UNIVERSITY OF CALIFORNIA, SAN DIEGO

Multivariate Analogues of Catalan Numbers, Parking Functions, and their Extensions

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2003

The dissertation of Nicholas Anthony Loehr is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

2003

DEDICATION

Cui dono lepidum novum libellum
arida modo pumice expolitum?
Antoni, tibi: namque tu solebas
meas esse aliquid putare nugas.

— Catullus I (adapted)

EPIGRAPH

Μηδὲν ἄγαν σπεύδειν πάντων μέσ’ ἄριστα.

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ACKNOWLEDGEMENTS

The author received financial support from a National Science Foundation Graduate Research Fellowship for a portion of the research described in this document.

The author thanks the following individuals and institutions for providing mathematical, technical, editorial, and/or financial assistance during various phases of the preparation of this work: Donald Allison, Edward Bender, the Center for Communications Research, William Doran IV, Dominique Foata, Adriano Garsia, Ronald Getoor, James Haglund, Mark Haiman, Alfred Hales, A. Mendes, James Munkres, the National Science Foundation, Jeffrey Remmel, Joseph Rotman, Glenn Tesler, Adrian Wadsworth, Joe Weening, and Ruth Williams.

§1.5 is a reprint of a section from the paper "A Conjectured Combinatorial Formula for the Hilbert Series for Diagonal Harmonics" by J. Haglund and N. Loehr, *Proceedings of FPSAC 2002*, Melbourne Australia, July 2002. The dissertation author was the primary investigator and single author of the section of the paper used.

An abridged version of Chapter 1 appears in the introduction of the paper "Conjectured Combinatorial Models for the Hilbert Series of Generalized Diagonal Harmonics Modules" by N. Loehr and J. Remmel, which is now in preparation for publication. The dissertation author was the primary investigator and single author of the section of the paper used.

Chapter 2 is essentially a reprint, with minor modifications, of the paper "Conjectured Statistics for the Higher q, t-Catalan Sequences" by N. Loehr, which has been submitted for publication in *Electronic Journal of Combinatorics.* The dissertation author was the primary investigator and sole author of this paper.

Chapter 3 is essentially a reprint, with minor modifications, of the paper "Trapezoidal Lattice Paths and Multivariate Analogues" by N. Loehr, which has been accepted for publication in *Advances in Applied Mathematics.* The dissertation author was the primary investigator and sole author of this paper.

Chapter 4 is essentially a reprint, with minor modifications, of the paper "Conjectured Combinatorial Models for the Hilbert Series of Generalized Diagonal Harmonics Modules" by N. Loehr and J. Remmel, which is now in preparation for publication. The dissertation author was the primary investigator and author of this paper.

Chapter 5 is now in preparation for publication in one or more papers by N. Loehr. The dissertation author was the primary investigator and author of this material.

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L. Heath and N. Loehr, "Efficient Algorithms for Finding Conway Polynomials," *Tenth Annual ACM-SIAM Symposium on Discrete Algorithms*, 1999.

P. Stockmeyer et al., "Exchanging Disks in the Tower of Hanoi," *Intl. Journal of Computer Mathematics* 59, 1995.

ABSTRACT OF THE DISSERTATION

Multivariate Analogues of Catalan Numbers, Parking Functions, and their Extensions

by

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Doctor of Philosophy in Mathematics

University of California San Diego, 2003

Professor Jeffrey Remmel, Chair

This document is concerned with the Catalan numbers and their generalizations. The Catalan numbers, which occur ubiquitously in combinatorics, are also connected to certain problems in representation theory, symmetric function theory, the theory of Macdonald polynomials, algebraic geometry, and Lie algebras. Garsia and Haiman introduced a bivariate analogue of the Catalan numbers, called the q,t-Catalan sequence, in this setting. This sequence counts multiplicities of the sign character in a certain doubly graded S_n-module called the diagonal harmonics module. Several classical q-analogues of the Catalan numbers can be obtained from this sequence by suitable specializations. Haglund and Haiman separately proposed combinatorial interpretations for this q,t-Catalan sequence by defining two statistics on Dyck paths. Garsia and Haglund later proved the correctness of these interpretations. Haglund, Haiman, and Loehr defined similar statistics on labelled Dyck paths, which are conjectured to give the Hilbert series of the diagonal harmonics module.

In this thesis, we introduce and analyze several conjectured combinatorial interpretations for the "higher" q,t-Catalan sequences of Garsia and Haiman. These interpretations involve pairs of statistics for unlabelled lattice paths staying inside certain triangles. Trivariate generating functions for these paths are also discussed. These constructions are then generalized to lattice paths contained in trapezoids. We study five-variable generating functions for these paths and derive their combinatorial properties.

Next, we consider multivariate generating functions for labelled lattice paths (parking functions) staying within various shapes. In the case of triangles, we obtain a conjectured combinatorial interpretation for the Hilbert series of higher-order analogues of the diagonal harmonics module.

Finally, we present some miscellaneous results connected to the various Catalan sequences. We present a variation of André's reflection principle to count paths contained in trapezoidal regions. We give a determinantal formula for the Carlitz-Riordan numbers that enumerate Dyck paths by area. We also give several ways to define the bivariate Catalan sequence in terms of classical permutation statistics.

1

Introduction to Catalan Sequences and Parking Functions

This document is concerned with the Catalan numbers and their generalizations. The Catalan numbers, which occur ubiquitously in combinatorics, are also connected to certain problems in representation theory, symmetric function theory, the theory of Macdonald polynomials, algebraic geometry, and Lie algebras. Garsia and Haiman introduced a bivariate analogue of the Catalan numbers, called the q,t-Catalan sequence, in this setting. This sequence counts multiplicities of the sign character in a certain doubly graded S_n-module called the diagonal harmonics module. Several classical q-analogues of the Catalan numbers can be obtained from this sequence by suitable specializations. Haglund and Haiman separately proposed combinatorial interpretations for this q,t-Catalan sequence by defining two statistics on Dyck paths. Garsia and Haglund later proved the correctness of these interpretations. Haglund, Haiman, and the present author defined similar statistics on labelled Dyck paths, which are conjectured to give the Hilbert series of the diagonal harmonics module.

We will study these sequences and their generalizations from a combinatorial standpoint. We define statistics on various collections of lattice paths (both unlabelled and labelled) that stay within special shapes such as triangles and trapezoids. We prove formulas, recursions, specializations, and bijections involving these statistics. In some cases, we have conjectured interpretations of the generating functions for these statistics in terms of representation theory or symmetric functions.

In this first chapter, we introduce the Catalan numbers and their classical q-analogues. We briefly describe the problems in representation theory that motivated the definition of the original q,t-Catalan sequence. We then discuss results of Garsia, Haiman, Haglund, et al. that give a combinatorial description of this bivariate sequence. Finally, we give a preview of the more general collections of objects that admit a similar combinatorial treatment.

1.1 Catalan Numbers

Definition 1.1. The *Catalan numbers* C_n are defined by the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \text{ for } n \geq 0.$$

C_n can also be defined recursively by setting

$$C_0 = 1, \quad C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} \text{ for } n > 0.$$

The first few Catalan numbers are

$$C_0 = 1, \quad C_1 = 1, \quad C_2 = 2, \quad C_3 = 5, \quad C_4 = 14, \quad C_5 = 42, \quad C_6 = 132, \quad C_7 = 429.$$

The Catalan number C_n counts the number of rooted, unlabelled binary trees with n vertices; the number of rooted, unlabelled planar trees with $n+1$ vertices; and many other combinatorial objects. See [29, 30] for a list of more than 95 collections of objects counted by the Catalan numbers. The objects of greatest interest for our purposes are the Dyck paths and the Catalan words.

Definition 1.2. A *Dyck path of order* n is a path in the xy-plane from $(0,0)$ to (n,n) consisting of n north steps and n east steps (each of length one), such that the path never goes strictly below the diagonal line $y = x$. Let $\mathcal{D}_n$ denote the collection of Dyck paths of order n.

Definition 1.3. A *Catalan word of order* n is a word $w = w_1 w_2 \ldots w_{2n}$ consisting of n zeroes and n ones, such that for all i, the number of zeroes in the prefix $w_1 w_2 \ldots w_i$ is greater than or equal to the number of ones in $w_1w_2\dots w_i$. Let $\mathcal{W}_n$ denote the collection of Catalan words of order n.

An example of a Dyck path appears in Figure 1.1. Suppose we encode the steps of the path, starting at $(0,0)$, by writing the symbol 0 for each north step and writing the symbol 1 for each east step. For the Dyck path in Figure 1.1, we obtain the word

$$w = 00010011110100100110011110011.$$

This word has n zeroes and n ones, since there are n north steps and n east steps in the Dyck path. Also, in any prefix of w, there are at least as many zeroes as ones, since otherwise the Dyck path would go below the diagonal line $y = x$. This process of encoding a Dyck path as a word is clearly reversible. Thus, we have a bijection between Dyck paths and Catalan words. If we write a left parenthesis instead of a zero and a right parenthesis instead of a one, a Catalan word turns into a string of n left and right parentheses where there are no unmatched parentheses. For example, the word w above becomes the string

$$\texttt{((()(())))()(()(())(()))(())}.$$

This construction gives another collection of objects counted by the Catalan number.

1.2 Classical q-analogues of the Catalan Numbers

We can obtain generalizations of the Catalan numbers by looking at collections of *weighted* objects. For instance, we can assign weights to Catalan words using the following classical statistics.

Definition 1.4. Let $w = w_1w_2\dots w_n$ be any word, where each w_i is an integer.
(1) Define the *inversions of* w by

$$inv(w) = \sum_{1\le i<j\le n} \chi(w_i > w_j). \tag{1.1}$$

Here and below, for any logical statement A we set $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false.

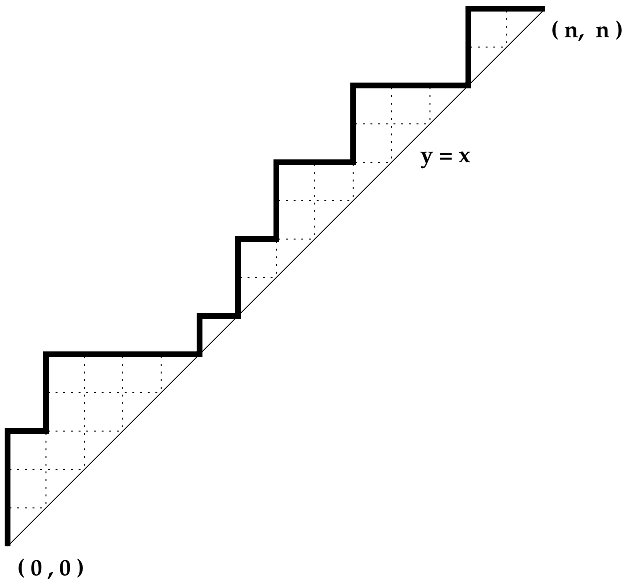


Figure 1.1: A Dyck path.

(2) Define the *major index of* w by

$$maj(w) = \sum_{i=1}^{n-1} i\chi(w_i > w_{i+1}).$$

For example, Table 1.1 lists the values of these statistics for all $w \in \mathcal{W}_3$, the Catalan words of order 3.

We now obtain "q-analogues" of the Catalan numbers by looking at generating functions for the Catalan words relative to the various statistics.

Definition 1.5. Let q be a formal variable.

(1) Define the *inversion q-analogue of the Catalan numbers* by

$$C_n^{inv}(q) = \sum_{w \in \mathcal{W}_n} q^{inv(w)}.$$

(2) Define the *major index q-analogue of the Catalan numbers* by

$$C_n^{maj}(q) = \sum_{w \in \mathcal{W}_n} q^{maj(w)}.$$

Table 1.1: Statistics for Catalan words of order 3.

Word w in $\mathcal{W}_3$	$inv(w)$	$maj(w)$
010101	3	6
001101	2	4
010011	2	2
001011	1	3
000111	0	0

For example, using Table 1.1, we compute

$$C_3^{inv}(q) = 1 + q + 2q^2 + q^3,$$

$$C_3^{maj}(q) = 1 + q^2 + q^3 + q^4 + q^6.$$

The sequence $C_n^{inv}(q)$ has been studied by Carlitz, Riordan, and other authors [7]. The sequence $C_n^{maj}(q)$ was first studied by MacMahon [25]. MacMahon proved the formula

$$C_n^{maj}(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q,$$

where we use the notation

$$[0]_q = 0,\ [m]_q = 1 + q + q^2 + \cdots + q^{m-1} = \frac{1 - q^m}{1 - q} \text{ for } m > 0, \text{ and} \tag{1.2}$$

$$\begin{bmatrix} a+b \\ a \end{bmatrix}_q = \begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \frac{\prod_{i=1}^{a+b}(1-q^i)}{\prod_{i=1}^{a}(1-q^i)\prod_{i=1}^{b}(1-q^i)}. \tag{1.3}$$

The sequence $C_n^{inv}(q)$ has no simple explicit formula like the one for $C_n^{maj}(q)$. However, we will discuss a determinantal formula for $C_n^{inv}(q)$ in §5.2.

We can also define statistics on geometric objects such as Dyck paths.

Definition 1.6. Let D be a Dyck path of order n.

(1) Define $area(D)$ to be the number of complete lattice cells between D and the diagonal line $y = x$.

(2) Define the *area q-analogue of the Catalan numbers* by

$$C_n^{area}(q) = \sum_{D \in \mathcal{D}_n} q^{area(D)}.$$

For instance, the path D in Figure 1.1 has $area(D) = 16$. By examining the five Dyck paths of order 3 in Figure 1.2, we compute

$$C_3^{area}(q) = 1 + 2q + q^2 + q^3.$$

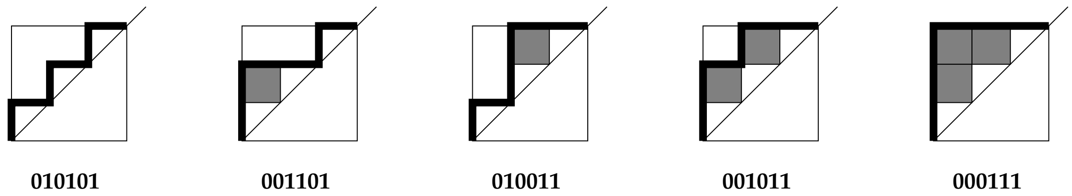


Figure 1.2: Dyck paths of order 3, and the associated Catalan words.

We remark that $C_3^{area}(1) = 5 = C_3$. In general, for any q-analogue of the Catalan number, we can recover the number C_n by setting $q = 1$.

Comparing Figure 1.2 to Table 1.1, we observe that

$$area(D) + inv(w) = 3 = \binom{3}{2},$$

where w is the Catalan word associated to the Dyck path D. This is a special case of the following lemma.

Lemma 1.7. *Let D be a Dyck path of order n, with associated Catalan word w. Then*

$$area(D) + inv(w) = \binom{n}{2}.$$

Proof. Consider the triangle bounded by the lines $y = x$, $y = n$, and $x = 0$. Let us count the number of complete lattice squares inside this triangle. There are $n - 1$ squares in the leftmost column, $n - 2$ squares in the next column, etc. The total number of squares is

$$(n-1) + (n-2) + \cdots + 1 + 0 = \binom{n}{2}.$$

Note that each square in this triangle is either below or above the Dyck path D. The number of squares below the path is $area(D)$. We claim the number of squares above the path and inside the triangle is $inv(w)$. To see this, let c_i be the number of zeroes in w that follow the i^{th} one in w. Since w consists only of zeroes and ones, the definition of

$inv(w)$ shows that $inv(w) = c_1 + c_2 + \cdots + c_n$. On the other hand, consider the number of squares above the path D in the i^{th} column from the left. These squares lie above the i^{th} horizontal segment in the path, which corresponds to the i^{th} one in w. The number of such squares is the same as the number of vertical segments following the i^{th} horizontal segment in the path, which is the same as the number of zeroes following the i^{th} one in w. Thus, there are c_i squares above the path in column i. Adding over all i, we see that the total area above the path is $inv(w)$, as claimed. □

As a consequence of this lemma, we compute

$$\begin{aligned} C_n^{inv}(q) &= \sum_{w \in \mathcal{W}_n} q^{inv(w)} = \sum_{D \in \mathcal{D}_n} q^{\binom{n}{2} - area(D)} = q^{\binom{n}{2}} \sum_{D \in \mathcal{D}_n} \left(\frac{1}{q}\right)^{area(D)} \\ &= q^{\binom{n}{2}} C_n^{area}(1/q). \end{aligned}$$

Thus, there is no essential difference between the sequences C_n^{inv} and C_n^{area}. For this reason, we may refer to either sequence as the *Carlitz-Riordan q-analogue of the Catalan numbers*. On the other hand, C_n^{maj} cannot be obtained from these sequences by a simple transformation of this kind.

Remarkably, it turns out that both sequences C_n^{area} and C_n^{maj} can be obtained as special cases of a *bivariate* sequence involving two variables q and t. This sequence, introduced by Garsia and Haiman, arose from problems in representation theory and symmetric function theory. We now interrupt the combinatorial discussion to describe the development of this bivariate sequence.

1.3 Diagonal Harmonics and Bivariate Catalan Sequences

We assume the reader is well-acquainted with basic facts about groups, rings, fields, vector spaces, modules, and algebras; see [26] or any algebra text for a detailed discussion of these concepts. This section also assumes some knowledge of the representation theory of finite groups and symmetric function theory, including Macdonald polynomials. Different aspects of this material are treated in [27, 29, 24]. For the reader's convenience, we summarize some needed definitions and notation in the next few subsections.

1.3.1 Partitions

We first review some standard terminology associated with integer partitions.

Definition 1.8. A *partition* is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ of weakly decreasing positive integers, called the *parts* of λ. The integer $N = \lambda_1 + \lambda_2 + \cdots + \lambda_k$ is called the *area* or *weight* of λ and denoted $|\lambda|$. In this case, λ is said to be a *partition of* N, written in symbols as $\lambda \vdash N$. The number of parts k is called the *length of* λ and denoted $\ell(\lambda)$. We often depict a partition λ by its *Ferrers diagram.* This diagram consists of k left-justified rows of boxes (called *cells*), where the i^{th} row from the top has exactly λ_i boxes.

Figure 1.3 shows the Ferrers diagram of $\lambda = (8, 7, 5, 4, 4, 2, 1)$, which is a partition of 31 having seven parts.

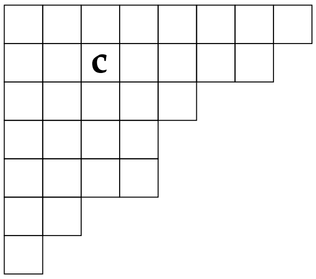


Figure 1.3: Diagram of a partition.

Definition 1.9. Given a partition λ, the *transpose* λ' of λ is the partition obtained by interchanging the rows and columns of the Ferrers diagram of λ.

For example, the transpose of the partition in Figure 1.3 is

$$\lambda' = (7, 6, 5, 5, 3, 2, 2, 1).$$

Definition 1.10. Let λ be a partition of N. Let c be one of the N cells in the diagram of λ.

(1) The *arm of* c, denoted $a(c)$, is the number of cells strictly right of c in the diagram of λ.

(2) The *coarm of* c, denoted $a'(c)$, is the number of cells strictly left of c in the diagram of λ.

(3) The *leg of* c, denoted $l(c)$, is the number of cells strictly below c in the diagram of λ.

(4) The *coleg of* c, denoted $l'(c)$, is the number of cells strictly above c in the diagram of λ.

For example, the cell labelled c in Figure 1.3 has $a(c) = 4$, $a'(c) = 2$, $l(c) = 3$, and $l'(c) = 1$.

Definition 1.11. We define the *dominance partial ordering* on partitions of N as follows. If λ and μ are partitions of N, we write $\lambda \geq \mu$ to mean that

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \text{ for all } i \geq 1.$$

For example, we have $(4, 3, 1) \geq (3, 3, 1, 1)$. As another example, $\lambda = (3, 1, 1, 1)$ and $\mu = (2, 2, 2)$ are not comparable to one another relative to the partial order $\geq$.

Definition 1.12. Fix a positive integer N and a partition μ of N. We introduce the following abbreviations to shorten upcoming formulas:

$$\begin{aligned}
h_\mu(q,t) &= \prod_{c\in\mu}(q^{a(c)} - t^{l(c)+1}) \\
h'_\mu(q,t) &= \prod_{c\in\mu}(t^{l(c)} - q^{a(c)+1}) \\
n(\mu) &= \sum_{c\in\mu} l(c) \\
B_\mu(q,t) &= \sum_{c\in\mu} q^{a'(c)}t^{l'(c)} \\
\Pi_\mu(q,t) &= \prod_{c\in\mu, c\neq(0,0)} (1 - q^{a'(c)}t^{l'(c)})
\end{aligned}$$

In all but the last formula above, the sums and products range over all cells in the diagram of μ. In the product defining $\Pi_\mu(q,t)$, the northwest corner cell of μ is omitted from the product. This is the cell c with $a'(c) = l'(c) = 0$; if we did not omit this cell, then $\Pi_\mu(q,t)$ would be zero. Note that

$$n(\mu') = \sum_{c\in\mu'} l(c) = \sum_{c\in\mu} a(c).$$

Example 1.13. Let $\mu = (3,2)$. Then

$$\begin{aligned}
h_{(3,2)}(q,t) &= (q^2-t^2)(q^1-t^2)(q^0-t^1)(q^1-t^1)(q^0-t^1) \\
h'_{(3,2)}(q,t) &= (t^1-q^3)(t^1-q^2)(t^0-q^1)(t^0-q^2)(t^0-q^1) \\
n_{(3,2)} &= 2+1+0+1+0 = 4 \\
B_{(3,2)}(q,t) &= q^0t^0 + q^1t^0 + q^2t^0 + q^0t^1 + q^1t^1 \\
\Pi_{(3,2)}(q,t) &= (1-q^1t^0)(1-q^2t^0)(1-q^0t^1)(1-q^1t^1) \\
n_{(3,2)'} &= n_{(2,2,1)} = 1+1+0+0+0 = 2.
\end{aligned}$$

1.3.2 Symmetric Functions

Next, we review notation for symmetric functions and various bases for the symmetric functions. For more information, see [27, 29, 24].

Let K be a field of characteristic zero, such as the rational numbers $\mathbb{Q}$ or the complex numbers $\mathbb{C}$. Let $\Lambda = \Lambda(K)$ denote the ring of symmetric functions in the indeterminates $x_1, x_2, \ldots, x_n, \ldots$ with coefficients in K. Let Λ^m denote the subring of homogeneous symmetric functions of degree m (including zero). For a formal algebraic definition of Λ^m and Λ, see [24], Chapter 1.

Definition 1.14. We now introduce some commonly occurring symmetric functions.

(1) For $n > 0$, define the *elementary symmetric function* e_n by

$$e_n = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Define $e_0 = 1$ and $e_j = 0$ for $j < 0$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, define $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_t}$.

(2) For $n > 0$, define the *complete symmetric function* h_n by

$$h_n = \sum_{i_1 \le i_2 \le \ldots \le i_n} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Define $h_0 = 1$ and $h_j = 0$ for $j < 0$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, define $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_t}$.

(3) For $n > 0$, define the *power-sum symmetric function* p_n by

$$p_n = \sum_{i \geq 1} x_i^n.$$

Define $p_0 = 1$ and $p_j = 0$ for $j < 0$.

