a different degeneration of genus g curves (intuition: if a non-hyperelliptic curve degenerates to a hyperelliptic one, this is the degeneration of the image of the canonical map)
- They are hyperplane sections of double structures on rational normal scrolls. More precisely, consider the projective space

$$\mathbb{P}^g = \mathbb{P}(\operatorname{Sym}^a \mathrm{k}^2 \oplus \operatorname{Sym}^{g-1-a} \mathrm{k}^2)$$

  with an a-uple RNC and $(g-1-a)$-uple RNC. Let B be the homog. coordinate ring of the corresponding scroll.
- There is an extension

$$0 \to \omega_B \to B' \to B \to 0$$

  where B' is the homog. coordinate ring of a **K3 carpet**. It is a double structure on the scroll.

- The ribbons are smoothable to curves of gonality $a+2$. Hence proving Green's conjecture for each ribbon proves it for most curves of each gonality (not just the maximal value ones).
- Can prove it holds in characteristic $p \geq a$. In generic case $a = (g-1)/2$, this beats $(g+2)/2$ from cuspidal curves and resolves a conjecture of Eisenbud–Schreyer.
- Coordinate ring A of $\mathrm{P}(\mathrm{Sym}^a \mathrm{k}^2 \oplus \mathrm{Sym}^{g-1-a} \mathrm{k}^2)$ is bigraded.
- The syzygies of ω_B and B are understood, so we again need to consider a long exact sequence.

The problem reduces to showing that the following map is surjective for $i < a$:

$$\begin{array}{ccc}
\mathrm{Tor}^A_{i+1}(B, \mathrm{k})_{i+2} & \longrightarrow & \mathrm{Tor}^A_i(\omega_B, \mathrm{k})_{i+2} \\
\| & & \| \\
\mathrm{D}^{i-1}\mathrm{k}^2 \otimes \bigwedge^{i+1}(\mathrm{Sym}^{a-1}\mathrm{k}^2 \oplus \mathrm{Sym}^{g-2-a}\mathrm{k}^2) & \overset{?}{\dashrightarrow} & \mathrm{Sym}^{g-3-i}\mathrm{k}^2 \otimes \bigwedge^{i}(\mathrm{Sym}^{a-1}\mathrm{k}^2 \oplus \mathrm{Sym}^{g-2-a}\mathrm{k}^2)
\end{array}$$

We can decompose the last map into bigraded components u, v, fix them, and sum over all a, g.

Again, both terms are f.g. modules over a symmetric algebra and the cokernel is a Koszul module.

It is not finite length, but we need to consider something different.

- Given vector spaces V_1, V_2 and $K \subset V_1 \otimes V_2 \subset \bigwedge^2(V_1 \oplus V_2)$, $W(V,K)$ is the middle homology of

$$\mathrm{Sym}(V_1 \oplus V_2) \otimes K \to \begin{matrix} \mathrm{Sym}(V_1 \oplus V_2) \otimes V_1(0,1) \\ \mathrm{Sym}(V_1 \oplus V_2) \otimes V_2(1,0) \end{matrix} \to \mathrm{Sym}(V_1 \oplus V_2)(1,1)$$

- In previous setting, $V_1 = \mathrm{D}^u \mathrm{k}^2$, $V_2 = \mathrm{D}^v \mathrm{k}^2$, $K = \mathrm{D}^{u+v-2}\mathrm{k}^2 + \mathrm{D}^{u+v}\mathrm{k}^2$.
- Raicu–Sam:
  - $K^\perp \subset V_1^* \otimes V_2^*$ contains no nonzero rank ≤ 2 matrix
  - $W(V,K)_{d,e} = 0$ for $d, e \gg 0$
  - $W(V,K)_{d,e} = 0$ for $d \geq \dim V_2 - 2$ and $e \geq \dim V_1 - 2$.

  As before, this proves Green's conjecture for ribbons.

- Tangential surface is locus of binary forms $\ell_0^{g-1}\ell_1$. Can do higher tangentials $\{\ell_0^{g-d}\ell_1\cdots\ell_d\}$ and get similar structure on sum over g, but Koszul complex replaced by

$$\mathrm{D}^{d(i+1)+i-1}\mathsf{k}^2 \otimes S(-d-1) \to \mathrm{D}^{d(i+1)}\mathsf{k}^2 \otimes S(-d) \to S$$

  where $S = \mathrm{Sym}(\mathrm{D}^{i+1}\mathsf{k}^2)$.
  How about other factorization patterns?
- Red sequence above has linear syzygies (ignoring torsion homology). Anything else in common with Koszul complex? Might suggest other interesting examples.
- Symmetric case? Could take $V_1 = V_2$ and ask that $K \subset \mathrm{D}^2(V_1) \subset V_1 \otimes V_2$. Didn't find example of this.
- Other natural examples of (bigraded) Koszul modules? $K \subset \bigwedge^2 V$ or $K \subset V_1 \otimes V_2$. In each case, most interesting when the orthogonal complement misses the rank 2 matrices.