1. Invertible matrix: definition. If $AB = BA = I$, then $B = A^{-1}$. How to check if an $n \times n$ matrix $A$ is invertible?
   (a) Echelon form: $n$ pivots; nonzero diagonal entries;
   (b) All columns are linearly independent;
   (c) All columns span $\mathbb{R}^n$;
   (d) $Ax = 0$ has only the trivial solution $x = 0$;
   (e) $Ax = b$ has a unique solution for any $b$;
   (f) $\det A \neq 0$;
   (g) $A^T$ is invertible.

2. Let $T$ be a one-to-one linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. Let $A$ be the matrix associated with $T$. Is $A$ invertible?

3. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}$. Find out the LU factorization of $A$ and use this to solve $Ax = b$ by solving two triangular systems.

4. Find all the co-factors, the adjugate matrix, and the determinant of the matrix

   \[
   \begin{bmatrix}
   1 & 1 & 2 & -1 \\
   0 & 2 & -4 & 1 \\
   -1 & 0 & 2 & 0 \\
   \end{bmatrix}
   \]

5. True or false:
   (a) If a square matrix $B$ is an echelon form of matrix $A$ then $\det A = \det B$;
   (b) The determinant of any elementary matrix is 1;
   (c) $\det(A + B) = \det A + \det B$, $\det(-A) = -\det A$, and $\det(AB) = (\det A)(\det B)$;
   (d) $\det A^T = \det A$ and $\det A^{-1} = (\det A)^{-1}$;
   (e) A square matrix is invertible if and only if its determinant is nonzero.

6. Calculate the determinant of each of the following matrices:

   \[
   \begin{bmatrix}
   5 & 0 & -1 \\
   1 & -3 & -2 \\
   0 & 5 & 3 \\
   \end{bmatrix};
   \begin{bmatrix}
   1 & 1 & 2 & -1 \\
   0 & 0 & -4 & 0 \\
   -1 & 3 & 12 & 0 \\
   \end{bmatrix};
   \begin{bmatrix}
   1 & 3 & 3 & -4 \\
   0 & 1 & 2 & 5 \\
   2 & 5 & 4 & -3 \\
   -3 & -7 & -5 & 2 \\
   \end{bmatrix}.
   \]

7. Show that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (a - b)(b - c)(c - a)$.

8. What is Cramer’s rule? Let $A$ be a $3 \times 3$ matrix and assume $\det A = 2$, the co-factors $C_{11} = 2$, $C_{12} = -4$, $C_{13} = 6$. What is the solution to $Ax = e_1$ with $e_1 = (1, 0, 0)^T$?

9. Since $A^{-1} = \frac{1}{\det A} \text{adj} A$, we have $A \text{adj} A = (\det A)I$. Correct?

10. Find the area of the parallelogram whose vertices are $(0, -2), (6, -1), (-3, 1), (3, 2)$.

11. Find the volume of the parallelepiped with one vertex at $(0, 0, 0)$ and its three adjacent vertices $(1, 4, 0), (-1, 2, -1), \text{and} (-2, -5, 2)$.

12. Show that the area of the triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is the absolute value of $\frac{1}{2} \det \begin{bmatrix}
   1 & 1 & 1 \\
   x_1 & x_2 & x_3 \\
   y_1 & y_2 & y_3 \\
   \end{bmatrix}$. 

13. Determine if $H$ is a subspace of $\mathbb{R}^3$:
   (a) $H$ consists of all the vectors in $\mathbb{R}^3$ with the product of all the components equal to 0;
   (b) $H$ consists of all the vectors in $\mathbb{R}^3$ with the first component equal to 1;
   (c) $H$ consists of all the vectors in $\mathbb{R}^3$ with the first component equal to 0;
   (d) $H$ consists of all the vectors \[
   \begin{bmatrix}
   2a - b \\
   2b - c \\
   2c - a
   \end{bmatrix}
   \] with $a, b, c$ all real numbers.

14. Is the set of all $2 \times 2$ diagonal matrices a subspace of the space of all $2 \times 2$ matrices?

15. Determine if
\[
\begin{bmatrix}
0 \\
1 \\
-2
\end{bmatrix}
\] is in the span by \[
\begin{bmatrix}
1 \\
1 \\
-2
\end{bmatrix}, \quad \begin{bmatrix}
-2 \\
0 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
4 \\
-4
\end{bmatrix}.
\]

16. Determine if \[
\begin{bmatrix}
2 \\
1
\end{bmatrix}
\] is in the column space and null space of \[
\begin{bmatrix}
-2 & 4 \\
-1 & 2
\end{bmatrix}.
\]

17. Let $v_1$ and $v_2$ be two vectors in a vector space. Show that any three vectors in Span $\{v_1, v_2\}$ must be linearly dependent.

18. Show that $p_0(t) = 1$, $p_1(t) = 1 + t$, and $p_2(t) = 1 + t + t^2$ form a basis of the vector space $\mathbb{P}_2$. Find the coordinates of $p(t) = 2 - t + 3t^2$ with respect to this basis.

19. Find a basis for the column space, the null space, and row space of
\[
\begin{bmatrix}
1 & -2 \\
-2 & 7 \\
3 & -9
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
1 & 0 & -1 & 5 \\
0 & 1 & 0 & 2 \\
2 & 0 & 2 & -1
\end{bmatrix}.
\]
Find also the rank of each of these two matrices.

20. True or false:
   (a) If $B$ is an echelon form of $A$, then $A$ and $B$ have the same null space, column space, and row space;
   (b) The row space of $A$ is the column space of $A^T$;
   (c) If the null space of a square matrix $A$ is $\{0\}$ then $A$ is invertible;
   (d) If $u$ and $v$ are two vectors in $\mathbb{R}^3$ then the rank of the matrix $uv^T$ is always 0 or 1.

21. What is the dimension of $\mathbb{R}^4$, $\mathbb{P}_4$, the space of all $4 \times 4$ matrices?

22. If $V$ is a vector space and dim $V = n$ then $n + 1$ vectors in $V$ must be linearly dependent. Correct? Why?

23. True or false:
   (a) rank $A + rank B = rank (A + B)$;
   (b) rank $(AB) = (rank A)(rank B)$;
   (c) dim Col $A +$ dim Nul $A =$ number of columns of $A$.

24. If $\lambda$ is an eigenvalue of $A$. Then $\lambda^2$ is an eigenvalue of $A^2$. Correct? Why?

25. Any three different eigenvectors of a matrix $A$ corresponding to three different eigenvalues must be linearly independent. Correct? Why?

26. Find all the eigenvalues and eigenvectors of \[
\begin{bmatrix}
9 & -2 \\
2 & 5
\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
6 & -2 & 0 \\
-2 & 9 & 0 \\
5 & 8 & 3
\end{bmatrix}.
\]