For a partition $\lambda = (\lambda_1, \ldots, \lambda_t)$, define $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_t}$.

(4) For any partition λ of weight n, define the *Schur function* s_λ by the Jacobi-Trudi formula

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}.$$

(There are several other equivalent ways of defining s_λ.)

Example 1.15. For $n = 1$, we have

$$e_1 = h_1 = p_1 = s_{(1)} = \sum_i x_i.$$

For $n = 2$, we have

$$\begin{aligned} e_2 &= \sum_{i<j} x_i x_j \\ h_2 &= \sum_i x_i^2 + \sum_{i<j} x_i x_j \\ p_2 &= \sum_i x_i^2 \end{aligned}$$

$$\begin{aligned} p_{(1,1)} &= \left(\sum_i x_i\right)^2 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j \\ s_{(2)} &= h_2 = \sum_i x_i^2 + \sum_{i<j} x_i x_j \\ s_{(1,1)} &= h_1^2 - h_0 h_2 = e_2 = \sum_{i<j} x_i x_j \\ e_{(1,1)} &= e_1 e_1 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j \end{aligned}$$

Comparing these expressions, we see that

$$\begin{aligned} p_{(2)} &= 1s_{(2)} - 1s_{(1,1)} \\ p_{(1,1)} &= 1s_{(2)} + 1s_{(1,1)} \\ s_{(2)} &= (1/2)p_{(2)} + (1/2)p_{(1,1)} \\ s_{(1,1)} &= (-1/2)p_{(2)} + (1/2)p_{(1,1)} \end{aligned} \tag{1.4}$$

These formulas will be used in later examples.

We can use the symmetric functions defined above to get several different bases for Λ, viewed as a vector space over K.

Theorem 1.16. *Let K be a field of characteristic zero.*

(1) The set $\{e_\lambda\}$, as λ ranges over all partitions, is a basis for Λ.
The set $\{e_\lambda : \lambda \vdash m\}$ is a basis for Λ^m.
The set $\{e_1, e_2, \ldots\}$ is algebraically independent and generates Λ as a K-algebra.

(2) The set $\{h_\lambda\}$, as λ ranges over all partitions, is a basis for Λ.
The set $\{h_\lambda : \lambda \vdash m\}$ is a basis for Λ^m.
The set $\{h_1, h_2, \ldots\}$ is algebraically independent and generates Λ as a K-algebra.

(3) The set $\{p_\lambda\}$, as λ ranges over all partitions, is a basis for Λ.
The set $\{p_\lambda : \lambda \vdash m\}$ is a basis for Λ^m.
The set $\{p_1, p_2, \ldots\}$ is algebraically independent and generates Λ as a K-algebra.

(4) The set s_λ, as λ ranges over all partitions, is a basis for Λ.
The set $\{s_\lambda : \lambda \vdash m\}$ is a basis for Λ^m.

Proof. See [24] Chapter 1, or [29] Chapter 7. □

Definition 1.17. Let f be an arbitrary element of Λ. Since $\{s_\lambda\}$ is a basis for Λ as a K-vector space, there is a unique expansion

$$f = \sum_\lambda c_\lambda s_\lambda, \quad c_\lambda \in K.$$

We define the *coefficient of* s_λ *in* f to be the scalar c_λ. In symbols, we write

$$f|_{s_\lambda} = c_\lambda.$$

Similar notation can be defined for the other bases of Λ mentioned above.

Example 1.18. Inspection of (1.4) shows that

$$p_{(2)}|_{s_{(2)}} = 1 \text{ and } p_{(2)}|_{s_{(1,1)}} = -1.$$

Example 1.19. Some care must be exercised when using this notation, since the value of $f|_g$ depends on the basis to which g belongs, not just on g itself. For instance, let $f = 3e_2 + 3e_{(1,1)}$ and $g = s_{(1,1)} = e_2$. Evidently, $f|_{e_2} = 3$. On the other hand, since $e_2 = s_{(1,1)}$ and $e_{(1,1)} = s_2 + s_{(1,1)}$ by Example 1.15, it follows that

$$f = 3s_{(2)} + 6s_{(1,1)},$$

so that $f|_{s_{(1,1)}} = 6$. Thus, the value of $f|_g$ depends on whether we consider g as belonging to the Schur basis (s_λ) or to the elementary basis (e_λ).

Theorem 1.20. *[Foundation for Plethystic Substitution]*
Let p_n denote the power-sum symmetric function, as usual. Let R be a K-algebra. If $\phi_0 : \{p_1, p_2, \ldots, p_n, \ldots\} \to R$ is any function, then there exists a unique K-algebra homomorphism $\phi : \Lambda(K) \to R$ that extends ϕ_0.

Proof. This follows immediately from the fact that the set $\{p_1, p_2, \ldots, p_n, \ldots\}$ is algebraically independent and generates Λ as a K-algebra. Specifically, to get a K-algebra homomorphism that extends ϕ_0, we must set

$$\phi(p_\lambda) = \phi(p_{\lambda_1} \cdots p_{\lambda_k}) = \phi(p_{\lambda_1}) \cdots \phi(p_{\lambda_k}) = \phi_0(p_{\lambda_1}) \cdots \phi_0(p_{\lambda_k}).$$

For any $f \in \Lambda$, we can write f uniquely as $f = \sum_\lambda c_\lambda p_\lambda$ where $c_\lambda \in K$. Then we must set

$$\phi(f) = \sum_\lambda c_\lambda \phi(p_\lambda).$$

This shows that ϕ, if it exists, must be unique. To show existence, define ϕ by the formulas just given. It is routine to check that ϕ is a well-defined K-algebra homomorphism from $\Lambda(K)$ to R. $\square$

Example 1.21. Let ϕ_0 be the function that sends each p_k to $(1-q^k)p_k$. Let ϕ be the corresponding map produced by the theorem. We have

$$\phi(p_1) = (1-q)p_1, \qquad \phi(p_2) = (1-q^2)p_2,$$

from which we deduce that

$$\phi(p_{(1,1)}) = \phi(p_1)\phi(p_1) = (1-q)^2 p_1 p_1 = (1-2q+q^2)p_{(1,1)}.$$

Next, since $s_{(2)} = (1/2)p_{(2)} + (1/2)p_{(1,1)}$ and $s_{(1,1)} = (-1/2)p_{(2)} + (1/2)p_{(1,1)}$, linearity of ϕ gives

$$\begin{aligned} \phi(s_{(2)}) &= \frac{1}{2}(1-q^2)p_{(2)} + \frac{1}{2}(1-2q+q^2)p_{(1,1)} \\ \phi(s_{(1,1)}) &= -\frac{1}{2}(1-q^2)p_{(2)} + \frac{1}{2}(1-2q+q^2)p_{(1,1)}. \end{aligned}$$

Using (1.4) to rewrite the p's in terms of Schur functions, we get

$$\begin{aligned} \phi(s_{(2)}) &= (1-q)s_{(2)} + (q^2-q)s_{(1,1)} \\ \phi(s_{(1,1)}) &= (q^2-q)s_{(2)} + (1-q)s_{(1,1)} \end{aligned} \tag{1.5}$$

We will use these calculations in the next subsection.

1.3.3 Modified Macdonald Polynomials and the Nabla Operator

In this section, we define the modified Macdonald polynomials, which form another useful basis for the ring of symmetric functions. We also define the nabla operator, a linear operator on Λ that has many important properties. The modified Macdonald polynomials were introduced by Garsia and Haiman [18] by modifying the definition in Macdonald's book [24]. The nabla operator was first introduced by F. Bergeron and Garsia [4].

Let K be the field $\mathbb{Q}(q,t)$ whose elements are quotients $g(q,t)/h(q,t)$ of polynomials in the variables q and t. We will be working in the ring $\Lambda = \Lambda(K)$ of symmetric functions with coefficients in K. Let $\alpha_0 : K \to K$ be the automorphism of K that interchanges q and t; thus, $\alpha_0(g(q,t)/h(q,t)) = g(t,q)/h(t,q)$. Extend α_0 to an automorphism $\alpha : \Lambda \to \Lambda$ by setting

$$\alpha\left(\sum_\lambda c_\lambda s_\lambda\right) = \sum_\lambda \alpha_0(c_\lambda)s_\lambda \qquad (c_\lambda \in K).$$

Sometimes we abuse notation and write an element $f \in \Lambda$ as $f(x;q,t)$. Then we would write $\alpha(f) = f(x;t,q)$.

Next, let $\phi_0 : \{p_1, p_2, \ldots\} \to \Lambda$ be the function given by $\phi_0(p_k) = (1-q^k)p_k \in \Lambda$. By Theorem 1.20, we obtain a unique K-algebra homomorphism $\phi : \Lambda \to \Lambda$ extending ϕ_0. (Authors who use plethystic notation would write $f[(1-q)X]$ instead of $\phi(f)$, for $f \in \Lambda$.) Now we can introduce the modified Macdonald polynomials.

Theorem 1.22. *Let K, α, and ϕ be defined as above. There exists a unique basis $\tilde{H}_\mu$ of $\Lambda(K)$ with the following properties:*

(1) $\phi(\tilde{H}_\mu) = \sum_{\lambda \geq \mu} c_{\lambda,\mu} s_\lambda$ *for certain scalars* $c_{\lambda,\mu} \in K$.

(2) $\alpha(\tilde{H}_\mu) = \tilde{H}_{\mu'}$.

(3) $\tilde{H}_\mu|_{s_{(n)}} = 1$.

Here, μ ranges over all partitions, and $\geq$ is the dominance partial order on partitions. Also, $\{\tilde{H}_\mu : \mu \vdash m\}$ is a basis of $\Lambda^m(K)$.
(Some authors write the three properties in the definition using different notation, as follows:

(1) $\tilde{H}_\mu[(1-q)X;q,t] = \sum_{\lambda \geq \mu} c_{\lambda,\mu}(q,t) s_\lambda(X)$ *for certain scalars* $c_{\lambda,\mu} \in K$.

(2) $\tilde{H}_\mu(X;q,t) = \tilde{H}_{\mu'}(X;t,q)$.

(3) $\langle \tilde{H}_\mu(X;q,t), s_{(n)}(X) \rangle = 1$.

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product on $\Lambda(K)$ defined by requiring that $\{s_\lambda\}$ be an orthonormal basis relative to the inner product.)

Proof. See, for instance, [18]. The proof for the original Macdonald polynomials is in [24]. □

Definition 1.23. The elements $\tilde{H}_\mu$ are called *modified Macdonald polynomials.*

Fix a partition $\mu \vdash n$. Since $\{s_\lambda : \lambda \vdash n\}$ is a basis for Λ^n, we can uniquely write

$$\tilde{H}_\mu = \sum_{\lambda \vdash n} \tilde{K}_{\lambda,\mu} s_\lambda$$

for certain coefficients $\tilde{K}_{\lambda,\mu} \in \mathbb{Q}(q,t)$. These coefficients are called the *modified Kostka-Macdonald coefficients*. The following theorem of Haiman resolves a long-standing conjecture of Macdonald regarding these coefficients.

Theorem 1.24. *[Haiman]*
For every $\lambda \vdash n$ and $\mu \vdash n$, $\tilde{K}_{\lambda,\mu}$ is a polynomial in q and t with nonnegative integer coefficients.

Proof. See [18]. □

In advance, one only knows that $\tilde{K}_{\lambda,\mu}$ is a *rational function* with *rational* coefficients. Haiman's proof uses sophisticated machinery from algebraic geometry. The proof provides an explicit interpretation for the coefficients of the polynomials $\tilde{K}_{\lambda,\mu}$. These coefficients count the multiplicities of irreducible modules in a certain doubly graded S_n-module.[1] In particular, the coefficients must be nonnegative integers.

Example 1.25. Let us compute $\tilde{H}_{(2)}$ and $\tilde{H}_{(1,1)}$. Each of these is a linear combination of the Schur functions $s_{(2)}$ and $s_{(1,1)}$. Using condition (3) of the definition, we can write

$$\begin{aligned} \tilde{H}_{(2)} &= 1s_{(2)} + a(q,t)s_{(1,1)} \\ \tilde{H}_{(1,1)} &= 1s_{(2)} + b(q,t)s_{(1,1)} \end{aligned}$$

for some unknown scalars $a(q,t)$ and $b(q,t)$ in K. By condition (2) of the definition, $b(q,t) = a(t,q)$ so it suffices to compute $a(q,t)$. To do this, we use condition (1) of the definition. On one hand, using (1.5) and linearity of ϕ, we have

$$\phi(\tilde{H}_{(2)}) = \left(1 - q + a(q,t)(q^2 - q)\right) s_{(2)} + \left((q^2 - q) + a(q,t)(1 - q)\right) s_{(1,1)}.$$

On the other hand, condition (1) in the definition of $\tilde{H}_\mu$ tells us that

$$\phi(\tilde{H}_{(2)}) = c(q,t)s_{(2)} + 0s_{(1,1)}$$

for some scalar $c(q,t)$. Comparing coefficients of $s_{(1,1)}$ gives us the equation

$$(q^2 - q) + a(q,t)(1 - q) = 0.$$

[1]This terminology is reviewed in the next subsection.

Solving this gives $a(q,t) = (q - q^2)/(1-q) = q$. Therefore, $b(q,t) = a(t,q) = t$, and we have

$$\begin{aligned} \tilde{H}_{(2)} &= 1s_{(2)} + qs_{(1,1)} \\ \tilde{H}_{(1,1)} &= 1s_{(2)} + ts_{(1,1)}. \end{aligned} \tag{1.6}$$

We can also solve for the Schur functions in terms of $\tilde{H}_\mu$:

$$\begin{aligned} s_{(2)} &= \tfrac{t}{t-q}\tilde{H}_{(2)} + \tfrac{-q}{t-q}\tilde{H}_{(1,1)} \\ s_{(1,1)} &= \tfrac{-1}{t-q}\tilde{H}_{(2)} + \tfrac{1}{t-q}\tilde{H}_{(1,1)}. \end{aligned} \tag{1.7}$$

We now define the nabla operator of F. Bergeron and Garsia. Some of the special properties of this operator are developed in [4, 5, 6].

Definition 1.26. The *nabla operator* ∇ is the linear operator on $\Lambda(K)$ that acts on the modified Macdonald basis as follows:

$$\nabla(\tilde{H}_\mu) = q^{n(\mu')}t^{n(\mu)}\tilde{H}_\mu.$$

If f is any element of $\Lambda(K)$, we can write f uniquely as

$$f = \sum_\mu c_\mu \tilde{H}_\mu, \quad c_\mu \in K.$$

By linearity, we then have

$$\nabla(f) = \sum_\mu c_\mu q^{n(\mu')}t^{n(\mu)}\tilde{H}_\mu.$$

Note that ∇ is a linear operator with eigenvalues $q^{n(\mu')}t^{n(\mu)}$ and corresponding eigenfunctions $\tilde{H}_\mu$.

Example 1.27. For partitions of 2, we have

$$\begin{aligned} \nabla(\tilde{H}_{(2)}) &= q\tilde{H}_{(2)} \\ \nabla(\tilde{H}_{(1,1)}) &= t\tilde{H}_{(1,1)}. \end{aligned} \tag{1.8}$$

Using the relations (1.6) and (1.7), we can determine the action of nabla on the Schur basis. Applying ∇ to both sides of (1.7) and using (1.8), we get

$$\begin{aligned} \nabla(s_{(2)}) &= \tfrac{qt}{t-q}\tilde{H}_{(2)} + \tfrac{-qt}{t-q}\tilde{H}_{(1,1)} \\ \nabla(s_{(1,1)}) &= \tfrac{-q}{t-q}\tilde{H}_{(2)} + \tfrac{t}{t-q}\tilde{H}_{(1,1)}. \end{aligned} \tag{1.9}$$

We then use (1.6) to express the answer in terms of Schur functions. After some routine calculations, we obtain:

$$\begin{array}{rclcl} \nabla(s_{(2)}) & = & 0s_{(2)} & + & (-qt)s_{(1,1)} \\ \nabla(s_{(1,1)}) & = & 1s_{(2)} & + & (q+t)s_{(1,1)}. \end{array}$$

Thus, the matrix of ∇ on $\Lambda^2(K)$ relative to the basis $(s_{(2)}, s_{(1,1)})$ is

$$[\nabla]_2 = \begin{pmatrix} 0 & 1 \\ -qt & q+t \end{pmatrix}. \tag{1.10}$$

Example 1.28. We can carry out similar calculations for s_λ with $|\lambda| = 3$. We invite the reader to verify the following formulas.

$$\begin{array}{rclclcl} \tilde{H}_{(3)} & = & 1s_{(3)} & + & (q+q^2)s_{(2,1)} & + & q^3 s_{(1,1,1)} \\ \tilde{H}_{(2,1)} & = & 1s_{(3)} & + & (q+t)s_{(2,1)} & + & qts_{(1,1,1)} \\ \tilde{H}_{(1,1,1)} & = & 1s_{(3)} & + & (t+t^2)s_{(2,1)} & + & t^3 s_{(1,1,1)}. \end{array} \tag{1.11}$$

Proceeding as in the last example, one can then compute the matrix of the linear map ∇ relative to the ordered basis $(s_{(3)}, s_{(2,1)}, s_{(1,1,1)})$. After some calculations, the result is:

$$[\nabla]_3 = \begin{pmatrix} 0 & 0 & 1 \\ q^2t^2 & -qt(q+t) & q+q^2+t+t^2+qt \\ q^2t^2(q+t) & -qt(q^2+qt+t^2) & q^3+q^2t+t^3+qt+qt^2 \end{pmatrix}. \tag{1.12}$$

The next theorem, due to Garsia and Haiman, gives an explicit formula for $\nabla(e_n) = \nabla(s_{1^n})$ as an expansion in terms of the basis $(\tilde{H}_\mu)$.

Theorem 1.29.

$$\nabla(e_n) = \nabla(s_{1^n}) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu t^{n(\mu)} q^{n(\mu')}(1-t)(1-q)\Pi_\mu(q,t)B_\mu(q,t)}{h_\mu(q,t)h'_\mu(q,t)}.$$

Proof. See [15, 14, 18]. □

For example, (1.9) illustrates the case $n = 2$ of this formula. The formula in this theorem has a representation theoretical interpretation, conjectured by Garsia and Haiman and later proved by Haiman. This interpretation is described in subsection §1.3.5 below. That subsection also gives specializations of this formula when $t = 1/q$ or $t = 1$.

1.3.4 Representation Theory of Symmetric Groups

In this subsection, we review some basic facts from the representation theory of the symmetric group. A good reference for this material is [27].

Definition 1.30. Fix $n \geq 1$.

(1) Let S_n denote the *symmetric group*, consisting of all bijections

$$\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

with group operation composition. We frequently write a permutation σ by listing its values:

$$\sigma = \sigma(1) \quad \sigma(2) \quad \cdots \quad \sigma(n).$$

(2) Let $\mathcal{A}(S_n) = \mathbb{C}S_n$ denote the *group algebra* of S_n. The elements of the group algebra are functions $f : S_n \to \mathbb{C}$. It is convenient to write elements $f \in \mathcal{A}(S_n)$ as formal sums

$$f = \sum_{\sigma \in S_n} f(\sigma)\sigma.$$

With this notation, the operations in the group algebra are defined by

$$\begin{aligned} f + g &= \sum_{\sigma \in S_n} (f(\sigma) + g(\sigma))\sigma; \\ f - g &= \sum_{\sigma \in S_n} (f(\sigma) - g(\sigma))\sigma; \\ cf &= \sum_{\sigma \in S_n} (cf(\sigma))\sigma \quad \text{for } c \in \mathbb{C}; \\ fg &= \sum_{\sigma \in S_n} \sum_{\substack{\alpha,\beta \in S_n \\ \alpha\beta = \sigma}} (f(\alpha)g(\beta))\sigma. \end{aligned}$$

(3) We can embed S_n in the group algebra $\mathcal{A}(S_n)$ by identifying $\sigma \in S_n$ with the function f_σ such that $f_\sigma(\sigma) = 1$ and $f_\sigma(\tau) = 0$ for $\tau \neq \sigma$. Then S_n is a basis for $\mathcal{A}(S_n)$, viewed as a $\mathbb{C}$-vector space.

(4) Let V be a complex vector space[2], We say that V is an S_n*-module* if and only if it is a module over the ring $\mathcal{A}(S_n)$. This is equivalent to having a *representation* of

[2] V could be infinite-dimensional. For simplicity, we deal only with finite-dimensional vector spaces V, or infinite-dimensional graded (or doubly graded) vector spaces V where each graded piece is finite-dimensional.

S_n on V, which is a group homomorphism $A : S_n \to GL(V)$ from S_n to the group of invertible linear transformations of V.

(5) Given an S_n-module V with associated representation $A : S_n \to GL(V)$, the *character* of V (or A) is the function $\chi_V = \chi_A : S_n \to \mathbb{C}$ that sends $\sigma \in S_n$ to the trace of the linear operator $A(\sigma)$. Thus, χ_V is an element of the group algebra $\mathcal{A}(S_n)$.

(6) An S_n-*submodule* of V is a subspace W of V that is mapped to itself under the action of each $\sigma \in S_n$. In symbols, $A(\sigma)(w) \in W$ for every $w \in W$ and $\sigma \in S_n$. This is equivalent to saying that W is an $\mathcal{A}(S_n)$-submodule of V.

(7) V is an *irreducible* S_n-module if and only if it is nonzero and has no S_n-submodules other than V itself and the zero submodule.

The next theorem summarizes some basic facts from representation theory.

Theorem 1.31. *Fix $n \geq 1$.*

(1) Every S_n-module V can be decomposed into a direct sum of irreducible S_n-modules.

(2) The isomorphism classes of irreducible S_n-modules correspond in a natural way to the partitions $\lambda \vdash n$. Thus, we may label these irreducible modules M_λ.

(3) An S_n-module V is determined (up to isomorphism) by its character χ_V.

(4) For any S_n-module V, the character χ_V belongs to the center of the group algebra $\mathcal{A}(S_n)$.

(5) The characters $\chi_\lambda \stackrel{def}{=} \chi_{M_\lambda}$ are a vector-space basis for the center of the group algebra.

(6) The center of the group algebra is isomorphic to the ring $\Lambda(\mathbb{C})^n$ of homogeneous symmetric functions of degree n under an isomorphism sending χ_λ to s_λ. This isomorphism is called the Frobenius map.

Proof. See [27]. ∎

Definition 1.32. Let V be an S_n-module. We can decompose V into a direct sum of irreducible submodules, say

$$V = \bigoplus_{\lambda \vdash n} c_\lambda M_\lambda \text{ (where } c_\lambda \in \mathbb{N}) \ .$$

The *Frobenius characteristic of* V is defined by

$$\mathcal{F}_V = \sum_{\lambda \vdash n} c_\lambda s_\lambda \in \Lambda^n.$$

Thus, $\mathcal{F}_V$ is a homogeneous symmetric function of degree n, and the coefficient of s_λ in this function is the multiplicity of the irreducible module M_λ in V.

A similar procedure is possible for graded S_n-modules and doubly graded S_n-modules, which we now define.

Definition 1.33. Fix $n \geq 1$.

(1) An S_n-module V is called a *graded* S_n-module if there is a direct sum decomposition

$$V = \bigoplus_{h \geq 0} V_h,$$

where each V_h is an S_n-submodule of V.

