Math 20B.
Midterm Exam 1
January 27, 2011

Turn off and put away your cell phone.
You are not allowed to use a calculator during this exam.
Read each question carefully, and answer each question completely.
Show all of your work; no credit will be given for unsupported answers.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

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1. (9 points) Evaluate the given expression:

(a) \[ \frac{d}{dx} \left( \int_2^x \left( \frac{2t}{1+t^3} \right)^{1/4} dt \right) \]

\[ = \left( \frac{2e^x}{1+(e^x)^3} \right)^{1/4} \cdot e^x \]

\[ = \left( \frac{2e^x}{1+e^{3x}} \right)^{1/4} \cdot e^x \]

FTC II

with Chain rule

(b) \[ \int_0^{\pi/2} \sin^6 x \cos x \, dx \]

\[ = \left[ \frac{u^7}{7} \right]_0^{\pi/2} \]

\[ = \left( \sin \frac{\pi}{2} \right)^7 - \left( \sin 0 \right)^7 \]

\[ = \frac{1}{7} \]
(c) \( \int \frac{dx}{x \ln(\sqrt{x})} \)

\[
= \int 2 \cdot \frac{1}{u} \, du
\]

\[
= 2 \ln |u| + C
\]

\[
= 2 \ln |\ln \sqrt{x}| + C
\]

**Check:**

\[ f(x) = 2 \ln |\ln \sqrt{x}| + C \]

\[ f'(x) = 2 \cdot \frac{1}{\ln \sqrt{x}} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2 \sqrt{x}} \]

\[
= \frac{1}{x \ln \sqrt{x}}
\]
2. (6 points) A particle is moving along a straight line with velocity \( v(t) = 6t^2 + 6t - 12 \) feet per second.

(a) Find the average velocity of the particle (i.e. the average value of the function \( v \)) on the interval \([0, 2]\).

\[
\text{Avg value} = \frac{1}{b-a} \int_a^b f(x) \, dx
\]

\[
\text{Avg value} = \frac{1}{2} \int_0^2 (6t^2 + 6t - 12) \, dt
\]

\[
= \frac{1}{2} (2t^3 + 3t^2 - 12t) \bigg|_0^2
\]

(b) Find the distance travelled by the particle on the interval \([0, 2]\).

\[
\text{distance} = \int_0^2 |v(t)| \, dt
\]

find roots
\[
6t^2 + 6t - 12 = 0
\]

\[
t^2 + t - 2 = 0
\]

\[
(t+2)(t-1) = 0
\]

\[
t = -2, \quad t = 1
\]

\[
\text{distance} = \int_0^1 -(6t^2 + 6t - 12) \, dt + \int_1^2 6t^2 + 6t - 12 \, dt
\]

\[
= -2t^3 - 3t^2 + 12t \bigg|_0^1 + 2t^3 + 3t^2 - 12t \bigg|_0^2
\]

\[
= (-2-3+12) - 0 + (16+12-24) - (2+3-12)
\]

\[
= \sqrt{19} \, \text{ft}
\]
3. (9 points) Let \( R \) be the region enclosed by the curves \( y = x^2 + 2, y = 0, x = 0 \) and \( x = 2 \), as shown in the figure to the right.

(a) Write down (but do not evaluate) an expression involving definite integrals that equals the volume of the solid obtained by rotating \( R \) about the \( x \)-axis.

\[
\int_0^2 \pi (x^2 + 2)^2 \, dx
\]

(b) Write down (but do not evaluate) an expression involving definite integrals that equals the volume of the solid obtained by rotating \( R \) about the \( y \)-axis.

\[
\int_0^2 \pi (2)^2 dy + \int_2^6 \pi (2)^2 - \pi \left( \sqrt{y - 2} \right)^2 dy
\]

(c) Write down (but do not evaluate) an expression involving definite integrals that equals the volume whose base is \( R \) and whose cross-sections perpendicular to the \( x \)-axis are semi-circles.

\[
\int_0^2 \pi \left( \frac{x^2 + 2}{2} \right)^2 \, dx
\]
4. (3 points) Write down (but do not evaluate) an expression involving definite integrals that equals the area of the region that lies inside the curve \( r = 3 + 2 \sin \theta \) and outside the circle \( r = 2 \).