(2) Let $V = \oplus_h V_h$ be a graded S_n-module. Decompose each V_h into irreducible submodules, say $V_h = \oplus_{\lambda \vdash n} c_h(\lambda) M_\lambda$. The *Frobenius series of* V is

$$\mathcal{F}_V(q) = \sum_{h \geq 0} \left(\sum_{\lambda \vdash n} c_h(\lambda) s_\lambda \right) q^h = \sum_{h \geq 0} \mathcal{F}_{V_h} q^h.$$

(3) Let $V = \oplus_h V_h$ be a graded S_n-module. The *Hilbert series of* V is

$$H_V(q) = \sum_{h \geq 0} dim_{\mathbb{C}}(V_h) q^h.$$

(4) An S_n-module V is called a *doubly graded* S_n-module if there is a direct sum decomposition

$$V = \bigoplus_{h \geq 0} \bigoplus_{k \geq 0} V_{h,k},$$

where each $V_{h,k}$ is an S_n-submodule of V.

(5) Let $V = \oplus_{h,k} V_{h,k}$ be a doubly graded S_n-module. Decompose each $V_{h,k}$ into irreducible submodules, say $V_{h,k} = \oplus_{\lambda \vdash n} c_{h,k}(\lambda) M_\lambda$. The *Frobenius series of* V is

$$\mathcal{F}_V(q,t) = \sum_{h \geq 0} \sum_{k \geq 0} \left(\sum_{\lambda \vdash n} c_{h,k}(\lambda) s_\lambda \right) q^h t^k = \sum_{h \geq 0} \sum_{k \geq 0} \mathcal{F}_{V_{h,k}} q^h t^k.$$

(6) Let $V = \oplus_{h,k} V_{h,k}$ be a doubly graded S_n-module. The *Hilbert series of* V is

$$H_V(q,t) = \sum_{h \geq 0} \sum_{k \geq 0} dim_{\mathbb{C}}(V_{h,k}) q^h t^k.$$

Given a doubly graded S_n-module V, there is a simple way to recover the Hilbert series of V from the Frobenius series of V. Specifically, let f_λ be the dimension of the irreducible S_n-module M_λ. A well-known theorem [27] states that f_λ is the number of standard tableaux of shape λ, which is $n!$ divided by the product of the hook lengths of λ. It is immediate from the definitions that

$$H_V(q,t) = [\mathcal{F}_V(q,t)]|_{s_\lambda = f_\lambda},$$

where this notation indicates that we should replace every s_λ by the integer f_λ.

Similarly, we can use the Frobenius series to obtain the generating function for the occurrences of any particular irreducible S_n-module inside V. For instance, M_{1^n} is the irreducible submodule that affords the sign character of S_n. Thus, to find the generating function for the doubly graded submodule of V that carries the sign representation, we would look at $\mathcal{F}_V(q,t)|_{s_{1^n}}$, the coefficient of s_{1^n} in the Frobenius series.

1.3.5 Diagonal Harmonics

We now define the module of diagonal harmonics, which is a crucial example of a doubly graded S_n-module. This module, studied by Garsia and Haiman in [15], is the source of the bivariate q,t-Catalan sequence to be defined below.

Fix a positive integer n. Consider the polynomial ring

$$R_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

in two sets of n independent variables. The ring R_n is an infinite-dimensional vector space over $\mathbb{C}$ with a basis given by the set of all monomials. We make R_n into an

S_n-module as follows. Given $\sigma \in S_n$, define an action of σ on the variables in R_n by setting

$$\sigma \cdot x_i = x_{\sigma(i)} \text{ and } \sigma \cdot y_i = y_{\sigma(i)}.$$

Extend this action to all monomials by multiplicativity, and then extend to all polynomials by linearity. Thus, for $f \in R_n$, we have

$$\sigma \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

This is called the *diagonal action* of S_n on R_n, since σ permutes the indices of the x-variables and the y-variables in the same way.

Definition 1.34. Define the *diagonal harmonics* in R_n by

$$DH_n = \left\{ f \in R_n : \sum_{i=1}^{n} \frac{\partial^h}{\partial x_i^h} \frac{\partial^k}{\partial y_i^k} f = 0 \text{ for } 1 \leq h + k \leq n \right\}.$$

It is easy to see that DH_n is a vector subspace of R_n. Since all subscripts appear symmetrically in the definition, $f \in DH_n$ implies $\sigma \cdot f \in DH_n$ for any $\sigma \in S_n$. This shows that DH_n is an S_n-submodule of R_n. Furthermore, DH_n is a doubly graded module: we can write

$$DH_n = \bigoplus_{h \geq 0} \bigoplus_{k \geq 0} V_{h,k}(n),$$

where $V_{h,k}(n)$ is the submodule of DH_n consisting of zero and those polynomials f that are homogeneous of degree h in the x-variables and homogeneous of degree k in the y-variables.

We can now form the Frobenius series $\mathcal{F}_{DH_n}(q,t)$, the Hilbert series $H_{DH_n}(q,t)$, and the generating function for the sign character $\mathcal{F}_{DH_n}(q,t)|_{s_{1^n}}$, as discussed in the last subsection. For notational convenience, we will henceforth denote these three generating functions by $F_n(q,t)$, $H_n(q,t)$, and $RC_n(q,t)$, respectively.

To understand the representation theory of diagonal harmonics, we would like to have more explicit formulas for $F_n(q,t)$, $H_n(q,t)$, and $RC_n(q,t)$. As pointed out earlier, it is sufficient to find a formula for the Frobenius series. Garsia and Haiman conjectured such a formula involving the modified Macdonald polynomials [15]. The formula was proved much later by Haiman using advanced machinery from algebraic geometry. The next theorem gives this formula.

Theorem 1.35.

$$F_n(q,t) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu t^{n(\mu)} q^{n(\mu')} (1-t)(1-q) \Pi_\mu(q,t) B_\mu(q,t)}{h_\mu(q,t) h'_\mu(q,t)}.$$

Proof. See [18, 21]. □

Combining this result with Theorem 1.29, we have

$$F_n(q,t) = \nabla(s_{1^n}).$$

Example 1.36. If we know the matrix of ∇ relative to the Schur basis for Λ^n, we can find the Frobenius series by simply reading off the entries in the column corresponding to s_{1^n}. For instance, equation (1.10) implies that

$$F_2(q,t) = 1 s_{(2)} + (q+t) s_{(1,1)}.$$

Equation (1.12) implies that

$$F_3(q,t) = 1 s_{(3)} + (q + q^2 + t + t^2 + qt) s_{(2,1)} + (q^3 + q^2 t + t^3 + qt + qt^2) s_{(1,1,1)}.$$

The following theorem of Garsia and Haiman can be used to compute the specializations $F_n(q,1)$ and $F_n(q,1/q)$ of the Frobenius series.

Theorem 1.37.

(1) For a Dyck path D of order n, define $a_i(D)$ to be the number of north steps taken by the path along the line $x = i$. Then

$$\nabla(e_n)\big|_{t=1} = \sum_{D \in \mathcal{D}_n} q^{area(D)} \prod_{i=0}^{n-1} e_{a_i(D)},$$

where e_j denotes an elementary symmetric function, as usual.

(2)

$$q^{n(n-1)/2} \nabla(e_n)\Big|_{t=1/q} = \sum_{\mu \vdash n} s_\mu \frac{s_{\mu'}(1, q, q^2, \ldots, q^n)}{[n+1]_q}.$$

Proof. See Theorem 1.2 and Corollary 2.5 in [15]. □

Recall that the Hilbert series of DH_n is given by $H_n(q,t) = F_n(q,t)|_{s_\lambda = f_\lambda}$. Haiman's work also implies the following specializations of the Hilbert series.

Theorem 1.38.

$$H_n(1,1) = (n+1)^{n-1}$$

$$q^{n(n-1)/2} H_n(q, 1/q) = [n+1]_q^{n-1}.$$

Proof. See [18, 21]. □

Note that the first statement just says that $dim(DH_n) = (n+1)^{n-1}$. Even this seemingly simple fact is very difficult to prove.

Example 1.39. We have $f_2 = f_{(1,1)} = f_3 = f_{(1,1,1)} = 1$ and $f_{(2,1)} = 2$. Hence, from the previous example,

$$H_2(q,t) = 1 + q + t,$$

$$H_3(q,t) = 1 + 2q + 2q^2 + q^3 + 2t + 2t^2 + t^3 + qt^2 + q^2t + 3qt.$$

Note that $H_2(1,1) = 3 = (2+1)^{2-1}$ and $H_3(1,1) = 16 = 4^2$. Moreover,

$$qH_2(q, 1/q) = 1 + q + q^2 = [3]_q^1,$$

and it is easy to check that

$$q^3 H_3(q, 1/q) = (1 + q + q^2 + q^3)^2 = [4]_q^2.$$

Next, consider $RC_n(q,t) = F_n(q,t)|_{s_{1^n}}$, the generating function for occurrences of the sign character in DH_n. Before Theorem 1.35 was proved, Garsia and Haiman [15] were able to compute the coefficient of s_{1^n} in the conjectured character formula

$$\sum_{\mu \vdash n} \frac{\tilde{H}_\mu t^{n(\mu)} q^{n(\mu')} (1-t)(1-q) \Pi_\mu(q,t) B_\mu(q,t)}{h_\mu(q,t) h'_\mu(q,t)}.$$

In light of Theorem 1.29, this coefficient is just $\nabla(s_{1^n})|_{s_{1^n}}$, the entry in the lower-right corner of the matrix representing nabla relative to the Schur basis. This coefficient is the original version of the q,t-Catalan number, as defined by Garsia and Haiman in [15].

Definition 1.40. For $n \geq 1$, define the *original q,t-Catalan sequence* by

$$OC_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2n(\mu)} q^{2n(\mu')} (1-t)(1-q)\Pi_\mu(q,t) B_\mu(q,t)}{h_\mu(q,t) h'_\mu(q,t)}.$$

Theorem 1.41. *For all n,*

$$OC_n(q,t) = \nabla(s_{1^n})|_{s_{1^n}}.$$

Proof. See [15]. □

Of course, it is immediate from Haiman's Theorem 1.35 that $OC_n(q,t) = RC_n(q,t)$. However, since this equality is very difficult to prove, it is useful to maintain separate notation for the two expressions.

Garsia and Haiman also proved the following specializations of $OC_n(q,t)$, which explain why they called it the q,t-Catalan sequence.

Theorem 1.42. *For all n,*

$$OC_n(1,1) = \frac{1}{n+1}\binom{2n}{n} = C_n$$

$$q^{n(n-1)/2} OC_n(q, 1/q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q = C_n^{maj}(q)$$

$$OC_n(1,q) = OC_n(q,1) = \sum_{D \in \mathcal{D}_n} q^{area(D)} = C_n^{area}(q)$$

Proof. See [15]. □

This theorem shows that the two classical q-analogues of the Catalan numbers from §1.2 can be derived from $OC_n(q,t)$ by appropriate substitutions.

Example 1.43. We calculated the matrix of ∇ on the space Λ^3 in Example 1.28. Examining the lower-right entry of this matrix, we find that

$$\nabla(s_{(1,1,1)})|_{s_{(1,1,1)}} = q^3 + q^2 t + t^3 + qt + qt^2.$$

On the other hand, computing $OC_3(q,t)$ from the definition, we obtain a sum of three rational functions that correspond to the three partitions μ of 3:

$$\begin{aligned} OC_3(q,t) &= \frac{t^0q^6(1-t)(1-q)(1-q)(1-q^2)(1+q+q^2)}{(q^2-t)(1-q^3)(q-t)(1-q^2)(1-t)(1-q)} \\ &+\frac{t^2q^2(1-t)(1-q)(1-q)(1-t)(1+q+t)}{(q-t^2)(t-q^2)(1-t)(1-q)(1-t)(1-q)} \\ &+\frac{t^6q^0(1-t)(1-q)(1-t)(1-t^2)(1+t+t^2)}{(t^2-q)(1-t^3)(t-q)(1-t^2)(1-t)(1-q)} \end{aligned}$$

As promised by the theorem, this messy sum does simplify to $q^3+q^2t+t^3+qt+qt^2$. Moreover,

$$\begin{aligned} OC_3(q,1) = OC_3(1,q) = 1+2q+q^2+q^3 = C_3^{area}(q); \\ q^{\binom{3}{2}}OC_3(q,1/q) = 1+q^2+q^3+q^4+q^6 = C_3^{maj}(q). \end{aligned}$$

In light of this last result, it is natural to ask if there is a purely combinatorial interpretation for the bivariate sequence $OC_n(q,t)$. In other words, we would like to have a second statistic on Dyck paths, say $tstat$, such that

$$OC_n(q,t) = \sum_{D\in\mathcal{D}_n} q^{area(D)}t^{tstat(D)}.$$

Two such statistics were separately conjectured by Haglund and Haiman [16, 19]. Later, Garsia and Haglund proved that these conjectures really do give $OC_n(q,t)$ [14]. Their proof is discussed briefly in §1.4.5 below.

Similarly, we would like to have combinatorial interpretations for the Hilbert series $H_n(q,t)$ and the Frobenius series $F_n(q,t)$ by introducing suitable pairs of statistics on some collection of objects. Haglund, Haiman, and the present author conjectured such statistics for the Hilbert series (see [17] and §1.5 below). At this time, it is an open problem to prove that these conjectured statistics are correct. It is also an open problem to give a combinatorial interpretation (even a conjectural one) for the full Frobenius series.

1.4 Combinatorial q,t-Catalan Sequences

In this section, we describe two different combinatorial versions of the bivariate Catalan sequence. These sequences are based on two statistics proposed by Haglund and Haiman, respectively. The idea is to take the sequence $C_n^{area}(q)$ and add a new parameter to keep track of the new statistic. The new sequences have the form

$$\sum_{D\in\mathcal{D}_n} q^{area(D)} t^{tstat(D)},$$

where *tstat* is Haglund's statistic or Haiman's statistic.

After describing these two statistics, we give a bijection demonstrating the equivalence of the two sequences. Then we derive a fundamental recursion (first proved by Haglund) characterizing the sequence. We briefly indicate how Garsia and Haglund used this recursion to prove that the combinatorial sequence was equal to the original sequence $OC_n(q,t)$. Finally, we use the recursion to prove specialized formulas where t is replaced by $1/q$.

1.4.1 Haglund's Combinatorial Catalan Sequence

This subsection describes the statistic proposed by Haglund in [16]. Let E be a Dyck path of order n. We first define a *bounce path* derived from E as follows. The bounce path begins at (n,n) and moves to $(0,0)$ via an alternating sequence of horizontal and vertical moves. Starting at (n,n), the bounce path proceeds west until it reaches the north step of the Dyck path going from height $n-1$ to height n. From there, the bounce path goes south until it reaches the main diagonal line $y=x$. This process continues recursively. When the bounce path has reached the point (i,i) on the main diagonal $(i>0)$, the bounce path goes west until it is blocked by the north step of the Dyck path going from height $i-1$ to height i. From there, the bounce path goes south until it hits the main diagonal. The bounce path terminates when it reaches $(0,0)$. See Figure 1.4 for an example.

Suppose the bounce path derived from E hits the main diagonal at the points

$$(n,n),\quad (i_1,i_1),\quad (i_2,i_2),\quad \ldots,\quad (i_s,i_s),\quad (0,0).$$

The bounce statistic for E is defined by

$$bounce(E)=\sum_{k=1}^{s} i_k.$$

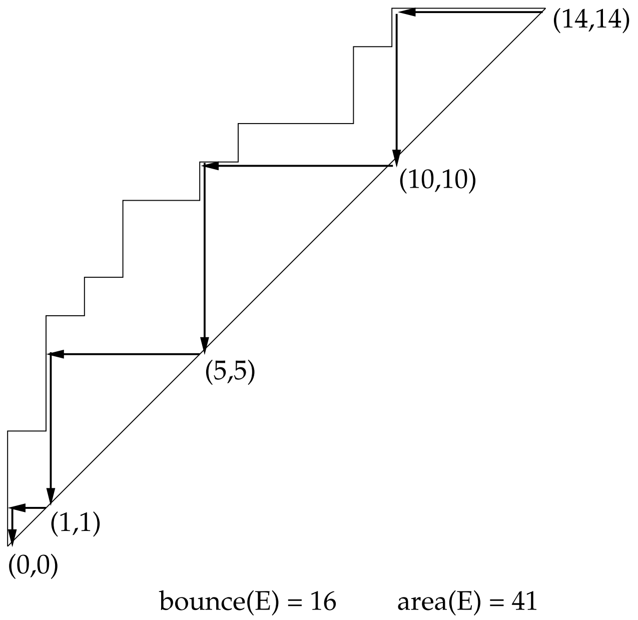


Figure 1.4: A Dyck path with its derived bounce path.

We also define the *bounce count* by

$$bcount(E) = s,$$

which is one less than the number of bounces. For example, in Figure 1.4, the bounce path for E hits the main diagonal at $(14,14)$, $(10,10)$, $(5,5)$, $(1,1)$, and $(0,0)$. Thus, $bounce(E) = 10+5+1 = 16$ and $bcount(E) = 3$ for this path.

Definition 1.44. We define *Haglund's combinatorial* q,t*-Catalan sequence* by

$$C_n(q,t) = \sum_{E\in\mathcal{D}_n} q^{area(E)} t^{bounce(E)} \text{ for } n \geq 1.$$

Example 1.45. Let $n = 3$, and consider the five paths in Figure 1.2. The statistics *area*, *bounce*, and *maj* for these five paths are given in Table 1.2. From this table, we read off

$$C_3(q,t) = t^3 + qt^2 + qt + q^2t + q^3 = OC_3(q,t).$$

Observe that $C_3(q,t) = C_3(t,q)$, and we have the specializations

$$C_3(q,1) = C_3(1,q) = 1 + 2q + q^2 + q^3 = C_3^{area}(q)$$

Table 1.2: Statistics for Dyck paths of order 3.

encoding of Dyck path D	$area(D)$	$bounce(D)$	$maj(D)$
010101	0	3	6
001101	1	2	4
010011	1	1	2
001011	2	1	3
000111	3	0	0

and

$$q^3 C_3(q, 1/q) = 1 + q^2 + q^3 + q^4 + q^6 = C_3^{maj}(q).$$

1.4.2 Haiman's Combinatorial Catalan Sequence

This subsection describes the statistic proposed by Haiman in [19]. Let D be a Dyck path of order n. We begin by giving an alternate method of describing the Dyck path D. For $0 \leq i < n$, define $\gamma_i(D)$ to be the number of complete cells strictly between the path and the main diagonal in the i^{th} row of the picture, where the bottom row is row zero. Define $\gamma(D)$ to be the vector $(\gamma_0(D), \ldots, \gamma_{n-1}(D))$. For example, for the path D shown in Figure 1.5, we have

$$\gamma(D) = (0, 1, 2, 2, 3, 0, 0, 1, 1, 2, 1, 2, 0, 1).$$

Since the total number of area cells is the sum of the area cells in each row, we have $area(D) = \sum_{i=0}^{n-1} \gamma_i(D)$.

Definition 1.46. We define Haiman's statistic $dinv$ by the formula

$$dinv(D) = \sum_{i<j} \left[\chi(\gamma_i(D) = \gamma_j(D)) + \chi(\gamma_i(D) = \gamma_j(D) + 1)\right]. \tag{1.13}$$

For example, we have $dinv(D) = 41$ for the path in Figure 1.5. Some of the 41 pairs of indices (i, j) that contribute to this count are:

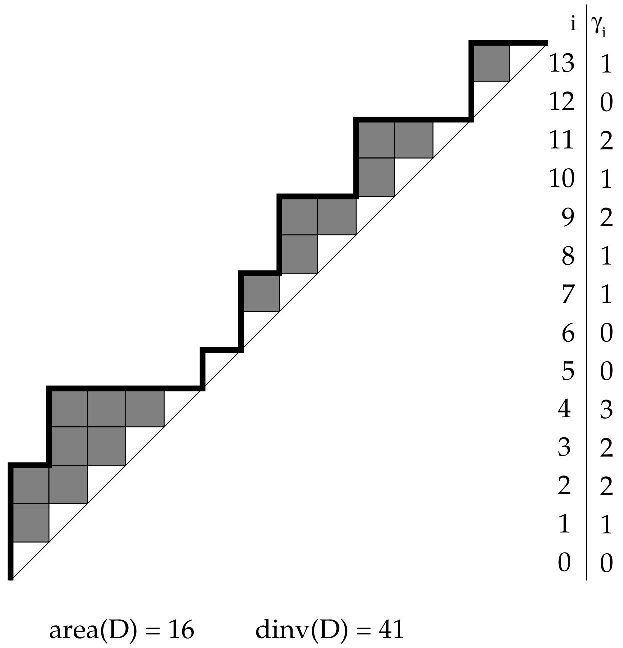


Figure 1.5: A Dyck path and the associated γ-vector.

- $i = 0$ and $j = 5$ (since $\gamma_0 = 0 = \gamma_5$)
- $i = 3$ and $j = 9$ (since $\gamma_3 = 2 = \gamma_9$)
- $i = 3$ and $j = 7$ (since $\gamma_3 = 2 = 1 + 1 = \gamma_7 + 1$)
- $i = 4$ and $j = 11$ (since $\gamma_4 = 3 = 2 + 1 = \gamma_{11} + 1$)

Note that $\gamma_7 = 1 = 0 + 1 = \gamma_6 + 1$. However, the pair $(i = 7, j = 6)$ does *not* contribute to the statistic because $i \not< j$.

Definition 1.47. We define *Haiman's combinatorial q,t-Catalan sequence* to be

$$HC_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{dinv(D)} t^{area(D)} \text{ for } n \geq 1.$$

Note that we use t, not q, to keep track of area in this sequence.

For later reference, we give the following characterization of the vectors $\gamma(D)$.

Theorem 1.48. *Let $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ be a vector of n integers. Then γ has the form $\gamma(D)$ for some Dyck path D of order n if and only if the following three properties hold:*

(P1) $\gamma_0 = 0$.

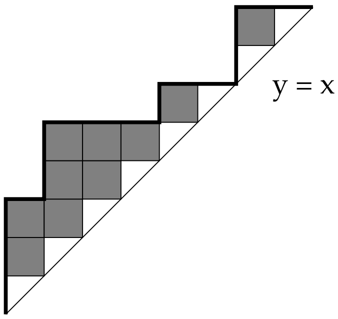
(P2) $\gamma_i \geq 0$ *for* $0 \leq i \leq n-1$.

(P3) $\gamma_{i+1} \leq \gamma_i + 1$ *for* $0 \leq i < n-1$.

Proof. Assume that $\gamma = \gamma(D)$ for some Dyck path D. In the lowest row (which is row zero), there is no room between the lines $x = 0$ and $y = x$ for any area cells. Hence, $\gamma_0(D) = 0$. The number of area cells in the i^{th} row must be nonnegative, so $\gamma_i(D) \geq 0$ for all i. Finally, given that there are $\gamma_i(D)$ area cells in row i, there can be at most $\gamma_i(D) + 1$ area cells in row $i+1$. For, to get more than $\gamma_i(D) + 1$ area cells in row $i+1$, the path D would need to take one or more steps west, which is not allowed. (Compare to Figure 1.6 below.)