Find intersection points:

\[
2 = 3 + 2 \sin \theta \\
\frac{-1}{2} = \sin \theta \\
\theta = \frac{-\pi}{6}, \frac{7\pi}{6}
\]

\[
\frac{1}{2} \int_{-\pi/6}^{\pi/6} (3+2\sin \theta)^2 \, d\theta - \frac{1}{2} \int_{-\pi/6}^{\pi/6} (2)^2 \, d\theta
\]
Math 20B
Midterm Exam 1
January 31, 2012

Version A

Instructions

1. No calculators or other electronic devices are allowed during this exam.
2. You may use one page of notes, but no books or other assistance during this exam.
3. Write your Name, PID, and Section on the front of your Blue Book.
4. Write the Version of your exam on the front of your Blue Book.
5. Write your solutions clearly in your Blue Book
   (a) Carefully indicate the number and letter of each question and question part.
   (b) Present your answers in the same order they appear in the exam.
   (c) Start each question on a new page.
6. Read each question carefully, and answer each question completely.
7. Show all of your work; no credit will be given for unsupported answers.

0. (1 point) Carefully read and complete the instructions at the top of this exam sheet.

1. (3 points) Compute the following derivative, where b is a constant: \( \frac{d}{dx} \int_{2012}^{x} \frac{dt}{\sqrt{t + 20b}} \).

2. (4 points) Evaluate \( \int x \sqrt{x + 2} \, dx \).

3. (4 points) Evaluate \( \int \sqrt{x} \ln(x) \, dx \).

4. (6 points) A particle initially at the origin moves along the x-axis with velocity \( v(t) = (2 - t)\sqrt{t} \).
   (a) Find the particle's position at time \( t = 4 \).
   (b) What is the total distance traveled by the particle during the time interval from \( t = 0 \) to \( t = 4 \)? (Be careful!)

5. (6 points) Find the volume of the solid obtained by revolving the region bounded by \( y = 2x - 2 \), \( y = -3x + 8 \), and the x-axis about the y-axis.
Solutions - prepared by Janine Tiefenbruck

1) \[ \frac{d}{dx} \int_{2012}^{x} \frac{dt}{\sqrt{t+20b}} = \frac{1}{\sqrt{x+20b}} \] by FTC II

2) \[ \int x\sqrt{x+2} \, dx \]
\[ = \int (u-2)(\sqrt{u}) \, du \]
\[ = \int (u - 2)u^{1/2} \, du \]
\[ = \int u^{3/2} - 2u^{1/2} \, du \]
\[ = \frac{2u^{5/2}}{5} - \frac{2.2u^{3/2}}{3} + C \]
\[ = \frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C \]

3) \[ \int 3\sqrt{x} \ln x \, dx \]
\[ u = \ln x \quad dv = \frac{3}{\sqrt{x}} \, dx \]
\[ du = \frac{1}{x} \, dx \quad v = \frac{3x^{1/2}}{4} \]
3, cont.) \[ uv - \int vdu \]
\[ \frac{3}{4} x^{4/3} \ln x - \int \frac{3}{4} x^{1/3} \, dx \]
\[ = \frac{3}{4} x^{4/3} \ln x - \frac{3}{4} \cdot \frac{3}{4} x^{4/3} + C \]
\[ = \frac{3}{4} x^{4/3} \ln x - \frac{9}{16} x^{4/3} + C \]

4) a) \[ s(t) = \text{position}, \quad v(t) = \text{velocity} \]
\[ v(t) = s'(t), \quad \text{so } s(t) \text{ is an antiderivative of } v(t) \]
\[ s(t) = \int v(t) \, dt \]
\[ = \int (2-t) \sqrt{t} \, dt \]
\[ = \int 2 t^{1/2} - t^{3/2} \, dt \]
\[ = 2 \cdot 2 t^{3/2} - \frac{2}{5} t^{5/2} + C \]
\[ \text{since it starts at origin, } s(t) = 0, \text{ so } C = 0 \]
\[ s(t) = \frac{4}{3} t^{3/2} - \frac{2}{5} t^{5/2} \]
4a, cont.) position at time \( t = 4 \) is
\[
s(4) = \frac{4}{3} (4)^{3/2} - \frac{2}{3} (4)^{5/2}
\]
\[
= \frac{4}{3} (2)^{3} - \frac{2}{3} (2)^{5}
\]
\[
= \frac{32}{3} - \frac{64}{5} \text{ units}
\]