Conversely, assume γ satisfies the three stated conditions. We construct D such that $\gamma = \gamma(D)$ as follows. Draw a picture by shading in γ_i complete lattice cells in row i, working from right to left from the diagonal line $y = x$. If $\gamma_i = 0$, shade in the vertical segment in row i closest to this diagonal. The westward and northward boundary of the collection of shaded cells and segments forms a path D. The path starts at $(0,0)$ with a north step, since $\gamma_0 = 0$. The path ends at (n,n), there are n rows in the picture. The path never goes right of the diagonal $y = x$, since $\gamma_i \geq 0$ for all i. Finally, the path takes only north and east steps (no west steps), since $\gamma_{i+1} \leq \gamma_i + 1$ and $\gamma_0 = 0$. Hence, the path D is a Dyck path of order n. It is clear from the construction that $\gamma(D) = \gamma$. ☐

For example, Figure 1.6 shows the paths constructed from the vectors $\gamma = (0,1,2,2,3,1,0,1)$ and $\delta = (0,3,1,2,1,1,0,2)$. The path for γ is a Dyck path, since γ satisfies the three conditions of the proposition. The path for δ is not a Dyck path — there are west steps at each row where condition (P3) fails for δ.



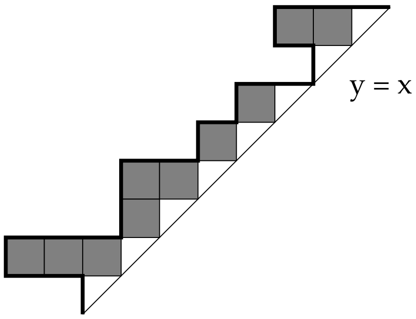


Figure 1.6: Paths constructed from two integer vectors.

1.4.3 Comparing the Two Combinatorial Sequences

In this subsection, we give a proof that $C_n(q,t) = HC_n(q,t)$. The proof constructs a bijection $\phi : \mathcal{D}_n \to \mathcal{D}_n$ with the property that

$$area(\phi(D)) = dinv(D) \text{ and } bounce(\phi(D)) = area(D).$$

It follows that

$$\sum_{D \in \mathcal{D}_n} q^{area(D)} t^{bounce(D)} = \sum_{D \in \mathcal{D}_n} q^{dinv(D)} t^{area(D)},$$

proving the equivalence of Haglund's sequence and Haiman's sequence.

The bijection ϕ (or its inverse ϕ^{-1}) appears to have been discovered independently by many authors in various forms, though not every author was aware of the connection to the q,t-Catalan sequence. In [2], Andrews, Krattenthaler, Orsina, and Papi give a bijection that is essentially equivalent to ϕ, although the statistics *dinv* and *bounce* are not mentioned. They show that the combinatorial generating function

$$\sum_{D \in \mathcal{D}_n} q^{bcount(D)} t^{area(D)}$$

is in fact equal to the generating function for *ad*-nilpotent ideals of a Borel subalgebra of $sl(n+1,\mathbb{C})$, enumerated by class of nilpotence and dimension. (This leads to the question of whether Haglund's statistic $bounce(D)$ has an interpretation in the Lie algebra setting.) The bijection ϕ also appears implicitly in the work of Iarrobino and Yameogo [22]. See also [16, 17]. Our discussion here is based on the description of ϕ in [17]. More general bijections containing this one as a special case will be given in later chapters.

Description of the map ϕ.
Fix a path $D \in \mathcal{D}_n$. We will construct a new path $E = \phi(D) \in \mathcal{D}_n$ such that $area(D) = bounce(E)$ and $dinv(D) = area(E)$.

Set $\gamma = \gamma(D) = (\gamma_0(D), \ldots, \gamma_{n-1}(D))$. From the last subsection, we know that $\gamma_i \geq 0$ for all i, $\gamma_0 = 0$, and $\gamma_{i+1} \leq \gamma_i + 1$ for all $i < n - 1$. Set $s = \max_{0 \leq i \leq n-1} \gamma_i$. For $0 \leq j \leq s$, let a_j be the number of occurrences of j in γ. Since the entries of γ begin at 0 and can increase by at most one, there can be no omitted values between 0 and s; in other words, $a_j > 0$ for all $j \leq s$. Moreover, $a_0 + \cdots + a_s = n$, since there are n symbols in γ.

To construct E, we first draw a bounce path B whose successive horizontal moves (starting from (n, n)) have lengths $a_0, \ldots, a_s$. See Figure 1.7 for an example of B. This bounce path, together with the main diagonal line $y = x$, creates a sequence of $s+1$ triangles which we shall call $T_0, \ldots, T_s$. For $1 \leq i \leq s$, there is an empty rectangular region R_i located north of triangle T_i and west of triangle T_{i-1}. Note that rectangle R_i has width a_i and height a_{i-1}.

The path E will have B as its derived bounce path. We now describe how to construct the portion of the path E located in rectangle R_i. Fix i between 1 and s, and let w_i be the word obtained from γ by deleting all symbols other than $i-1$ and i. Then w_i consists of a_{i-1} occurrences of $i - 1$ and a_i occurrences of i. By the conditions on γ, the first symbol in w_i must be $i-1$. This is clear if $i = 1$, since $\gamma_0 = 0$. For $i > 1$, recall that the entries of γ increase by at most one when read left to right. Thus, since $\gamma_0 < i$, the first occurrence of the symbol i in γ must be immediately preceded by the symbol $i - 1$.

Read the symbols in w_i from left to right. Starting at the northwest tip of triangle T_{i-1} and proceeding southwest, draw a horizontal step when the symbol i is read; draw a vertical step when the symbol $i-1$ is read. The first symbol in w_i is $i-1$, so this partial path must begin with a vertical step.

After filling all the rectangular regions in this way, we obtain the Dyck path

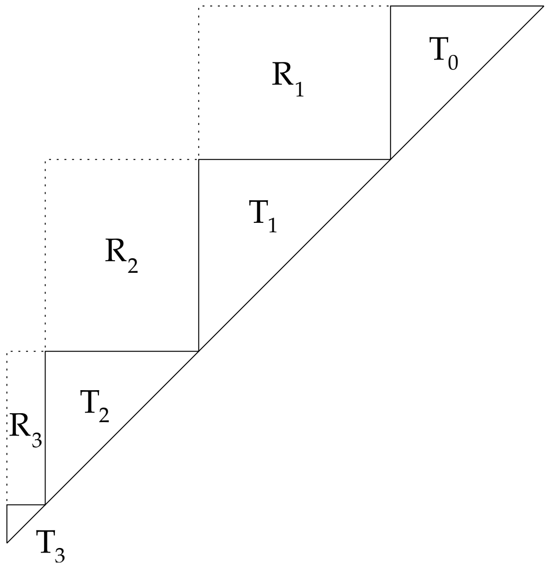


Figure 1.7: A bounce path, with associated triangles and rectangles.

E. Observe that, because the paths within each R_i have a vertical step at the northeast corner, B will be the bounce path derived from E. Note that B touches the main diagonal at the points

$$(n,n),\ (n-a_0,n-a_0),\ (n-a_0-a_1,n-a_0-a_1),\ \ldots,(0,0),$$

by definition of the triangles T_i. Therefore,

$$\begin{aligned}
bounce(E) &= (n-a_0)+(n-a_0-a_1)+\cdots+(n-a_0-a_1-\cdots-a_s) \\
&= n(s+1)-(s+1)a_0-sa_1-\cdots-(s+1-j)a_j-\cdots-1a_s \\
&= (s+1)(n-a_0-\cdots-a_s)+\sum_{j=0}^{s} ja_j \\
&= \sum_{j=0}^{s} ja_j = \sum_{i=0}^{n-1} \gamma_i \quad \text{(since } \gamma \text{ has } a_j \text{ copies of } j) \\
&= area(D).
\end{aligned}$$

Recall from (1.13) that

$$dinv(D) = \sum_{i<j} \left[\chi(\gamma_i(D)=\gamma_j(D)) + \chi(\gamma_i(D)=\gamma_j(D)+1)\right].$$

Consider the contribution to $dinv(D)$ arising from all nonzero summands of the form

$$\chi(\gamma_i(D) = \gamma_j(D)).$$

For each k with $0 \le k \le s$, there are a_k symbols in γ that are equal to k. Hence, there are $\binom{a_k}{2}$ pairs of indices $i < j$ with $\gamma_i = k = \gamma_j$. Thus, we get a total contribution of $\sum_{k=0}^{s} \binom{a_k}{2}$.

Next, consider the contribution to $dinv(D)$ arising from the nonzero summands of the form

$$\chi(\gamma_i(D) = \gamma_j(D) + 1).$$

For each k with $1 \le k \le s$, we can find all the pairs $i < j$ such that $\gamma_j(D) = k - 1$ and $\gamma_i(D) = \gamma_j(D) + 1 = k$ by looking at the subword w_k. This subword contains all occurrences of $k-1$ and k, in the same order that they occurred in γ. A pair $i < j$ with $\gamma_i(D) = k$ and $\gamma_j(D) = k-1$ corresponds exactly to an *inversion* between the same two symbols in the subword w_k. (See equation (1.1).) All the inversions of w_k arise in this way. Hence, adding over all k, the total contribution is $\sum_{k=1}^{s} inv(w_k)$.

To summarize this discussion, we have found that

$$dinv(D) = \sum_{k=0}^{s} \binom{a_k}{2} + \sum_{k=1}^{s} inv(w_k).$$

Now $\binom{a_k}{2} = \frac{a_k(a_k-1)}{2}$ is the number of complete lattice cells in the triangle T_k. It is easy to see that $inv(w_k)$ is the number of lattice cells beneath the path E in the rectangle R_k; the proof is just like that of Lemma 1.7. We conclude that $dinv(D) = area(E)$.

Example 1.49. Let D be the path shown in Figure 1.5. Recall that $area(D) = 16$ and $dinv(D) = 41$. For this path, $n = 14$ and

$$\gamma = (0, 1, 2, 2, 3, 0, 0, 1, 1, 2, 1, 2, 0, 1).$$

In this vector, there are four zeroes, five ones, four twos, and one three. We therefore set $s = 3$, $a_0 = 4$, $a_1 = 5$, $a_2 = 4$, and $a_3 = 1$. Next, we draw a bounce path B with successive bounce lengths 4, 5, 4, and 1. This is exactly the bounce path shown in Figure 1.7. Letting $i = 1, 2, 3$, we obtain the following subwords of γ:

$$w_1 = 010011101; \quad w_2 = 122112121; \quad w_3 = 22322.$$

Accordingly, we fill in rectangle R_1, from northeast to southwest, with steps

$$VHVVHHHVH$$

(where V means vertical step and H means horizontal step). We fill rectangle R_2 with the subpath $VHHVVHVHV$. We fill rectangle R_3 with the subpath $VVHVV$. The resulting path E is the path shown in Figure 1.4. This path is redrawn in Figure 1.8 with the area cells shaded. We have $bounce(E) = 16 = area(D)$ and $area(E) = 41 = dinv(D)$, as required.

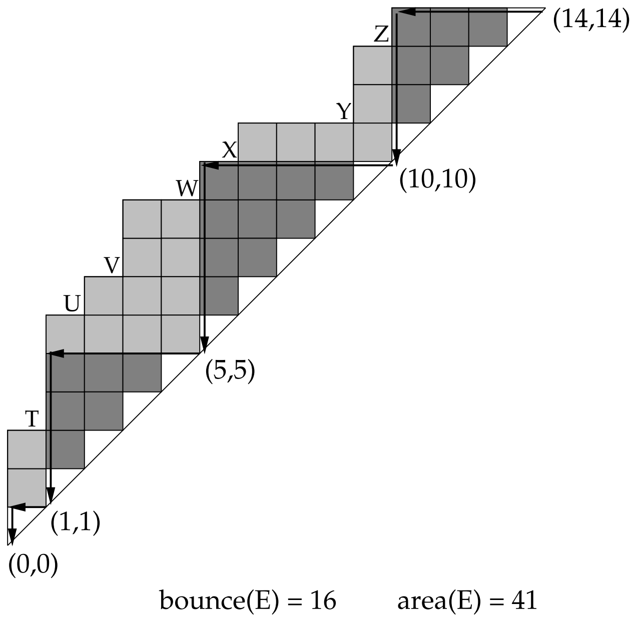


Figure 1.8: The image of D under the bijection ϕ.

Note that $inv(w_1) = 6$; accordingly, there are 6 area cells in rectangle R_1. These six cells correspond to the following six pairs $i < j$ such that $\gamma_j(D) = 0$ and $\gamma_i(D) = 1 = \gamma_j(D) + 1$:

$$(i, j) = (1, 5),\ (1, 6),\ (1, 12),\ (7, 12),\ (8, 12),\ (10, 12).$$

Similarly, note that triangle T_2 has six area cells, where $6 = \binom{4}{2}$. These six cells correspond to the following six pairs $i < j$ such that $\gamma_i(D) = 2 = \gamma_j(D)$:

$$(i, j) = (2, 3),\ (2, 9),\ (2, 11),\ (3, 9),\ (3, 11),\ (9, 11).$$

Before describing the inverse of ϕ, we record one further property of the bijection that will be used later. First, we make the following definitions. For any word $v = v_1 v_2 \cdots v_n$, an *ascent of* v is an index $i < n$ such that $v_i < v_{i+1}$. For any lattice path E, an *inner corner of* E is a point (x, y) such that the horizontal segment from $(x-1, y)$ to (x, y) and the vertical segment from (x, y) to $(x, y+1)$ are both steps of E.

Given a Dyck path D, let $\gamma = \gamma(D)$ and $E = \phi(D)$. The new property of ϕ is that the ascents of γ correspond bijectively to the inner corners of E. To prove this, first note that the ascents of γ correspond bijectively to the ascents of the subwords w_i. This follows from the condition $\gamma_j \leq \gamma_{j+1} + 1$, which says that any ascent in γ must involve the adjacent symbols $i-1$ followed by i. Such an ascent will also be an ascent of the subword w_i, and conversely. When drawing E according to the definition of ϕ, an ascent in w_i translates into a vertical step followed by a horizontal step in rectangle R_i (following the path from northeast to southwest). This is exactly the same as an inner corner of E.

For example, in Figure 1.8, the inner corners of E are marked by capital letters. The inner corner marked T corresponds to the single ascent at position 2 in $w_3 = 22322$, which in turn corresponds to the ascent at position 3 in γ between $\gamma_3 = 2$ and $\gamma_4 = 3$. The inner corner marked Y corresponds to the ascent at position 4 in $w_1 = 010011101$, which in turn corresponds to the ascent at position 6 in γ between $\gamma_6 = 0$ and $\gamma_7 = 1$.

Description of ϕ^{-1}. To show that ϕ is a bijection, we describe the inverse map. Given a Dyck path E, we need to reconstruct the path $D = \phi^{-1}(E)$ such that $\phi(D) = E$. Here is an algorithm to find D:

(1) Draw the bounce path B derived from E. Let $a_0, \ldots, a_s$ be the lengths of the successive horizontal moves in this bounce path.

(2) Let $R_1, \ldots, R_s$ denote the rectangular regions above the bounce path, as described earlier. For $1 \leq k \leq s$, traverse the subpath of E going from the northeast corner of R_k to the southwest corner of R_k. Construct the word w_k by writing the symbol k for each horizontal step and writing the symbol $k-1$ for each vertical step. By definition of the bounce path, the subpath in rectangle R_k must start with a vertical step. Hence, every constructed word w_k starts with $k-1$.

(3) Build up an integer vector γ satisfying properties (P1)—(P3) from Theorem 1.48 as follows. Initially, γ is a string of a_0 zeroes. For $k = 1, 2, \ldots, s$, insert a_k copies of k into the current string γ according to w_k. More explicitly, read w_k from left to right. When a $k-1$ is encountered in w_k, scan γ from left to right for the next occurrence of $k-1$. When a k is encountered in w_k, place a k in the gap immediately to the right of the symbol currently being scanned in γ. Continue scanning w_k and γ until all copies of k have been inserted. It is clear that this is the only way to insert the k's into the current γ vector such that (P3) holds and such that the subword of γ consisting of the k's and $(k-1)$'s is precisely w_k.

(4) Use γ to draw a Dyck path D with $\gamma(D) = \gamma$, as discussed earlier. We have $D = \phi^{-1}(E)$.

Example 1.50. Consider the path E shown in Figure 1.8. To compute $\phi^{-1}(E)$, we first draw the bounce path for E (also shown in the figure). We have $s = 3$, $a_0 = 4$, $a_1 = 5$, $a_2 = 4$, and $a_3 = 1$. By decoding the subpaths in each rectangle R_i, we recover the words

$$w_1 = 010011101; \quad w_2 = 122112121; \quad w_3 = 22322$$

from the previous example. Let us use these words to build up γ in stages.

- Initially, $\gamma = 0000$ since $a_0 = 4$.
- Using word w_1 to insert five ones, we get $\gamma = 010011101$.
- Using word w_2 to insert 2's in gaps following the 1's, we get $\gamma = 0122001121201$.
- Using word w_3 to insert 3's in gaps following the 2's, we get $\gamma = 01223001121201$.

Finally, we use the completed γ vector to recover the Dyck path D from Figure 1.5.

1.4.4 Lattice Paths and q-Binomial Coefficients

In the next subsection, we are going to prove a recursion due to Haglund [16] characterizing the combinatorial sequence $C_n(q,t)$. The proof of the recursion requires a well-known combinatorial interpretation of the q-binomial coefficient $\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q$, which we discuss in this subsection.

Let $\mathcal{P}_{a,b}$ denote the set of lattice paths that fit inside a rectangle of height a and width b. In the proof of the next theorem, we assume the paths go from the northwest corner to the southeast corner of the rectangle. Of course, analogous results will hold for the set of paths going from the northeast corner to the southwest corner. For $P \in \mathcal{P}_{a,b}$, let $area(P)$ be the number of cells in the rectangle below the path P. Let $area'(P)$ be the number of cells in the rectangle above the path P. We have the following basic properties of the q-binomial coefficients.

Theorem 1.51. *Let C, D be nonnegative integers.*

(1) $\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q = \begin{bmatrix} C+D \\ D,C \end{bmatrix}_q$.

(2) $\begin{bmatrix} C \\ C,0 \end{bmatrix}_q = 1$.

(3) We have the recurrence

$$\begin{bmatrix} C+D+1 \\ C,D+1 \end{bmatrix}_q = \begin{bmatrix} C+D \\ C,D \end{bmatrix}_q + q^{D+1} \begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q.$$

(4) We have the recurrence

$$\begin{bmatrix} C+D+1 \\ C,D+1 \end{bmatrix}_q = q^C \begin{bmatrix} C+D \\ C,D \end{bmatrix}_q + \begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q.$$

(5) The q-binomial coefficients are uniquely determined by conditions (1), (2), (3). More specifically, if g is a function of two nonnegative integers satisfying

$$\begin{aligned} g(C,D) &= g(D,C) \\ g(C,0) &= 1 \\ g(C,D+1) &= g(C,D) + q^{D+1} g(C-1,D+1), \end{aligned}$$

then $g(C,D) = \begin{bmatrix} C+D \\ C,D \end{bmatrix}_q$ for all C, D.

(6) We have

$$\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q = \sum_{P \in \mathcal{P}_{C,D}} q^{area(P)} = \sum_{P \in \mathcal{P}_{C,D}} q^{area'(P)}.$$

Proof. Recall from (1.3) that

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \frac{\prod_{i=1}^{a+b}(1-q^i)}{\prod_{i=1}^{a}(1-q^i)\prod_{i=1}^{b}(1-q^i)}. \tag{1.14}$$

Items (1) and (2) are immediate from this definition. Similarly, (3) and (4) follow from an easy manipulation of the definition. For example, (3) is proved by calculating

$$\begin{aligned} \begin{bmatrix} C+D \\ C,D \end{bmatrix}_q + q^{D+1}\begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q &= \frac{\prod_{i=1}^{C+D}(1-q^i)}{\prod_{i=1}^{C}(1-q^i)\prod_{i=1}^{D}(1-q^i)} \\ &\quad + q^{D+1}\frac{\prod_{i=1}^{C+D}(1-q^i)}{\prod_{i=1}^{C-1}(1-q^i)\prod_{i=1}^{D+1}(1-q^i)} \\ &= \frac{[1-q^{D+1}+q^{D+1}(1-q^C)]\prod_{i=1}^{C+D}(1-q^i)}{\prod_{i=1}^{C}(1-q^i)\prod_{i=1}^{D+1}(1-q^i)} \\ &= \frac{\prod_{i=1}^{C+D+1}(1-q^i)}{\prod_{i=1}^{C}(1-q^i)\prod_{i=1}^{D+1}(1-q^i)} \\ &= \begin{bmatrix} C+D+1 \\ C,D+1 \end{bmatrix}_q. \end{aligned}$$

The proof of (4) is similar; it uses the fact that

$$q^C(1-q^{D+1}) + (1-q^C) = 1-q^{C+D+1}.$$

Statement (5) follows by induction on $C+D$, since $g(C,D)$ and the q-binomial coefficient $\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q$ satisfy the same recurrence (3) and initial conditions (1) and (2).

To prove (6), we invoke (5) with $g(C,D) = \sum_{P\in\mathcal{P}_{C,D}} q^{area'}(P)$. (The proof for *area* is similar.) The relation $g(C,D) = g(D,C)$ follows since we can rotate a path inside a $C \times D$ rectangle to obtain a path inside a $D \times C$ rectangle with the same number of cells above it. The relation $g(C,0) = g(0,C) = 1$ is immediate from the definition. We now give combinatorial arguments to prove that recurrences (3) and (4) hold for g.

Recall that the power of q in the generating function $g(C,D+1)$ records the area above a path that goes from northwest to southeast in a rectangle of height C and width $D+1$. See Figure 1.9. Let us classify such paths by their *initial* step at the northwest corner. If this step is horizontal, the remainder of the path lies in a rectangle of height C and width D, giving the term $g(C,D)$. If this step is vertical, the remainder of the path lies in a rectangle of height $C-1$ and width $D+1$, giving the term $g(C-1,D+1)$.

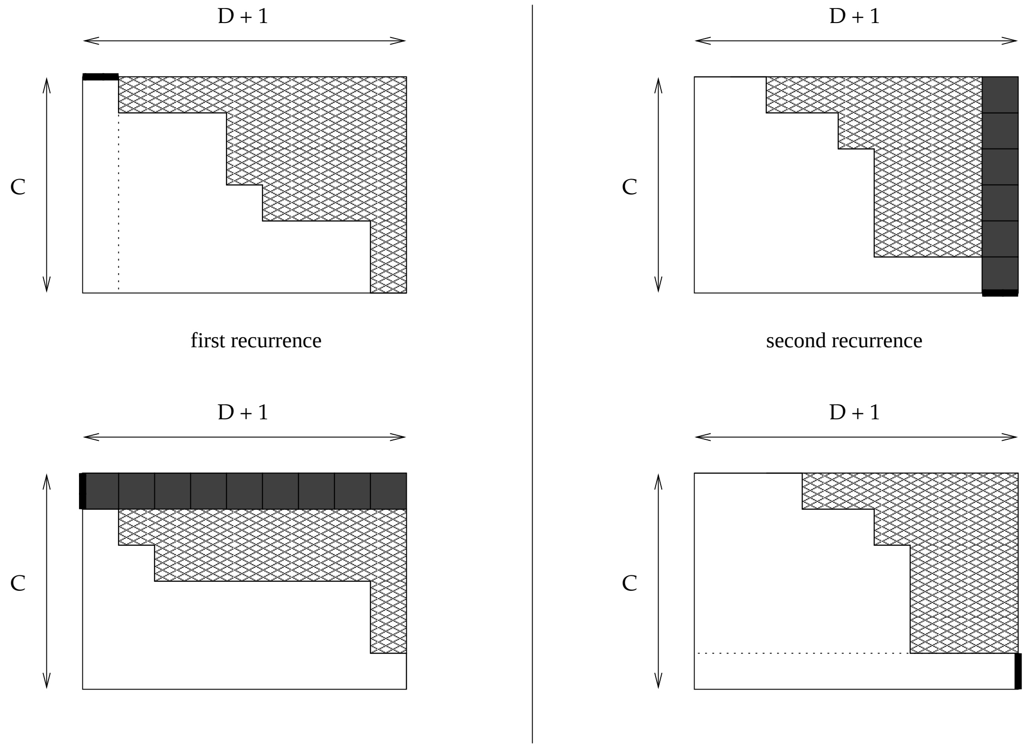


Figure 1.9: Picture used to prove Theorem 1.51(6).

However, we must also multiply by q^{D+1} to account for the $D+1$ area cells in the top row of the original rectangle. This proves that $g(C, D+1) = g(C, D) + q^{D+1}g(C-1, D+1)$.

To get the other recurrence, let us classify the paths by their *final* step at the southeast corner. If this step is horizontal, the remainder of the path lies in a rectangle of height C and width D, giving the term $g(C, D)$. However, we must also multiply by q^C to account for the C area cells in the rightmost column of the original rectangle. If the final step is vertical, the remainder of the path lies in a rectangle of height $C-1$ and width $D+1$, giving the term $g(C-1, D+1)$. This proves that $g(C, D+1) = q^C g(C, D) + g(C-1, D+1)$. □

1.4.5 Haglund's Recursion

We now state and prove Haglund's recursion, which characterizes the combinatorial q,t-Catalan sequence. Fix a positive integer n. Let $\mathcal{F}_{n,s}$ denote the set of Dyck paths of height n that terminate in exactly s horizontal steps. For such a path, the length of the first horizontal bounce move will be s. See Figure 1.11 below. Define

$$F_{n,s}(q,t) = \sum_{D \in \mathcal{F}_{n,s}} q^{area(D)} t^{bounce(D)}.$$

These generating functions are related to $C_n(q,t)$ by the identities

$$C_n(q,t) = \sum_{s=1}^{n} F_{n,s}(q,t)$$

and

$$t^n C_n(q,t) = F_{n+1,1}(q,t).$$

The first identity follows by classifying Dyck paths of height n by the number s of east steps in the topmost row. To prove the second identity, augment the diagram of a Dyck path of height n by adding a new top row with no area cells. The result is a Dyck path of height $n+1$ terminating in one east step preceded by one north step. All elements of $\mathcal{F}_{n+1,1}$ arise uniquely in this way. The bounce path derived from this augmented Dyck path starts with a bounce of size 1 contributing n to the bounce statistic, and afterwards bounces in the same way that the original bounce path did. See Figure 1.10, and compare to Figure 1.4.

Theorem 1.52. *The generating functions $F_{n,s}$ satisfy the recursion*

$$F_{n,s}(q,t) = t^{n-s} q^{s(s-1)/2} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q F_{n-s,r}(q,t) \text{ for } 1 \leq s < n \tag{1.15}$$

with initial condition $F_{n,n}(q,t) = q^{n(n-1)/2}$.

Remark 1.53. Note that the initial condition and recursion uniquely determine the polynomials $F_{n,s}(q,t)$ and allow these polynomials to be computed rapidly.

Proof. Consider the initial condition first. If $D \in \mathcal{F}_{n,n}$, then D is a Dyck path of height n terminating in exactly n east steps in the top row. This can only happen if D is the path consisting of n north steps followed by n east steps. Then $area(D) = n(n-1)/2$, since $\gamma(D) = (0, 1, \ldots, n-1)$; and $bounce(D) = 0$, since the only bounce hits the diagonal at $(0,0)$. So, $F_{n,n}(q,t) = q^{n(n-1)/2} t^0$ as claimed.

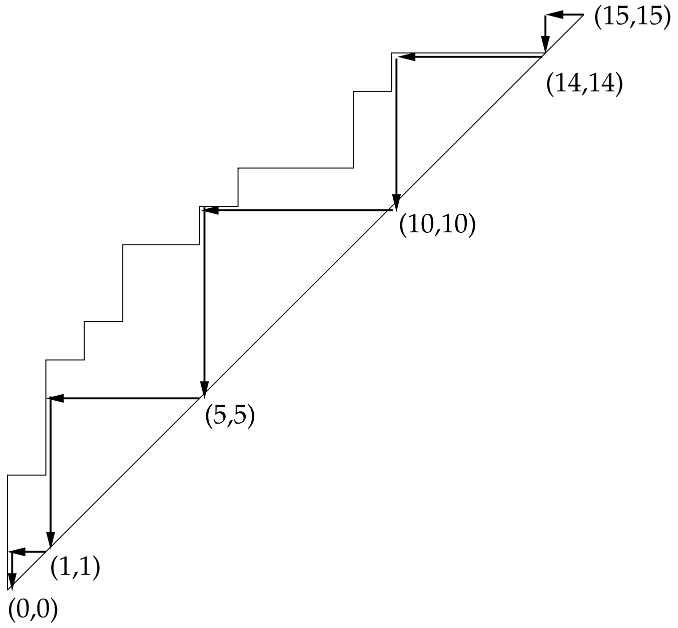


Figure 1.10: Adding an empty top row to a Dyck path.

The recursion for $F_{n,s}$ follows by "removing the first bounce" from a Dyck path to obtain a smaller Dyck path of height $n-s$. More precisely, let $D \in \mathcal{F}_{n,s}$. Then D ends in s east steps, so the derived bounce path starts with a bounce of size s ending at $(n-s, n-s)$. See Figure 1.11. If we ignore the top s rows of the figure, we see a smaller Dyck path D' of height $n-s$. Observe that the derived bounce path of D' is just the bounce path of D with the first bounce removed.

We can uniquely construct a path $D \in \mathcal{F}_{n,s}$ as follows. Choose a number $r \in \{1, 2, \ldots, n-s\}$. Given r, build D by making a sequence of choices. First, choose a path $D' \in \mathcal{F}_{n-s,r}$. The generating function for this choice is $F_{n-s,r}(q,t)$. Second, draw a vertical and horizontal segment to create a triangle with vertices $(n-s, n-s)$ and $(n-s, n)$ and (n, n). This triangle adds $s(s-1)/2$ area cells to the path being constructed, giving a factor $q^{s(s-1)/2}$. Also, the path D will have a new bounce going from (n, n) to $(n-s, n-s)$, so we get a contribution of t^{n-s} as well. Third, draw a subpath ending with a north step in the rectangular region above the top row of D' and

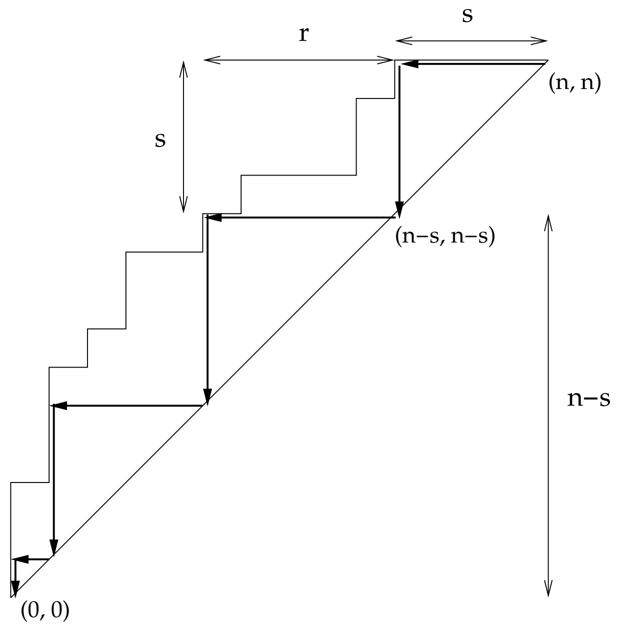


Figure 1.11: Proving the recursion by removing the first bounce.

left of the triangle just drawn. Equivalently, draw an *arbitrary* path in the rectangle of width r and height $s-1$ above the top row of D', and append one north step at the end. This subpath does not change the bounce statistic (since it ends in a north step), but the area increases by the number of cells beneath the subpath in its rectangle. The generating function for this choice is thus $\begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q$. The recursion follows immediately from the sum and product rules for generating functionons ([3], Ch. 10). $\square$

In the next two subsections, we give two applications of this recursion.

1.4.6 Comparison of $C_n(q,t)$ and $OC_n(q,t)$

In [14], Garsia and Haglund used the recursion (1.15) to prove that $C_n(q,t) = OC_n(q,t)$ for all n. More specifically, they defined

$$Q_{n,s}(q,t) = t^{n-s} q^{s(s-1)/2} \nabla(e_{n-s}[X(1+q+\cdots+q^{s-1})])\Big|_{s_{1^{n-s}}}.$$

Here, X is a formal infinite alphabet $X = x_1 + x_2 + \cdots$, and the square brackets denote plethystic substitution; in particular $e_n[X] = e_n$. To write this formula without plethystic notation, invoke Theorem 1.20 to obtain a unique K-algebra homomorphism ϕ of $\Lambda(K)$ extending the function ϕ_0 such that

$$\phi_0(p_k) = (1 + q^k + q^{2k} + \cdots + q^{(s-1)k})p_k.$$

Then

$$Q_{n,s}(q,t) = t^{n-s}q^{s(s-1)/2}\nabla(\phi(e_{n-s}))\Big|_{s_{1^{n-s}}}.$$

Garsia and Haglund showed that

$$Q_{n,s}(q,t) = t^{n-s}q^{s(s-1)/2}\sum_{r=1}^{n-s}\begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q Q_{n-s,r}(q,t) \text{ and } Q_{n,n}(q,t) = q^{n(n-1)/2}.$$

In other words, $Q_{n,s}$ satisfies the same recursion and initial condition that $F_{n,s}$ does. By uniqueness, $Q_{n,s}(q,t) = F_{n,s}(q,t)$ for all n and s. In particular,

$$C_n(q,t) = F_{n+1,1}(q,t)/t^n = Q_{n+1,1}(q,t)/t^n = \nabla(e_n)|_{s_{1^n}} = OC_n(q,t).$$

1.4.7 The Specializations $F_{n,s}(q,1/q)$ and $C_n(q,1/q)$

In [16], Haglund proved the formula

$$q^{n(n-1)/2}F_{n,s}(q,1/q) = \frac{[s]_q}{[n]_q}\begin{bmatrix} 2n-s-1 \\ n-s, n-1 \end{bmatrix}_q q^{(s-1)n}.$$

Using this formula, Haglund computed $q^{\binom{n}{2}}C_n(q,1/q)$ and showed that this specialization is equal to $C_n^{maj}(q)$.

In this subsection, we will prove the formula

$$F_{n,s}(q,1/q) = q^{-(n^2+n)/2+ns}\left(\begin{bmatrix} 2n-s-1 \\ n-s, n-1 \end{bmatrix}_q - q^s\begin{bmatrix} 2n-s-1 \\ n-s-1, n \end{bmatrix}_q\right). \tag{1.16}$$

Routine algebraic manipulations confirm that this formula is equivalent to Haglund's formula. Our proof is similar to Haglund's, but is more combinatorial. The basic idea of both proofs is to show that the given formulas satisfy the same initial condition and recursion that $F_{n,s}(q,1/q)$ does. Haglund did this in [16] using q-series identities. By rewriting Haglund's formula as a difference of q-binomial coefficients, as done in (1.16), we will be able to show the same thing using two simple combinatorial lemmas.

Lemma 1.54. *For all nonnegative integers C, D, and E,*

$$\sum_{i=0}^{D-E} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \begin{bmatrix} D-i \\ E, D-i-E \end{bmatrix}_q q^{(E+1)i} = \begin{bmatrix} C+D+1 \\ D-E, C+1+E \end{bmatrix}_q . \tag{1.17}$$

Proof. Recall from Theorem 1.51(6) that

$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_q = \sum_{P \in \mathcal{P}_{a,b}} q^{area(P)} = \sum_{P \in \mathcal{P}_{a,b}} q^{area'(P)},$$

where $\mathcal{P}_{a,b}$ is the set of lattice paths contained in a rectangle of height a and width b. Using this fact, we can prove (1.17) by drawing a picture. See Figure 1.12. (In this figure, the paths go from the northwest corner to the southeast corner of the rectangle. As usual, we could also consider paths going from the northeast corner to the southwest corner.)

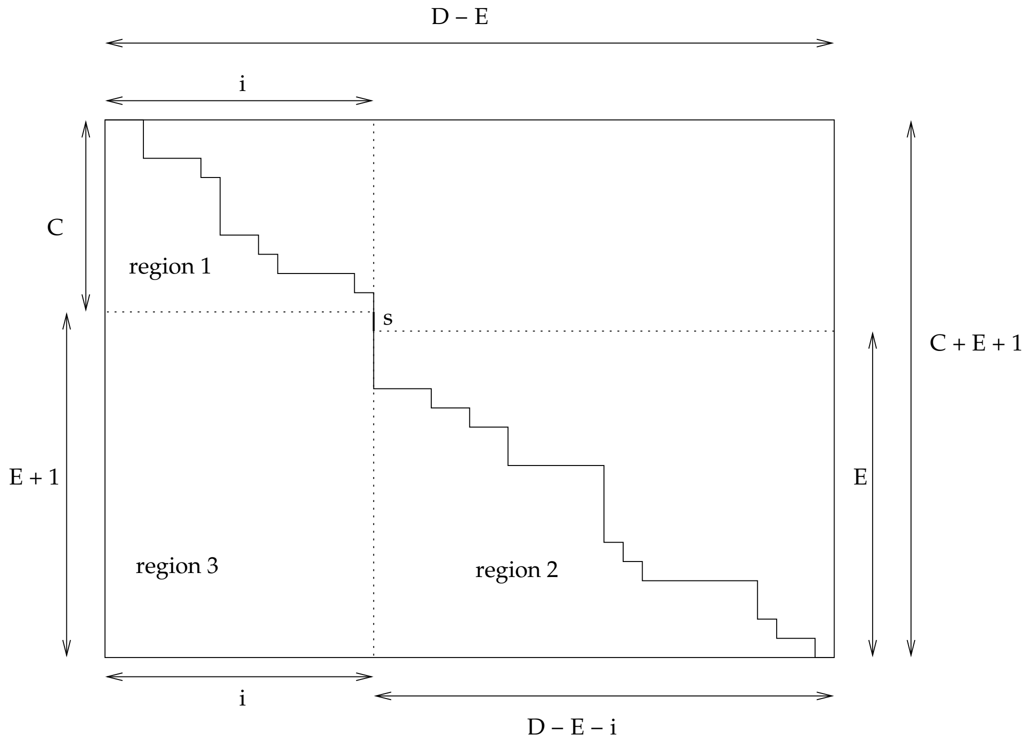


Figure 1.12: Picture used to prove (1.17).

We classify paths P contained in a rectangle of height $C+E+1$ and width $D-E$ based on what happens in row $(C+1)$ from the top. This row contains exactly one vertical step s of P; let i denote the distance of this vertical step from the left edge. Evidently, $0 \le i \le D - E$. Given i, we can uniquely construct such a path P as follows. First, choose a subpath P_1 in the rectangle R_1 northwest of s, which has height C and width i. Second, choose a subpath P_2 in the rectangle R_2 southeast of s, which has height E and width $D - E - i$. Then P is the concatenation of P_1 and the vertical step s and P_2.

Assume that the power of q records the area below the path P. This area is the sum of the area below P_1 inside R_1, the area below P_2 inside R_2, and the full area of the southwest rectangle of height $E+1$ and width i. These three pieces of the area are accounted for by the factors $\begin{bmatrix} C+i \\ C,i \end{bmatrix}_q$, $\begin{bmatrix} D-i \\ E, D-i-E \end{bmatrix}_q$, and $q^{(E+1)i}$, respectively. Adding over all choices of i, we immediately obtain (1.17). ☐

Lemma 1.55. *For nonnegative integers C and D,*

$$q^{-C}\left(\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q - \begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q\right) = \begin{bmatrix} C+D \\ C,D \end{bmatrix}_q - q^{D-C+1}\begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q. \quad (1.18)$$

Proof. This identity is equivalent to the relation

$$\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q + q^{D+1}\begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q = q^{C}\begin{bmatrix} C+D \\ C,D \end{bmatrix}_q + \begin{bmatrix} C+D \\ C-1,D+1 \end{bmatrix}_q, \quad (1.19)$$

which follows immediately from Theorem 1.51, since both sides are equal to $\begin{bmatrix} C+D+1 \\ C,D+1 \end{bmatrix}_q$. ☐

We are now ready to prove our formula for the specialization of $F_{n,s}$ at $t = 1/q$.

Theorem 1.56. *For all $n \ge 1$ and $1 \le s \le n$,*

$$F_{n,s}(q,1/q) = q^{-(n^2+n)/2+ns}\left(\begin{bmatrix} 2n-s-1 \\ n-s,n-1 \end{bmatrix}_q - q^{s}\begin{bmatrix} 2n-s-1 \\ n-s-1,n \end{bmatrix}_q\right). \quad (1.20)$$

Proof. Setting $t = 1/q$ in Theorem 1.52, we see that $F_{n,s}(q,1/q)$ is a solution to the recurrence

$$F_{n,s}(q,1/q) = q^{s(s-1)/2+s-n}\sum_{r=1}^{n-s}\begin{bmatrix} r+s-1 \\ r,s-1 \end{bmatrix}_q F_{n-s,r}(q,1/q) \text{ for } 1 \le s < n$$

with initial condition

$$F_{n,n}(q,1/q) = q^{n(n-1)/2}.$$

This recurrence and initial condition *uniquely* determine the quantities $F_{n,s}(q,1/q)$ for $n \geq 1$ and $1 \leq s \leq n$. Hence, it suffices to show that the claimed formulas for $F_{n,s}(q,1/q)$ satisfy the same recurrence and initial condition.

If $s = n$, the right side of (1.20) is

$$q^{-(n^2+n)/2+n^2} \cdot (1-0) = q^{(n^2-n)/2} = q^{n(n-1)/2},$$

so the initial condition is correct. Assume that $1 \leq s < n$ now. We must confirm that the expression

$$q^{-(n^2+n)/2+ns}\left(\begin{bmatrix} 2n-s-1 \\ n-s, n-1 \end{bmatrix}_q - q^s \begin{bmatrix} 2n-s-1 \\ n-s-1, n \end{bmatrix}_q\right)$$

is equal to the expression

$$\begin{gathered} q^{s(s-1)/2+s-n} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q q^{-((n-s)^2+(n-s))/2+(n-s)r} \times \\ \left(\begin{bmatrix} 2n-2s-r-1 \\ n-s-r, n-s-1 \end{bmatrix}_q - q^r \begin{bmatrix} 2n-2s-r-1 \\ n-s-r-1, n-s \end{bmatrix}_q\right) \end{gathered} \tag{1.21}$$

obtained by replacing $F_{n-s,r}(q,1/q)$ by the formula given in (1.20).

The second expression decomposes naturally into two pieces, namely

$$q^{s(s-1)/2+s-n} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q q^{-((n-s)^2+(n-s))/2+(n-s)r} \begin{bmatrix} 2n-2s-r-1 \\ n-s-r, n-s-1 \end{bmatrix}_q$$

and

$$-q^{s(s-1)/2+s-n} \sum_{r=1}^{n-s} \begin{bmatrix} r+s-1 \\ r, s-1 \end{bmatrix}_q q^{-((n-s)^2+(n-s))/2+(n-s)r+r} \begin{bmatrix} 2n-2s-r-1 \\ n-s-r-1, n-s \end{bmatrix}_q.$$

We will use the Lemma 1.54 to evaluate each piece separately. The first piece can be written

$$q^{-n^2/2+ns-3n/2+s} \sum_{i=0}^{n-s} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \begin{bmatrix} D-i \\ D-E-i, E \end{bmatrix}_q q^{(E+1)i},$$

where

$$C = s-1,\ D = 2n-2s-1,\ E = n-s-1,\ D-E = n-s.$$

Using the lemma, the first piece becomes

$$q^{s-n}\left(q^{-(n^2+n)/2+ns} \begin{bmatrix} 2n-s-1 \\ n-s, n-1 \end{bmatrix}_q\right).$$

Similarly, the second piece evaluates to

$$q^{s-n}\left(-q^{-(n^2+n)/2+ns}\begin{bmatrix}2n-s-1\\n-s-1,n\end{bmatrix}_q\right).$$

The sum of these two quantities *does* equal the desired expression

$$q^{-(n^2+n)/2+ns}\left(\begin{bmatrix}2n-s-1\\n-s,n-1\end{bmatrix}_q - q^s\begin{bmatrix}2n-s-1\\n-s-1,n\end{bmatrix}_q\right),$$

as can be seen by invoking Lemma 1.55 with $C = n - s$ and $D = n - 1$. This completes the proof. □

Corollary 1.57.

$$q^{\binom{n}{2}}C_n(q,1/q) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n,n\end{bmatrix}_q.$$

Proof. Recall that

$$q^{n(n-1)/2}C_n(q,1/q) = q^{n(n-1)/2}q^nF_{n+1,1}(q,1/q).$$

The result follows by substituting the expression for $F_{n+1,1}(q,1/q)$ provided by the theorem and simplifying. □

Remark 1.58. MacMachon [25] showed that

$$C_n^{maj}(q,1/q) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n,n\end{bmatrix}_q.$$

Garsia and Haiman [15] showed that

$$q^{\binom{n}{2}}OC_n(q,1/q) = \frac{1}{[n+1]_q}\begin{bmatrix}2n\\n,n\end{bmatrix}_q.$$

Thus, we conclude that

$$C_n(q,1/q) = OC_n(q,1/q).$$

Of course, this follows from the result $C_n(q,t) = OC_n(q,t)$ quoted above, but the latter result is much more difficult to prove.

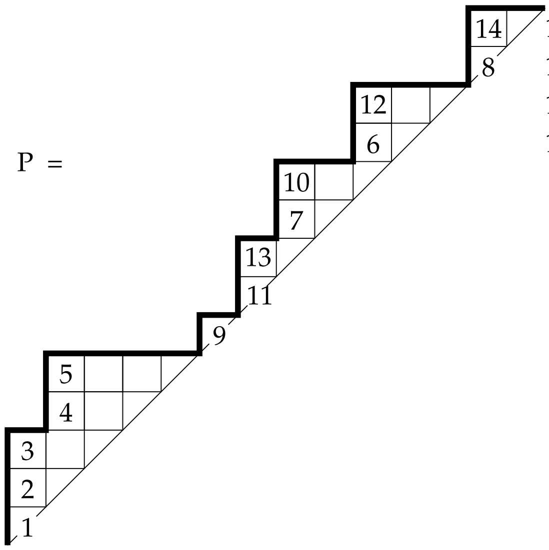
1.5 Combinatorial Hilbert Series

In this section, we describe conjectured combinatorial interpretations for the Hilbert series $H_n(q,t)$ of diagonal harmonics. These interpretations are due to Haglund, Haiman, and the present author [17].