b) distance = \( \int_{a}^{b} |v(t)| \, dt \)
\[
distance = \int_{0}^{4} |(2-t) \sqrt{t}| \, dt
\]
\[
= \int_{0}^{4} |2-t| \cdot |\sqrt{t}| \, dt < \text{property of abs. value}
\]
\[
= \int_{0}^{4} |2-t| \sqrt{t} \, dt < \sqrt{t} \text{ always positive for } 0 \leq t \leq 4
\]
\[
= \int_{0}^{2} (2-t) \sqrt{t} \, dt + \int_{2}^{4} -(2-t) \sqrt{t} \, dt
\]
Since \( 2-t \) is negative when \( t > 2 \)
\[ 4b, \text{cont} \]
\[ = \int_{0}^{2} (2-t) \sqrt{t} \, dt - \int_{2}^{4} (2-t) \sqrt{t} \, dt \]
\[ = \left[ \frac{4}{3} t^{3/2} - \frac{2}{5} t^{5/2} \right]_{0}^{2} - \left( \frac{4}{3} t^{3/2} - \frac{2}{5} t^{5/2} \right)_{2}^{4} \]

Using the antiderivative found in part (a)

\[ = \frac{4}{3} (2)^{3/2} - \frac{2}{5} (2)^{5/2} - \left( \frac{32}{3} - \frac{64}{5} \right) - \left( \frac{4}{3} (2)^{3/2} - \frac{2}{5} (2)^{5/2} \right) \]

\[ = \left\{ \frac{4}{3} (2)^{3/2} - \frac{2}{5} (2)^{5/2} \right\} - \left( \frac{32}{3} - \frac{64}{5} \right) \]

5) \[ y = 2x - 2 \]
\[ y = -3x + 8 \]
\[ x = 0 \text{ at } y \text{-axis} \]

Intersection

\[ 2x - 2 = -3x + 8 \]
\[ 5x = 10 \]
\[ x = 2 \]
\[ y = 2 \]

Roots
\[ 0 = 2x - 2 \]
\[ x = 1 \]
\[ 0 = -3x + 8 \]
\[ x = \frac{8}{3} \]
5, (cont.) \( r: x \) in the line \( y = 2x - 2 \)
\[
\begin{align*}
\chi &= \frac{y+2}{2} \\
\chi &= \frac{1}{2}y + 1 \\
\tau &= \frac{1}{2}y + 1
\end{align*}
\]

\( R: x \) in the line \( y = -3x + 8 \)
\[
\begin{align*}
\chi &= \frac{y-8}{-3} \\
\chi &= -\frac{1}{3}y + \frac{8}{3} \\
\tau &= -\frac{1}{3}y + \frac{8}{3}
\end{align*}
\]

\[ V = \pi \left( \int_{0}^{2} \left( \left( -\frac{1}{3}y + \frac{8}{3} \right)^{2} - \left( \frac{1}{2}y + 1 \right)^{2} \right) dy \right) \]
\[
V = \pi \left( \int_{0}^{2} \left( \frac{1}{9}y^{2} - \frac{16}{9}y + \frac{64}{9} - \left( \frac{1}{4}y^{2} + y + 1 \right) \right) dy \right)
\]
\[
V = \pi \left( \frac{1}{27}y^{3} - \frac{8}{9}y^{2} + \frac{64}{9}y - \frac{1}{12}y^{3} - \frac{1}{2}y^{2} - y \right)_{0}^{2}
\]
\[ V = \pi \left( \frac{8}{27} - \frac{32}{9} + \frac{128}{9} - \frac{8}{12} - \frac{4}{2} - 2 \right) \]

\[ V = \pi \left( \frac{8}{27} + \frac{32}{3} - \frac{2}{3} - 4 \right) \]

\[ V = \pi \left( \frac{8}{27} + 6 \right) \]