1.5.1 Labelled Dyck Paths — First Version

We start by defining a collection $\mathcal{P}_n$ of *labelled* Dyck paths of order n. Labelled Dyck paths are equivalent to *parking functions*, which are discussed in §1.5.5 below. To construct a typical object $P \in \mathcal{P}_n$, we attach labels to a path $D \in \mathcal{D}_n$ according to the following rules. Let $p_0 p_1 \cdots p_{n-1}$ be a permutation of the labels $\{1, 2, \ldots, n\}$. Place each label p_i in the i^{th} row of the diagram for D, in the cell just right of the unique vertical step of D in that row. There is one restriction: the labels in each column must *increase* when read from bottom to top. See Figure 1.13 for an example.

P =



i	γ_i	p_i
13	1	14
12	0	8
11	2	12
10	1	6
9	2	10
8	1	7
7	1	13
6	0	11
5	0	9
4	3	5
3	2	4
2	2	3
1	1	2
0	0	1

area(P) = 16 dinv(P) = 18 dinv(D(P)) = 41

Figure 1.13: A labelled Dyck path (version 1).

Given such a labelled path P constructed from the ordinary Dyck path $D = D(P)$, define $area(P)$ to be $area(D(P))$, the usual area of the ordinary path D. Also define

$$dinv(P) = \sum_{i<j} \left[\chi(\gamma_i(D(P)) = \gamma_j(D(P)) \text{ and } p_i < p_j) \right.$$
$$\left. +\chi(\gamma_i(D(P)) = \gamma_j(D(P) + 1) \text{ and } p_i > p_j)\right].$$

Comparing this formula to the one for $dinv(D(P))$, we see that $dinv(P) \leq dinv(D(P))$ for all P, and strict inequality can occur.

Define

$$CH_n(q,t) = \sum_{P \in \mathcal{P}_n} q^{area(P)} t^{dinv(P)}. \tag{1.22}$$

It is conjectured that $CH_n(q,t) = H_n(q,t)$. This says that the combinatorial generating function just defined is the same as the Hilbert series of the diagonal harmonics module.

1.5.2 Labelled Dyck Paths — Second Version

We now define another collection $\mathcal{Q}_n$ of labelled Dyck paths of order n. To construct a typical object $Q \in \mathcal{Q}_n$, we attach labels to a path $D \in \mathcal{D}_n$ according to the following rules. Let $q_0 q_1 \cdots q_{n-1}$ be a permutation of the labels $\{1, 2, \ldots, n\}$. Place each label q_i in the i^{th} row of the diagram for D in the *main diagonal cell*. There is one restriction: for each inner corner in the Dyck path consisting of an east step followed by a north step, the label q_i appearing due east of the north step must be less than the label q_j appearing due south of the east step. See Figure 1.14 for an example. In the figure, capital letters mark the inner corners in the Dyck path. Since $4 < 5$, $6 < 12$, $7 < 10$, $2 < 3$, $8 < 14$, $11 < 13$, and $1 < 2$, the labelled path shown does belong to $\mathcal{Q}_{14}$.

Given a labelled path Q constructed from the ordinary Dyck path $D = D(Q)$, define $dmaj(Q)$ to be $bounce(D(Q))$, which was defined earlier. Also define $area'(Q)$ to be the number of cells c in the diagram for Q such that:

1. Cell c is strictly between the Dyck path D and the main diagonal.

2. The label on the main diagonal directly east of c is less than the label on the main diagonal directly south of c.

In Figure 1.14, only the shaded cells satisfy both conditions and hence contribute to $area'(Q)$. Evidently, $area'(Q) \leq area(D(Q))$ for all Q, and strict inequality can occur.

Define

$$CH'_n(q,t) = \sum_{Q \in \mathcal{Q}_n} q^{dmaj(Q)} t^{area'(Q)}. \tag{1.23}$$

It is conjectured that $CH'_n(q,t) = H_n(q,t)$. We will prove in the next subsection that $CH_n(q,t) = CH'_n(q,t)$, so this conjecture is equivalent to the previous one.

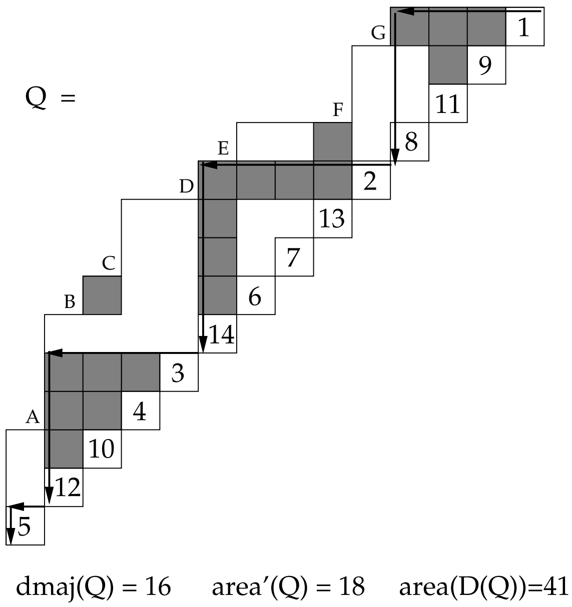


Figure 1.14: A labelled Dyck path (version 2).

1.5.3 Bijections between Labelled Paths

We now give a bijective proof that $CH_n(q,t) = CH'_n(q,t)$. The bijection builds on the bijection ϕ for unlabelled paths used to show $C_n(q,t) = HC_n(q,t)$. (See §1.4.3.)

We will define a bijection from $\mathcal{P}_n$ to $\mathcal{Q}_n$ that sends $area$ to $dmaj$ and sends $dinv$ to $area'$. This bijection proves that

$$\sum_{P\in\mathcal{P}_n} q^{area(P)}t^{dinv(P)} = \sum_{Q\in\mathcal{Q}_n} q^{dmaj(Q)}t^{area'(Q)}.$$

Fix $P \in \mathcal{P}_n$. We shall construct $Q \in \mathcal{Q}_n$ with $dmaj(Q) = area(P)$ and $area'(Q) = dinv(P)$. As an example, the labelled path P in Figure 1.13 will map to the labelled path Q in Figure 1.14.

Let $D = D(P)$ denote the underlying unlabelled Dyck path of P. Let E be the unlabelled Dyck path produced by the bijection ϕ, with $bounce(E) = area(D)$ and $area(E) = dinv(D)$. E will be the underlying unlabelled path for Q (i.e., $D(Q) = E$).

We obtain Q by attaching labels to E, as follows. Scan each of the diagonals of P, from southwest to northeast, starting with the main diagonal and proceeding upward.

Enter the labels of P, in the order in which they are encountered, on the main diagonal of Q going from northeast to southwest. For instance, in Figure 1.13, P has the labels 1, 9, 11, 8 on the main diagonal, followed by the labels 2, 13, 7, 6, 14 on the first superdiagonal, etc. Hence, as shown in Figure 1.14, the labels on the main diagonal of Q are 1, 9, 11, 8, 2, 13, 7, 6, 14, ... starting from (n, n). Clearly, we can recover the labelling of P from the labelling of Q.

Here is an equivalent way of describing the relation between the labels in P and Q. Recall that $E = D(Q)$ can be dissected into triangles $T_0, \ldots, T_s$ and rectangles $R_1, \ldots, R_s$. For $0 \leq j \leq s$, the a_j labels on the main diagonal of Q inside triangle T_j (read from top to bottom) are the labels appearing in the leftmost cells of the a_j rows of $D = D(P)$ for which $\gamma_i(D) = j$ (read from bottom to top).

Recall that the labels of P in a given column must increase from bottom to top. To check the validity of a given labelling, it clearly suffices to check that *adjacent* labels in the same column are always properly ordered. Suppose that the labels p_i and p_{i+1} in rows i and $i+1$ both occur in column j. This occurs if and only if $\gamma_{i+1} = \gamma_i + 1$ if and only if there is an *ascent* of γ at position i (recall that $\gamma_{i+1} \leq \gamma_i + 1$ for all i). We observed earlier that the ascents of $\gamma(D)$ correspond bijectively to the inner corners of $E = D(Q)$. It is easy to verify that label p_i appears in Q directly east of the inner corner corresponding to the ascent $\gamma_i < \gamma_{i+1}$, and the label p_{i+1} appears in Q directly south of this inner corner. Hence, the labelling restrictions on P imply the corresponding labelling restrictions on Q, and conversely.

Clearly, $dmaj(Q) = dmaj(E) = area(D) = area(P)$. We now show that $area'(Q) = dinv(P)$. Consider a typical area cell c of the path $E = D(Q)$. Suppose first that c is inside triangle T_k. Let $x_1, x_2, \ldots, x_{a_k}$ be the labels on the diagonal of Q inside T_k, from top to bottom. As noted above, the labels $x_1, x_2, \ldots, x_{a_k}$ are just the numbers p_i in all positions i for which $\gamma_i = k$. Thus, the cells in T_k that contribute to $area'(Q)$ correspond precisely to coinversions in the word $x_1, x_2, \ldots, x_{a_k}$. (A *coinversion* of a word $v_1 \ldots v_n$ is a pair (i, j) with $i < j$ and $v_i < v_j$.) We obtain a bijection between the contributing cells in T_k and the nonzero summands

$$\chi(\gamma_i = \gamma_j \text{ and } p_i < p_j)$$

for which $\gamma_i = \gamma_j = k$ in the definition of $dinv(P)$.

A similar argument applies to a cell c in rectangle R_k. The horizontal position of the cell determines a unique p_i such that $\gamma_i = k$, and the vertical position of the cell determines a unique p_j such that $\gamma_j = k-1$. As k ranges from 1 to s, all pairs (i,j) for which $\gamma_i = \gamma_j + 1$ are accounted for exactly once in this fashion. Now, we get a contribution to $dinv(P)$ if and only if $i < j$ and $p_i > p_j$; this occurs precisely when the associated cell c satisfies the two conditions for contributing to $area'(Q)$. We conclude that the number of contributing cells in all the rectangular regions R_k is exactly

$$\sum_{i<j} \chi(\gamma_i = \gamma_j + 1 \text{ and } p_i > p_j).$$

Combining this result with the one in the preceding paragraph, we see that $area'(Q) = dinv(P)$. This completes the proof.

1.5.4 Formula for $CH_n(q,t)$

We now describe an explicit formula for $CH_n(q,t)$ as a summation over permutations $\sigma \in S_n$. First, we need some notation. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, a *descent* of σ is an index $i < n$ such that $\sigma_i > \sigma_{i+1}$. Suppose σ has descents $i_1, i_2, \ldots, i_s$, where $i_1 < i_2 < \cdots < i_s$. Then we call the lists of elements

$$\sigma_1\sigma_2\cdots\sigma_{i_1}; \quad \sigma_{i_1+1}\cdots\sigma_{i_2}; \quad \cdots; \quad \sigma_{i_s+1}\cdots\sigma_{i_n}$$

the *ascending runs* of σ. For example, if $\sigma = 4,7,1,5,8,3,2,6$, then the ascending runs of σ are $4,7$ and $1,5,8$ and 3 and $2,6$. We can display the runs more concisely by writing

$$\sigma = 4,7 > 1,5,8 > 3 > 2,6.$$

For $1 \leq i \leq n$, define a number $w_i(\sigma)$ as follows. Let R_j be the ascending run of σ containing σ_i. Let R_{j+1} be the next ascending run of σ, if there is one. The number $w_i(\sigma)$ is the number of items in R_j that are larger than σ_i, plus the number of items in R_{j+1} that are smaller than σ_i if R_{j+1} exists, plus one if R_{j+1} does not exist (i.e., if R_j is the last ascending run of σ). For example, given $\sigma = 4,7 > 1,5,8 > 3 > 2,6$, we obtain

$$(w_1(\sigma), \ldots, w_8(\sigma)) = (2,2,2,2,1,1,2,1).$$

Then

$$CH_n(q,t) = \sum_{\sigma \in S_n} q^{maj(\sigma)} \prod_{i=1}^{n} [w_i(\sigma)]_t. \tag{1.24}$$

This formula is proved in [17].

We will continue the combinatorial study of labelled paths in Chapter 4. In that chapter, we prove the formula above as a corollary of a more general formula. We also give yet another pair of statistics on labelled paths whose generating function is $CH_n(q,t)$.

1.5.5 Parking Functions

This subsection introduces parking functions and discusses their connection to labelled Dyck paths. Parking functions were introduced by Konheim and Weiss [23] and have been subsequently studied and generalized by many authors (see, e.g., [10, 12, 28, 31, 32]). Further discussion of parking functions and their generalizations appears in Chapter 4.

Definition 1.59. A *parking function* or *preference function* of order n is a function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$|\{x : f(x) \le i\}| \ge i \text{ for } 1 \le i \le n.$$

Let $\mathcal{P}'_n$ denote the collection of parking functions of order n.

As in [23], we think of the elements x in the domain of f as cars that wish to park on a one-way street with parking spots labelled $1, 2, \ldots, n$ (in that order). The number $f(x)$ represents the spot where car x prefers to park. In the *standard parking policy*, cars 1 through n arrive at the beginning of the street in increasing numerical order. Each car drives forward to the spot $f(x)$ it prefers. If this spot is available, the car parks there. If not, the car continues driving forward and parks in the next available spot. It can be shown that a function f is a parking function if and only if all n cars are able to park following this policy.

We can identify a parking function f with a labelled Dyck path P as follows. Let $S_i = \{x : f(x) = i\}$ be the set of cars preferring spot i. Starting in the bottom row of an n by n grid of lattice cells, place the elements of S_1 in increasing order in the first column of the diagram, one per row. Starting in the next empty row, place the elements of S_2 in increasing order in the second column of the diagram, one per row. Continue similarly: after listing all elements x with $f(x) < i$, start in the next empty row and place the elements of S_i in increasing order in column i. Finally, draw a lattice path from $(0,0)$ to (n,n) by drawing vertical steps immediately left of each label, and then drawing the necessary horizontal steps to get a connected path. It can be shown that the resulting labelled lattice path is a labelled Dyck path if and only if f is a parking function. Furthermore, given a labelled Dyck path P, we can recover the parking function f by setting $f(i) = j$ if and only if label i occurs in column j. Thus, from now on, we will identify the set of parking functions $\mathcal{P}'_n$ with the set of labelled Dyck paths $\mathcal{P}_n$.

Example 1.60. Let $n = 8$, and define a function f by

$$f(1) = 2,\ f(2) = 3,\ f(3) = 5,\ f(4) = 4,$$

$$f(5) = 1,\ f(6) = 4,\ f(7) = 2,\ f(8) = 6.$$

It is easy to check that f is a parking function. The labelled path $P \in \mathcal{P}_8$ corresponding to f is shown in Figure 1.15. Note that $area(P) = 9$.

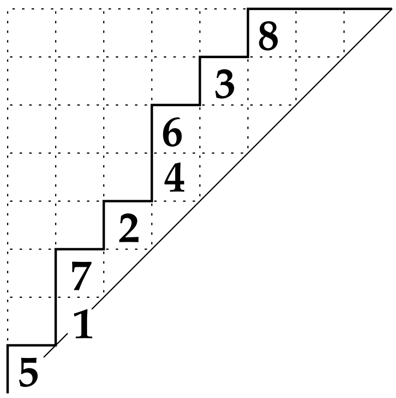


Figure 1.15: Diagram for a parking function.

If P is the diagram for a parking function f, we can compute $area(P)$ as follows. Note that the triangle bounded by the lines $x = 0$, $y = n$, and $x = y$ contains $n(n-1)/2$

complete lattice cells. Since label i occurs somewhere in column $f(i)$, there are $f(i)-1$ lattice cells inside the triangle and left of label i. These lattice cells lie outside the Dyck path associated to f. Subtracting, we find that

$$area(P) = n(n-1)/2 - \sum_{i=1}^{n}[f(i)-1] = n(n+1)/2 - \sum_{i=1}^{n} f(i). \qquad (1.25)$$

For instance, in the example above we have

$$area(P) = 36 - (2+3+5+4+1+4+2+6) = 9.$$

1.6 Extensions of Dyck Paths and Parking Functions

In this section, we introduce generalizations of the representation theoretical and combinatorial objects considered in the preceding sections. In later chapters, we conduct a detailed study of these generalized combinatorial objects.

Recall that the Frobenius series of diagonal harmonics is given by the formula $F_n(q,t) = \nabla(e_n)$. It is natural to ask what happens if we apply the nabla operator more than once. Specifically, for each positive integer m, let $F_n^{(m)}(q,t) = \nabla^m(e_n)$. For each $m \geq 1$, $F_n^{(m)}(q,t)$ is the Frobenius series of a doubly-graded S_n-module, which is described in [15]. We can consider the Hilbert series for this module,

$$H_n^{(m)}(q,t) = F_n^{(m)}(q,t)|_{s_\lambda = f_\lambda}$$

or the generating function for the sign character in this module,

$$RC_n^{(m)}(q,t) = F_n^{(m)}(q,t)|_{s_{1^n}}.$$

The following problems immediately suggest themselves.

Problems: Let $m \geq 1$.

(1) Find a combinatorial interpretation for the Frobenius series $F_n^{(m)}(q,t)$.

(2) Find a combinatorial interpretation for the Hilbert series $H_n^{(m)}(q,t)$.

(3) Find a combinatorial interpretation for the sign character $RC_n^{(m)}(q,t)$.

When $m = 1$, solutions to problem (3) and conjectured solutions to problem (2) have been described earlier in this chapter. In later chapters, we will provide conjectured solutions to problems (2) and (3) for all m. Finding conjectured statistics for problem (1) is an open question even when $m = 1$.

Our proposed solution to problem (3) will involve unlabelled lattice paths similar to Dyck paths, which stay inside a triangular region with vertices $(0,0)$ and $(0,n)$ and (mn,n). Similarly, our proposed solution to problem (2) will involve labelled paths staying within this same region. We will define pairs of statistics on each of these collections, whose corresponding generating functions are conjectured to give $RC_n^{(m)}(q,t)$ and $H_n^{(m)}(q,t)$, respectively. While we cannot prove this representation theoretical conjecture, we do prove various combinatorial facts about these generating functions. In particular, we will prove bijections, recursions, summation formulas, and specializations analogous to those presented above when $m = 1$. We believe that the recursions provided by the combinatorics will be the key to proving the validity of the representation theoretical conjectures.

There are further combinatorial extensions that arise. Having looked at statistics for lattice paths staying in triangular regions, one can consider similar statistics for lattice paths (unlabelled or labelled) staying in certain trapezoidal regions. The generating functions for these statistics have combinatorial properties similar to those considered in the last paragraph. This leads to the question of finding representation theoretical interpretations corresponding to the trapezoidal constructs. This is still an open problem.

Another open problem involves a third statistic that occurs in the combinatorial setting. We will define a trivariate Catalan sequence

$$C_n(q,t,r) = \sum_{D \in \mathcal{D}_n} q^{area(D)} t^{bounce(D)} r^{area^*(D)},$$

where $area^*$ is a modified version of the usual area statistic. This trivariate sequence reduces to the previous one when $r = 1$. It has various symmetry properties; for instance,

$$C_n(q,t,r) = C_n(r,t,q),$$

$$C_n(q,1,1) = C_n(1,q,1) = C_n(1,1,q).$$

There are similar trivariate sequences for the other classes of paths staying inside other shapes. The open question is to find a meaning for the new r-variable in the setting of representation theory.

The rest of this document is organized as follows. In Chapter 2, we discuss several conjectured combinatorial interpretations for $RC_n^{(m)}(q,t)$ involving unlabelled lattice paths inside certain triangles. Trivariate generating functions for these paths are also discussed. In Chapter 3, we generalize the preceding constructions to lattice paths lying in certain trapezoids. We study several five-variable generating functions for these paths and derive their combinatorial properties. In Chapter 4, we consider labelled lattice paths (generalized parking functions) staying within triangles and trapezoids. In the case of triangles, we obtain a conjectured combinatorial interpretation for the higher Hilbert series $H_n^{(m)}(q,t)$. We also introduce trivariate generating functions for the labelled paths. Finally, Chapter 5 contains some miscellaneous results connected to the various Catalan sequences. In particular, we give a determinantal formula for the Carlitz-Riordan numbers $C_n^{area}(q)$. We also give several ways to define the bivariate sequence $C_n(q,t)$ in terms of classical permutation statistics.

1.7 Summary of Notational Conventions

We conclude this chapter by reviewing notation that will be used constantly in the sequel.

Definition 1.61. (1) If A is any logical statement, define $\chi(A)=1$ if A is true, and $\chi(A)=0$ if A is false.

(2) If $w=w_1w_2\dots w_n$ is a word, where each w_i is an integer, define the *inversions of* w by

$$inv(w)=\sum_{1\le i<j\le n}\chi(w_i>w_j).$$

(3) Given a word w as in (2), define the *coinversions of* w by

$$coinv(w)=\sum_{1\le i<j\le n}\chi(w_i<w_j).$$

(4) Given a word w as in (2), define the *major index of* w by

$$maj(w) = \sum_{i=1}^{n-1} i\chi(w_i > w_{i+1}).$$

(5) Let x be any indeterminate. Define $[0]_x = 0$ and

$$[j]_x = 1 + x + x^2 + \cdots + x^{j-1} \quad \text{for integers } j > 0.$$

Also define

$$[n]!_x = \prod_{j=1}^{n} [j]_x$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{[n]!_x}{[k]!_x [n-k]!_x}$$

$$\begin{bmatrix} n \\ n_1, \ldots, n_s \end{bmatrix}_x = \frac{[n]!_x}{[n_1]!_x \cdots [n_s]!_x} \text{ when } n_1 + \cdots + n_s = n.$$

These are the "x-analogues" of the number j, the factorial $n!$, the binomial coefficient $\binom{n}{k}$, and the multinomial coefficient $\binom{n}{n_1,\ldots,n_s}$, respectively. It is easy to check that these formulas agree with those given in (1.2) and (1.3).

(6) Let x, y be indeterminates. Define $[0]_{x,y} = 0$ and

$$[j]_{x,y} = \sum_{k=0}^{j-1} x^k y^{j-1-k} \quad \text{for integers } j > 0.$$

Also define

$$[n]!_{x,y} = \prod_{j=1}^{n} [j]_{x,y}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{x,y} = \frac{[n]!_{x,y}}{[k]!_{x,y} [n-k]!_{x,y}}$$

$$\begin{bmatrix} n \\ n_1, \ldots, n_s \end{bmatrix}_{x,y} = \frac{[n]!_{x,y}}{[n_1]!_{x,y} \cdots [n_s]!_{x,y}} \text{ when } n_1 + \cdots + n_s = n.$$

These are the "homogeneous x, y-analogues" of numbers, factorials, binomial coefficients, and multinomial coefficients.

(7) Given $c \geq 0$ and $d \geq 0$, define $R(0^c1^d)$ to be the set of rearrangements of c zeroes and d ones. Define $\mathcal{R}(c,d)$ to be the set of lattice paths going from the southwest corner to the northeast corner of a rectangle of height c and width d. Given such a path P, let $a(P)$ be the number of cells below the path and inside the rectangle. Let $\tilde{a}(P)$ be the number of cells above the path and inside the rectangle. There is a bijection between $R(0^c1^d)$ and $\mathcal{R}(c,d)$ obtained by replacing zeroes by vertical steps and ones by horizontal steps. If $w \in R(0^c1^d)$ corresponds to $P \in \mathcal{R}(c,d)$ under this bijection, then $coinv(w) = a(P)$ and $inv(w) = \tilde{a}(P)$. We also have

$$\begin{bmatrix} c+d \\ c \end{bmatrix}_{x,y} = \begin{bmatrix} c+d \\ c,d \end{bmatrix}_{x,y} = \sum_{w\in R(0^c1^d)} x^{inv(w)}y^{coinv(w)} = \sum_{P\in\mathcal{R}(c,d)} x^{\tilde{a}(p)}y^{a(p)}.$$

(This follows from Theorem 1.51 part (6) if we replace q by y/x there and multiply everything by x^{cd}.) We prefer to use the multinomial coefficient rather than the binomial coefficient, so that both dimensions of the rectangle appear explicitly in the formula. By convention, the multinomial coefficient is zero if c or d is negative.

Acknowledgements: §1.5 is a reprint of a section from the paper "A Conjectured Combinatorial Formula for the Hilbert Series for Diagonal Harmonics" by J. Haglund and N. Loehr, *Proceedings of FPSAC 2002*, Melbourne Australia, July 2002. The dissertation author was the primary investigator and single author of the section of the paper used.

An abridged version of Chapter 1 appears in the introduction of the paper "Conjectured Combinatorial Models for the Hilbert Series of Generalized Diagonal Harmonics Modules" by N. Loehr and J. Remmel, which is now in preparation for publication. The dissertation author was the primary investigator and single author of the section of the paper used.

2

Higher-Order Catalan Sequences

In Chapter 1, we introduced several bivariate analogues of the Catalan numbers. These analogues are summarized in the following definition.

Definition 2.1. Let n be a positive integer.

(1) The *original q,t-Catalan number*, introduced by Garsia and Haiman in [15], is

$$OC_n(q,t) = \sum_{\mu \vdash n} \frac{t^{2n(\mu)} q^{2n(\mu')}(1-t)(1-q)\Pi_\mu(q,t)B_\mu(q,t)}{h_\mu(q,t)h'_\mu(q,t)}.$$

Here, we sum over all partitions of n. The quantities on the right side are defined in Definition 1.12.

(2) Consider the diagonal harmonics module DH_n, as discussed in §1.3.5. Let $s(h,k,n)$ denote the multiplicity of the sign character in the (h,k)-component of DH_n. Explicitly, $s(h,k,n)$ is the dimension of the subspace of polynomials $f \in DH_n$ such that f has degree h in the x-variables, f has degree k in the y-variables, and $\sigma \cdot f = sgn(\sigma) f$ for all $\sigma \in S_n$. The *representation-theoretical q,t-Catalan number* is

$$RC_n(q,t) = \sum_{h \geq 0} \sum_{k \geq 0} s(h,k,n) q^h t^k.$$

(3) The *symmetric-function q,t-Catalan number* is

$$SC_n(q,t) = \nabla(s_{1^n})|_{s_{1^n}},$$

where ∇ denotes the nabla operator from §1.3.3.

(4) The *first combinatorial q,t-Catalan number*, introduced by Haglund in [16], is

$$C_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{area(D)} t^{bounce(D)},$$

where $\mathcal{D}_n$ is the set of Dyck paths of order n, and *bounce* is the statistic defined in §1.4.1.

(5) The *second combinatorial q,t-Catalan number*, introduced by Haiman in [19], is

$$HC_n(q,t) = \sum_{D \in \mathcal{D}_n} q^{dinv(D)} t^{area(D)},$$

where *dinv* is the statistic defined in §1.4.2.

Theorem 2.2. *For all $n \geq 1$,*

$$RC_n(q,t) = OC_n(q,t) = SC_n(q,t) = C_n(q,t) = HC_n(q,t).$$

Proof. Garsia and Haiman proved (with different notation) that $OC_n(q,t) = SC_n(q,t)$ in [15]. The equality $RC_n(q,t) = OC_n(q,t)$ follows from this result and Haiman's proof of the full character formula for DH_n (see [18, 21]). As discussed in §1.4.6, Garsia and Haglund proved that $SC_n(q,t) = C_n(q,t)$ by showing that both formulas satisfied the same recursion [14]. Finally, a combinatorial proof that $C_n(q,t) = HC_n(q,t)$ was given in §1.4.3. □

In this chapter, we will study "higher-order" bivariate Catalan numbers that generalize those defined above. Garsia and Haiman introduced algebraic versions of these higher-order sequences in [15]. After discussing these algebraic sequences, we give several conjectured combinatorial interpretations for them analogous to $C_n(q,t)$ and $HC_n(q,t)$. These interpretations are based on statistics for lattice paths that never go east of the line $x = my$, where m is a positive integer. We prove explicit summation formulas, bijections, and recursions involving the new statistics. We show that specializations of the combinatorial sequences obtained by setting $t = 1$ or $q = 1$ or $t = 1/q$ agree with the corresponding specializations of the algebraic sequences. A third statistic occurs naturally in the combinatorial setting, leading to the introduction of trivariate Catalan sequences and higher-order sequences.

2.1 Algebraic Higher-order Catalan Sequences

Let m be a positive integer. This section introduces "algebraic Catalan sequences of order m" that generalize the sequences $OC_n(q,t)$, $SC_n(q,t)$, and $RC_n(q,t)$ defined above. When $m = 1$, the higher-order sequences reduce to the corresponding original sequences. All the definitions presented here are due to Garsia and Haiman [15].

Definition 2.3. The *original higher q,t-Catalan sequence of order m* is defined by

$$OC_n^{(m)}(q,t) = \sum_{\mu\vdash n} \frac{t^{(m+1)n(\mu)}q^{(m+1)n(\mu')}(1-t)(1-q)\Pi_\mu(q,t)B_\mu(q,t)}{h_\mu(q,t)h'_\mu(q,t)} \text{ for } n \geq 1. \quad (2.1)$$

This formula is the same as the one in Definition (1.40), except that the factors $t^{2n(\mu)}q^{2n(\mu')}$ in $OC_n(q,t)$ have been replaced by $t^{(m+1)n(\mu)}q^{(m+1)n(\mu')}$. Clearly, $OC_n^{(1)}(q,t) = OC_n(q,t)$.

Definition 2.4. The *symmetric function version* of the higher q,t-Catalan sequence of order m is defined by

$$SC_n^{(m)}(q,t) = \nabla^m(s_{1^n})|_{s_{1^n}} \text{ for } n \geq 1, \quad (2.2)$$

where ∇^m means apply the nabla operator m times in succession.

To calculate $SC_n^{(m)}(q,t)$ for a particular m and n, one should express $s_{1^n} = e_n$ as a linear combination of the modified Macdonald basis $\{\tilde{H}_\mu(q,t)\}$, multiply the coefficient of each $\tilde{H}_\mu$ by $t^{mn(\mu)}q^{mn(\mu')}$, express the result in terms of the Schur basis $\{s_\mu\}$, and extract the coefficient of s_{1^n}. Garsia and Haiman proved in [15] (using different notation) that $OC_n^{(m)}(q,t) = SC_n^{(m)}(q,t)$ using symmetric function identities.

Definition 2.5. The *representation-theory version* of the higher q,t-Catalan sequence of order m is defined as follows (cf. [15]). As in §1.3.5, let S_n act on the polynomial ring $R_n = \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]$ via the diagonal action. Let A_n denote the ideal in R_n generated by all polynomials $P \in R_n$ for which

$$\sigma \cdot P = \mathrm{sgn}(\sigma)P \text{ for all } \sigma \in S_n.$$

Let A_n^m denote the ideal in R_n generated by all products $P_1P_2\cdots P_m$, where each $P_i \in A_n$. Let M_n denote the ideal in R_n generated by all the variables x_i and y_i for $1 \leq i \leq n$.

Finally, define

$$\Gamma_n^{(m)} = A_n^m / M A_n^m.$$

One can check that $\Gamma_n^{(m)}$ is a doubly-graded S_n-module that is a direct sum of one-dimensional submodules M_{1^n} corresponding to the sign character. We let $RC_n^{(m)}(q,t)$ be the Hilbert series of $\Gamma_n^{(m)}$ relative to its double grading.

A problem mentioned but not solved in [15] is to give a combinatorial interpretation for the sequences $OC_n^{(m)}(q,t)$. That paper does give a simple interpretation for $OC_n^{(m)}(q,1)$, which we now describe.

Definition 2.6. Given positive integers m and n, define an *m-Dyck path of height n* to be a path in the xy-plane from $(0,0)$ to (mn,n) consisting of n north steps and mn east steps (each of length one), such that the path never goes strictly below the slanted line $x = my$. Let $\mathcal{D}_n^{(m)}$ denote the collection of m-Dyck paths of height n. For $D \in \mathcal{D}_n^{(m)}$, let $area(D)$ be the number of complete lattice squares strictly between the path D and the line $x = my$.

See Figure 2.1 for an example of an m-Dyck path of height n with $m = 3$ and $n = 8$. For this path, we have $area(D) = 23$.

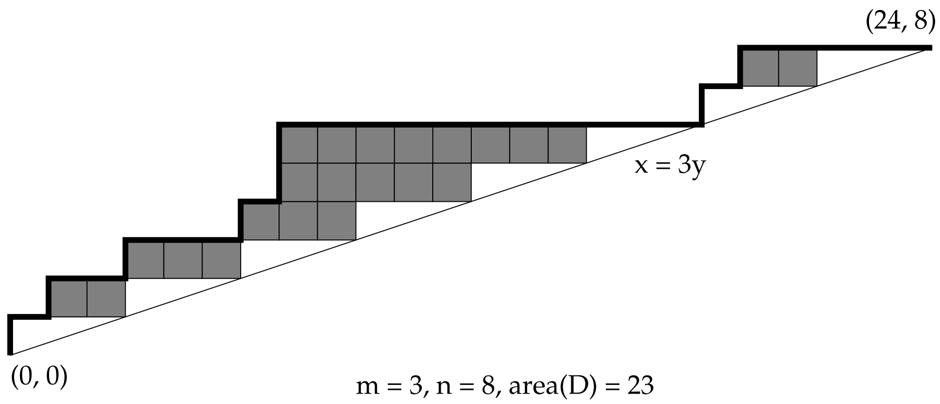


Figure 2.1: A 3-Dyck path of height 8.

Garsia and Haiman proved the following specializations of the higher-order Catalan sequences.

Theorem 2.7.

$$OC_n^{(m)}(q,1) = OC_n^{(m)}(1,q) = \sum_{D\in\mathcal{D}_n^{(m)}} q^{area(D)}.$$

$$q^{mn(n-1)/2} OC_n^{(m)}(q,1/q) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n\end{bmatrix}_q.$$

Proof. See [15]. □

2.2 Conjectured Combinatorial Models for the Higher q,t-Catalan Sequences

Fix a positive integer m. We next describe two statistics defined on m-Dyck paths that each have the same distribution as the area statistic. The first statistic generalizes Haiman's statistic for Dyck paths; the second statistic generalizes Haglund's bounce statistic. We conjecture that either statistic, when paired with area and summed over m-Dyck paths of height n, will give a generating function that equals $OC_n^{(m)}(q,t)$. In support of this conjecture, we show that these combinatorial generating functions have the same specializations as $OC_n^{(m)}$ when $q=1$ or $t=1$ or $t=1/q$ (see Theorem 2.7 above).

2.2.1 Haiman's Statistic for m-Dyck Paths

The statistic discussed here was derived from a statistic communicated to the author by M. Haiman [20].

Definition 2.8. Let $D\in\mathcal{D}_n^{(m)}$ be an m-Dyck path of height n.
(1) Define $\gamma_i(D)$ to be the number of cells in the i^{th} row that are completely contained in the region between the path D and the diagonal $x=my$, for $0\le i<n$. Here, the lowest row is row zero. Note that $area(D)=\sum_{i=0}^{n-1}\gamma_i(D)$.
(2) Define a statistic $h(D)$ by

$$h(D) = \sum_{0\le i<j<n}\sum_{k=0}^{m-1}\chi\left(\gamma_i(D)-\gamma_j(D)+k\in\{0,1,\ldots,m\}\right). \tag{2.3}$$

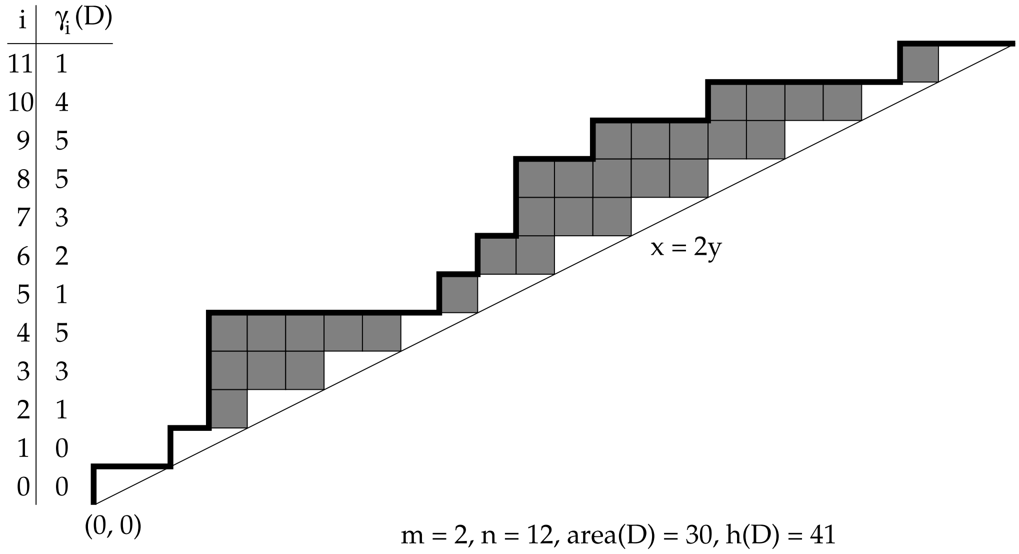


Figure 2.2: Defining the generalized Haiman statistic for a 2-Dyck path.

See Figure 2.2 for an example.

It is easy to see that $h(D)$ reduces to the statistic $dinv(D)$ from §1.4.2 when $m = 1$. Here is another formula for $h(D)$ which will be useful later. Define a function $sc_m : \mathbb{Z} \to \mathbb{Z}$ by

$$sc_m(p) = \begin{cases} m+1-p & \text{if } 1 \le p \le m; \\ m+p & \text{if } -m \le p \le 0; \\ 0 & \text{for all other } p. \end{cases}$$

Note that, given the value of a particular difference $\gamma_i(D) - \gamma_j(D)$ for a fixed i and j, we can evaluate the inner sum $\sum_{k=0}^{m-1} \chi(\gamma_i(D) - \gamma_j(D) + k \in \{0, 1, \ldots, m\})$ in (2.3). By checking the various cases, one sees that the value of this sum is exactly $sc_m(\gamma_i(D) - \gamma_j(D))$. For instance, if $\gamma_i(D) - \gamma_j(D)$ is 0 or 1, then we get a contribution for each of the m values of k, in agreement with the fact that $sc_m(0) = sc_m(1) = m$. Similarly, if $\gamma_i(D) - \gamma_j(D)$ is $-(m-1)$, then only the summand with $k = m-1$ will cause a contribution, in agreement with the fact that $sc_m(-(m-1)) = 1$. The remaining cases are checked similarly. We conclude that

$$h(D) = \sum_{0 \le i < j < n} sc_m(\gamma_i(D) - \gamma_j(D)). \tag{2.4}$$

Definition 2.9. Define the *first conjectured combinatorial version* of the higher q,t-Catalan sequence of order m by

$$HC_n^{(m)}(q,t) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{h(D)} t^{area(D)} \text{ for } n \geq 1.$$

In §2.2.5, we will prove that $HC_n^{(m)}(q,1) = HC_n^{(m)}(1,q)$. This says that the statistic h has the same univariate distribution as the area statistic.

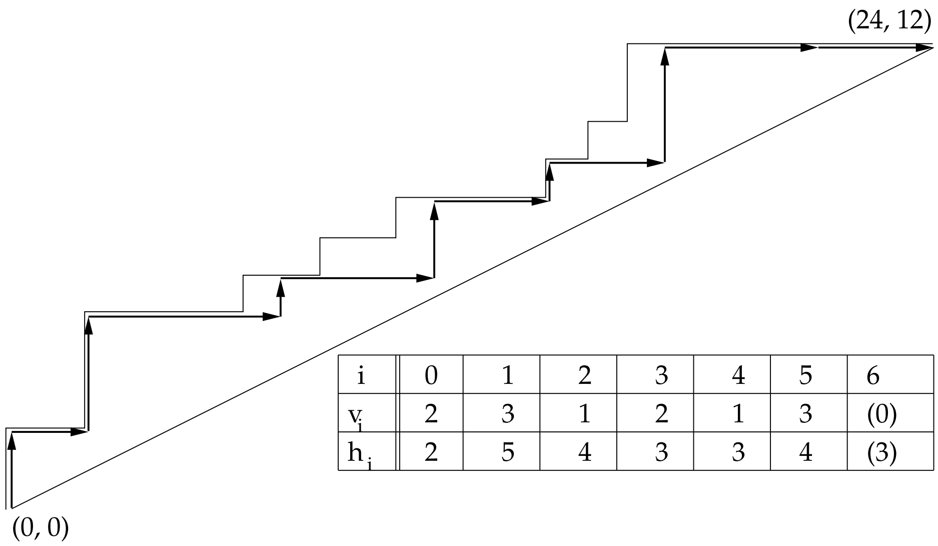
2.2.2 Bounce Statistics for m-Dyck paths

We now discuss how to define a bounce statistic for m-Dyck paths that generalizes Haglund's statistic on ordinary Dyck paths. To define this statistic, we must first define the *bounce path* derived from a given m-Dyck path D.

In Chapter 1, we obtained the bounce path derived from a Dyck path by starting at (n,n) and moving southwest towards $(0,0)$ according to certain rules (see Figure 1.4). We could have drawn an analogous bounce path that starts at $(0,0)$ and bounces northeast towards (n,n). At each stage, this new bounce path would move north until blocked by an east step of the Dyck path, and then move east to the line $y = x$. It turns out that this latter bounce path construction is easier to generalize to the case of m-Dyck paths.

Fix an integer $m \geq 2$, and fix an m-Dyck path D. The bounce path derived from D will consist of a sequence of alternating *vertical moves* and *horizontal moves.* We begin at $(0,0)$ with a vertical move, and eventually end at (mn,n) after a horizontal move. Let $v_0, v_1, \ldots$ denote the lengths of the successive vertical moves in the bounce path, and let $h_0, h_1, \ldots$ denote the lengths of the successive horizontal moves. These lengths are calculated as follows. (Refer to Figures 2.3 and 2.4 for examples.)

To find v_0, move due north from $(0,0)$ until you reach an east step of the m-Dyck path; the distance traveled is v_0. Next, move due east v_0 units (so $h_0 = v_0$). Next, move north from the current position until you reach an east step of the m-Dyck path; let v_1 be the distance traveled. Next, move due east $v_0 + v_1$ units (so $h_1 = v_0 + v_1$). In general, for $i < m$, we move north v_i units from our current position until we are blocked by an east step of the m-Dyck path, and then move east $h_i = v_0 + v_1 + \cdots + v_i$ units.



i	0	1	2	3	4	5	6
v_i	2	3	1	2	1	3	(0)
h_i	2	5	4	3	3	4	(3)

m = 2, n = 12, area(E) = 41, b(E) = 30

Figure 2.3: Defining the bounce statistic for a 2-Dyck path.

For $i \geq m$, the rules change. At stage i, we still move north v_i units until we are blocked by an east step of D. But we then move east $h_i = v_i + v_{i-1} + v_{i-2} + v_{i-(m-1)}$ units. In other words, the length of the next horizontal move is the sum of the m preceding vertical moves.

If we define $v_i = 0$ and $h_i = 0$ for all negative indices i, we can state a single rule that works for all the bounces. Start at $(0, 0)$. Assuming inductively that $v_j = v_j(D)$ and $h_j = h_j(D)$ have been determined for all $j < i$ (where $i \geq 0$), move north from the current position until you are blocked by an east step of the m-Dyck path; define the distance traveled to be v_i. Then move east $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ units. We continue bouncing until we reach (mn, n). (In fact, it suffices to stop once we reach the top rim of the figure, which is the horizontal line $y = n$.) Finally, we define the *bounce statistic* $b(D)$ to be

$$b(D) = \sum_{k \geq 0} k \cdot v_k(D), \tag{2.5}$$

a weighted sum of the lengths of the vertical segments in the bounce path derived from

D. For example, in Figure 2.3, we have

$$b(E) = 0 \cdot 2 + 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 2 + 4 \cdot 1 + 5 \cdot 3 = 30.$$

Remark 2.10. When $m = 1$, the new rule says that $h_i = v_i$ for all i. In other words, we move north until we hit a horizontal step of the Dyck path, and then move east the same distance, bringing us back to the main diagonal $y = x$. Thus, we obtain the modified bounce path construction on Dyck paths in which the bounce path begins at the origin. In this case, there is another formula for $b(D)$. Let s be the number of vertical moves needed to reach the top rim. Then $v_0 + v_1 + \cdots + v_{s-1} = n$, where n is the height of D. We claim that

$$b(D) = \sum_{k=0}^{s-1} (n - v_0 - v_1 - \cdots - v_k). \tag{2.6}$$

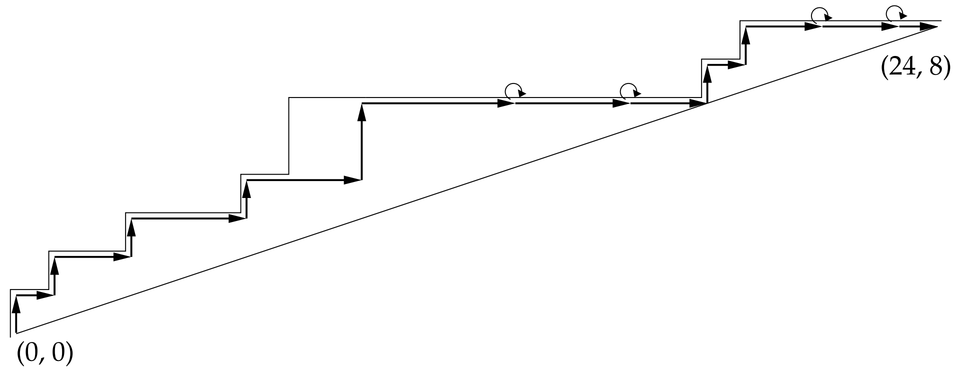
To see this, replace n by $v_0 + \cdots + v_{s-1}$ in (2.6) and simplify the resulting sum. We get

$$b(D) = (v_1+v_2+v_3+\cdots+v_{s-1})+(v_2+v_3+\cdots+v_{s-1})+(v_3+\cdots+v_{s-1})+\cdots = \sum_{k\geq 0} k \cdot v_k,$$

which is formula (2.5).

Remark 2.11. Consider the involution $\alpha : \mathcal{D}_n \to \mathcal{D}_n$ that sends a Dyck path P to its reflection in the line $y = n - x$. It is easy to see that $\alpha(P)$ is another Dyck path with the same area as P. Moreover, if we also reflect the bounce path starting at the origin about the line $y = n - x$, we obtain the original bounce path for $\alpha(P)$ starting at (n, n). Finally, it is easy to check that the numbers $n - v_0(P) - v_1(P) - \cdots - v_k(P)$ in (2.6) are the same as the numbers $i_k(\alpha(P))$ in the original definition of $bounce(\alpha(P))$ (see §1.4.1). Therefore, $b(P) = bounce(\alpha(P))$.

Note that, for $m > 1$, the bounce path does not necessarily return to the diagonal $x = my$ after each horizontal move. Consequently, it may occur that we cannot move north at all after making a particular horizontal move. This situation occurs for the bounce path shown in Figure 2.4, which is derived from the 3-Dyck path shown in Figure 2.1. In this case, we define the next v_i to be zero, and compute the next $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ just as before. In other words, vertical moves of length zero can occur, and are treated the same as nonzero vertical moves when computing the h_i's and the b statistic.



i	0	1	2	3	4	5	6	7	8	9	10
v_i	1	1	1	1	2	0	0	1	1	(0)	(0)
h_i	1	2	3	3	4	3	2	1	2	(2)	(1)

Figure 2.4: A bounce path with vertical moves of length zero.

The possibility now arises that the bounce path could get "stuck" in the middle of the figure. To see why, suppose that m consecutive vertical moves $v_i, \ldots, v_{i+m-1}$ in the bounce path had length zero. Then the next horizontal move h_{i+m-1} would be zero also. As a result, our position in the figure at stage $i+m$ is exactly the same as the position at the beginning of stage $i+m-1$, since $v_{i+m-1} = h_{i+m-1} = 0$. From the bouncing rules, it follows that $v_{i+m} = 0$ also. But then $v_j = h_j = 0$ for all $j \geq i+m$, so that the bouncing path is stuck at the current position forever.

We now argue that the situation described in the last paragraph will never occur. Since the m-Dyck path must start with a north step, we have $v_0 > 0$, and so we do not get stuck at $(0,0)$. The evolving bounce path will continue to make progress eastward with each horizontal step, unless $h_i = 0$ for some $i \geq 0$. Note that $h_i = 0$ if and only if $v_i + v_{i-1} + \cdots + v_{i-(m-1)} = 0$. Fix such an i, and consider the situation just after making the vertical move of length v_{i-1} and the horizontal move of length h_{i-1}. Let (x_0, y_0) denote the position of the bounce path at this instant. Then $y_0 = v_0 + v_1 + \cdots + v_{i-1}$ is the total vertical distance moved so far. Since $v_{i-1} = \cdots = v_{i-(m-1)} = 0$, we have $y_0 = v_0 + \cdots + v_{i-m}$. On the other hand, the total horizontal distance moved so far is

$x_0 = h_0 + h_1 + \cdots + h_{i-1}$. From the definition of the h_j's and the fact that $v_{i-1} = \cdots = v_{i-(m-1)} = 0$, it follows that $x_0 = mv_0 + mv_1 + \cdots + mv_{i-m}$. In more detail, note that the last nonzero v_j, namely v_{i-m}, contributes to the m horizontal moves $h_{i-m}, \ldots, h_{i-1}$. Similarly, for $j < i - m$, v_j has contributed to m horizontal moves that have already occurred at the end of stage $i-1$. Since $v_j = 0$ for $i - m < j \leq i - 1$, the stated formula for x_0 accounts for all the horizontal motion so far. Comparing the formulas for x_0 and y_0 gives $x_0 = my_0$, so that the bounce path has returned to the bounding diagonal $x = my$. If $y_0 = n$, the bounce path has reached its destination. If $y_0 < n$, the m-Dyck path continues above height y_0. But now $v_i > 0$ is forced; otherwise, the m-Dyck path must have gone east from (my_0, y_0), violating the requirement of always staying weakly above the line $x = my$. This argument is illustrated by the path in Figure 2.4.

Thus, the bounce path does not get stuck. The argument at the end of the last paragraph can be modified to show that the bounce path (like the m-Dyck path itself) never goes below the line $x = my$. For, after moving $v_0 + \cdots + v_{i-1}$ steps vertically at some time, we will have gone at most $mv_0 + \cdots + mv_{i-1}$ steps horizontally. Therefore, our position is on or above the line $x = my$.

Now that we know the bounce path is always well-defined, we can make the following definition.

Definition 2.12. Define the *second conjectured combinatorial version* of the higher q,t-Catalan sequence of order m by

$$C_n^{(m)}(q,t) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{area(D)} t^{b(D)} \text{ for } n \geq 1.$$

The involution α in Remark 2.11 shows that

$$\sum_{D \in \mathcal{D}_n} q^{area(D)} t^{b(D)} = \sum_{D \in \mathcal{D}_n} q^{area(D)} t^{bounce(D)},$$

which says that $C_n^{(1)}(q,t) = C_n(q,t)$.

In §2.2.5, we will give a bijective proof that $HC_n^{(m)}(q,t) = C_n^{(m)}(q,t)$. Setting $t = 1$ or $q = 1$ here shows that both new statistics (h and b) have the same distribution on m-Dyck paths of height n as *area*.

Conjecture 2.13. *For all m and n, we have*

$$OC_n^{(m)}(q,t)HC_n^{(m)}(q,t).$$

This equality has been verified for small values of m and n by computer. A possible approach to proving this conjecture will be indicated in §2.3.

2.2.3 Formula for $C_n^{(m)}(q,t)$

In this subsection, we give an explicit algebraic formula for $C_n^{(m)}(q,t)$ by analyzing bounce paths. This formula, while messy, is obviously a polynomial in q and t with nonnegative integer coefficients, unlike the formula defining $OC_n^{(m)}(q,t)$. A disadvantage of the new formula is that the (conjectured) symmetry $C_n^{(m)}(q,t) = C_n^{(m)}(t,q)$ is not evident from inspection of the formula. In the next subsection, we will prove that the same formula holds for $HC_n^{(m)}(q,t)$, which implies that $C_n^{(m)}(q,t) = HC_n^{(m)}(q,t)$.

Recall from Chapter 1 the following combinatorial interpretations for q-binomial coefficients:

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{P\in\mathcal{P}_{a,b}} q^{area(P)} = \sum_{P\in\mathcal{P}_{a,b}} q^{ab-area(P)}.$$

$$\begin{bmatrix} a+b \\ a,b \end{bmatrix}_q = \sum_{w\in R(0^a1^b)} q^{inv(w)} = \sum_{w\in R(0^a1^b)} q^{coinv(w)}. \tag{2.7}$$

We are now ready to state the formula for $C_n^{(m)}(q,t)$. Let $\mathcal{V}_n^{(m)}$ denote the set of all sequences $v = (v_0, v_1, v_2, \ldots, v_s)$ such that: each v_i is a nonnegative integer; $v_0 > 0$; $v_s > 0$; $v_0 + v_1 + v_2 + \cdots + v_s = n$; and there is never a string of m or more consecutive zeroes in v. As usual, let $v_i = 0$ for all negative i.

Theorem 2.14. *With $\mathcal{V}_n^{(m)}$ defined as above, we have:*

$$C_n^{(m)}(q,t) = \sum_{v\in\mathcal{V}_n^{(m)}} t^{\sum_{i\geq 0} iv_i} q^{pow} \prod_{i\geq 1} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q, \tag{2.8}$$

where

$$pow = m\sum_{i\geq 0}\binom{v_i}{2} + \sum_{i\geq 1} v_i \sum_{j=1}^{m} (m-j)v_{i-j}.$$

Equivalently, we may sum over all compositions v of n with zero parts allowed, if we identify compositions that differ only in trailing zeroes.

Remark 2.15. When $m = 1$, this formula reduces to a formula for $C_n(q,t)$ given by Haglund in [16].

Proof. Let $D \in \mathcal{D}_n^{(m)}$ be a typical object counted by $C_n^{(m)}(q,t)$. We can classify D based on the sequence $v(D) = (v_0, v_1, \ldots, v_s)$ of vertical moves in the bounce path derived from D. Call this sequence the *bounce composition* of D. By the discussion in the preceding section, the vector $v = v(D)$ belongs to $\mathcal{V}_n^{(m)}$. To prove the formula for $C_n^{(m)}(q,t)$, it suffices to show that

$$\sum_{D:\ v(D)=v} q^{area(D)} t^{b(D)} =$$
$$t^{\sum_{i\geq 0} iv_i} q^{m\sum_{i\geq 0}\binom{v_i}{2}} \prod_{i=1}^{s} q^{v_i \sum_{j=1}^{m}(m-j)v_{i-j}} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i,\, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q$$

for each $v = (v_0, \ldots, v_s) \in \mathcal{V}_n^{(m)}$. By our conventions for q-binomial coefficients, the right side of this expression is zero if any m consecutive v_i's are zero (in particular, this occurs if $v_0 = 0$). Thus, it does no harm in (2.8) to sum over all compositions v of n with zero parts allowed, not just the compositions v belonging to $\mathcal{V}_n^{(m)}$.

Now, fix $v \in \mathcal{V}_n^{(m)}$ and consider only the m-Dyck paths of height n having bounce composition v. By definition of the bounce statistic, every such path D will have the same t-weight, namely

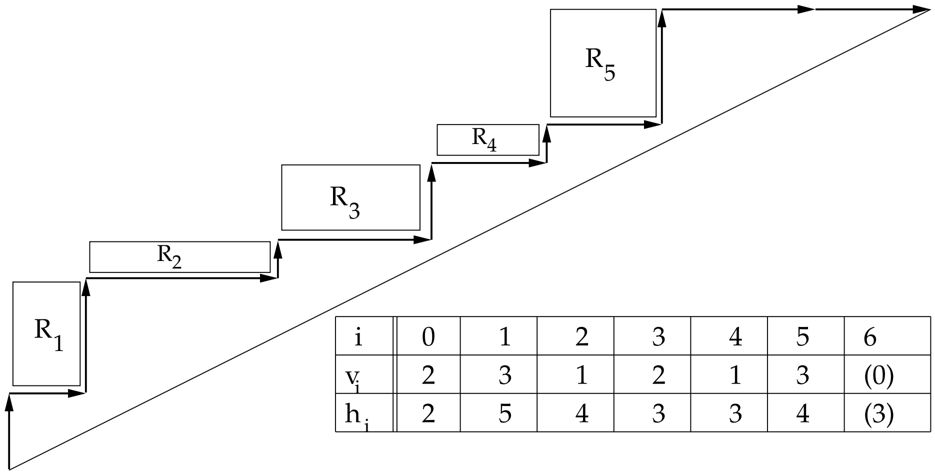
$$t^{b(D)} = t^{\sum_{i\geq 0} iv_i}.$$

To analyze the q-weights, note that we can construct all m-Dyck paths of height n having bounce composition v as follows.

1. Starting with an empty diagram, draw the bounce path with vertical segments $v_0, \ldots, v_s$. There is exactly one way to do this, since the horizontal moves h_i are completely determined by the vertical moves.

2. Having drawn the bounce path, there are now s empty rectangular areas just northwest of the "left-turns" in the bounce path. See Figure 2.5 for an example. Label these rectangles $R_1, \ldots, R_s$, as shown. By definition of the bounce path, rectangle R_i has height v_i and width $h_{i-1} = v_{i-1} + \cdots + v_{i-m}$ for each i. To complete the m-Dyck path, draw a path in each rectangle R_i from the southwest

corner to the northeast corner, *where each path begins with at least one east step.* The first east step in R_i must be present, by definition of v_{i-1}.



i	0	1	2	3	4	5	6
v_i	2	3	1	2	1	3	(0)
h_i	2	5	4	3	3	4	(3)

m = 2, n = 12.

Figure 2.5: Rectangles above the bounce path.

We can rephrase the second step as follows. Let R'_i denote the rectangle of height v_i and width $h_i = v_{i-1} + \cdots + v_{i-m} - 1$ obtained by ignoring the leftmost column of R_i. Then we can uniquely construct the path D by filling each shortened rectangle R'_i with an *arbitrary* path going from the southwest corner to the northeast corner.

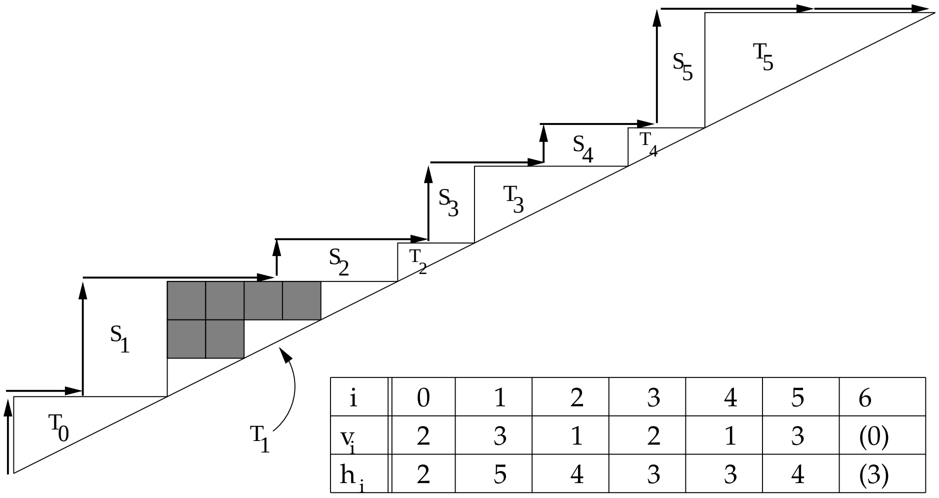
The generating function for the number of ways to perform this second step, where the exponent of q records the total area above the bounce path, is

$$\prod_{i=1}^{s} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q$$

by the preceding discussion of q-binomial coefficients.

We still need to multiply by a power of q that records the area under the bounce path, which is independent of the choices in the second step. We claim that this area is

$$m \sum_{i=0}^{s} \frac{1}{2} v_i(v_i - 1) + \sum_{i=1}^{s} \left(v_i \sum_{j=1}^{m} (m-j) v_{i-j} \right),$$



i	0	1	2	3	4	5	6
v_i	2	3	1	2	1	3	(0)
h_i	2	5	4	3	3	4	(3)

m = 2, n = 12.

Figure 2.6: Dissecting the area below the bounce path.

which will complete the proof.

To establish the claim, dissect the area below the bounce path as shown in Figure 2.6. There are $s+1$ triangular pieces T_i, where the i^{th} triangle contains $0+m+2m+\cdots+(v_i-1)m = m\frac{v_i(v_i-1)}{2}$ complete cells. In Figure 2.6, for instance, where $v_1 = 3$, we have shaded the $0+2+4=6$ cells in T_1 that contribute to the area statistic. The total area coming from the triangles is

$$m\sum_{i=0}^{s}\frac{1}{2}v_i(v_i-1).$$

There are also s rectangular slabs S_i (for $1 \le i \le s$). The height of slab S_i is v_i. What is the width of S_i? To answer this question, fix i, let (a,c) be the coordinates of the southeast corner of S_i, and let (b,c) be the coordinates of the southwest corner of S_i. First note that $c = v_0+v_1+\cdots+v_{i-1}$, the sum of the vertical steps prior to step i. Therefore,

$$a = mc = m(v_0+\cdots+v_{i-1}) = mv_{i-1}+mv_{i-2}+\cdots+mv_{i-m}+mv_{i-m-1}+\cdots$$

since the southeast corner of S_i lies on the line $x = my$. Next, $b = h_0+h_1+\cdots+h_{i-1}$, the sum of the horizontal steps prior to step i. Recall that each h_j is the sum of the

m preceding v_i's (starting with $i = j$). Substituting into the expression for b gives $b = 1v_{i-1} + 2v_{i-2} + \cdots + mv_{i-m} + mv_{i-m-1} + mv_{i-m-2} + \cdots$. We conclude that the width of S_i is

$$a - b = (m-1)v_{i-1} + (m-2)v_{i-2} + \cdots + (m-m)v_{i-m} + 0 + 0 + \cdots.$$

Finally, the area of S_i is the height times the width, which is

$$v_i(a-b) = v_i \sum_{j=1}^{m} (m-j)v_{i-j}.$$

Adding over all i gives the term

$$\sum_{i=1}^{s} \left(v_i \sum_{j=1}^{m} (m-j)v_{i-j} \right),$$

completing the proof of the theorem. □

2.2.4 Proving the Formula for $HC_n^{(m)}(q,t)$

We now prove that $HC_n^{(m)}(q,t)$ is given by the same formula as $C_n^{(m)}(q,t)$.

Theorem 2.16. *With $\mathcal{V}_n^{(m)}$ defined as above, we have:*

$$HC_n^{(m)}(q,t) = \sum_{v \in \mathcal{V}_n^{(m)}} t^{\sum_{i \geq 0} i v_i} q^{pow} \prod_{i \geq 1} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q, \tag{2.9}$$

where

$$pow = m \sum_{i \geq 0} \binom{v_i}{2} + \sum_{i \geq 1} v_i \sum_{j=1}^{m} (m-j) v_{i-j}.$$

Equivalently, we may sum over all compositions v of n with zero parts allowed, if we identify compositions that differ only in trailing zeroes.

Proof. Recall that an m-Dyck path D can be represented by a vector

$$\gamma(D) = (\gamma_0(D), \ldots, \gamma_{n-1}(D)),$$

where $\gamma_i(D)$ is the number of area cells between the path and the diagonal in the i^{th} row from the bottom. Clearly, the path D is uniquely recoverable from the vector γ. Also, a vector $\gamma = (\gamma_0, \ldots, \gamma_{n-1})$ represents an element $D \in \mathcal{D}_n^{(m)}$ if and only if the following three conditions hold:

1. $\gamma_0 = 0$.

2. $\gamma_i \geq 0$ for all i.

3. $\gamma_{i+1} \leq \gamma_i + m$ for all $i < n-1$.

The first condition reflects the fact that the lowest row cannot have any area cells. The second condition is clear, since γ_i is counting cells. The third condition follows since the path is not allowed to take any west steps.

Let $\mathcal{G}_n^{(m)}$ denote the set of all n-long vectors γ satisfying these three conditions. Then the preceding remarks show that

$$HC_n^{(m)}(q,t) = \sum_{\gamma \in \mathcal{G}_n^{(m)}} q^{h(\gamma)} t^{\sum_{i\geq 0} \gamma_i},$$

where $\sum_{i\geq 0} \gamma_i$ is the area of the path D corresponding to γ, and where we set

$$h(\gamma) = \sum_{0 \leq i < j < n} \sum_{k=0}^{m-1} \chi(\gamma_i - \gamma_j + k \in \{0, 1, \ldots, m\}),$$

so that $h(\gamma)$ is the h-statistic of the path D.

Given a vector $\gamma \in \mathcal{G}_n^{(m)}$, let $v_i(\gamma)$ be the number of times i occurs in the sequence $(\gamma_0, \ldots, \gamma_{n-1})$ for each $i \geq 0$. Let $v(\gamma) = (v_0(\gamma), v_1(\gamma), \ldots, v_s(\gamma))$ where s is the largest entry appearing in γ. We call $v(\gamma)$ the *composition of* γ. From the definitions of $\mathcal{G}_n^{(m)}$ and $v(\gamma)$, we see that $v_0 > 0$, $v_s > 0$, $v_0 + \cdots + v_s = n$, and there is never a string of m consecutive zeroes in v (lest $\gamma_{i+1} > \gamma_i + m$ for some i). In other words, v belongs to $\mathcal{V}_n^{(m)}$.

We now classify the objects γ in $\mathcal{G}_n^{(m)}$ based on their composition. To prove the summation formula for $HC_n^{(m)}(q,t)$, it suffices to show that

$$\begin{aligned} &\sum_{\gamma:\ v(\gamma)=v} q^{h(\gamma)} t^{\sum_{i\geq 0} \gamma_i} = \\ &t^{\sum_{i\geq 0} i v_i} q^{m \sum_{i \geq 0} \frac{1}{2} v_i (v_i - 1)} \prod_{i=1}^{s} q^{v_i \sum_{j=1}^{m} (m-j) v_{i-j}} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q \end{aligned} \tag{2.10}$$

for each $v = (v_0, \ldots, v_s) \in \mathcal{V}_n^{(m)}$. It is clear that the powers of t on each side of this equation agree, since v_i is the number of occurrences of the value i in the summation $\sum_{i\geq 0} \gamma_i$.

Before considering the powers of q, note that we can uniquely construct all vectors $\gamma \in \mathcal{G}_n^{(m)}$ having composition v as follows.

1. Initially, let γ be a string of v_0 zeroes.

2. Next, insert v_1 ones in the gaps to the right of these zeroes. There can be any number of ones in each gap, but no 1 may appear to the left of the leftmost zero.

3. Continue by inserting v_2 twos into valid locations, then v_3 threes, etc. The general step is to insert v_i copies of the symbol i into valid locations in the current string. Here, a "valid" location is one such that inserting i in that location will not cause a violation of the three conditions in the definition of $\mathcal{G}_n^{(m)}$.

How many ways are there to perform the i^{th} step of this insertion process, for $i > 0$? To answer this, note that a new symbol $i > 0$ can only be placed in a gap immediately to the right of the existing symbols $i-1, i-2, \ldots, i-m$ in the current string. There are $v_{i-1}+v_{i-2}+\cdots+v_{i-m}$ such symbols, and hence the same number of gaps. Since multiple copies of i can be placed in each gap, the number of ways to insert the v_i new copies of the symbol i is $\binom{v_i+v_{i-1}+\cdots+v_{i-m}-1}{v_i,\, v_{i-1}+\cdots+v_{i-m}-1}$. (To see this, represent a particular way of inserting the new i's by a string of v_i "stars" representing the i's and $v_{i-1}+\cdots+v_{i-m}-1$ "bars" that separate the $v_{i-1}+\cdots+v_{i-m}$ available gaps.) Multiplying these expressions as i ranges from 1 to s, we see that formula (2.10) is correct when $q = 1$.

It remains to see that the power of q is correct as well. We prove this by induction on the largest symbol s appearing in γ. If $s = 0$, then $v = (n)$, and γ must consist of a string of n zeroes. From the definition, we see that $h(\gamma) = mn(n-1)/2$. This is the same as the power of q on the right side of (2.10), since $v_0 = n$ and $v_i = 0$ for $i > 0$.

Now assume that $s > 0$. Fix $v = (v_0, \ldots, v_s) \in \mathcal{V}_n^{(m)}$. Let $v' = (v_0, \ldots, v_{s-1})$, which is an element of $\mathcal{V}_{n-v_s}^{(m)}$ (ignore trailing zeroes in v' if necessary). Our induction hypothesis says that

$$\sum_{\delta: v(\delta)=v'} q^{h(\delta)} = q^{m\sum_{i=0}^{s-1} v_i(v_i-1)/2} \prod_{i=1}^{s-1} q^{v_i \sum_{j=1}^{m}(m-j)v_{i-j}} \begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i,\, v_{i-1}+\cdots+v_{i-m}-1 \end{bmatrix}_q;$$

note that any trailing zeroes in v' just contribute extra factors of 1 to the right side, which are harmless. We want to establish the analogous formula for

$$\sum_{\gamma : v(\gamma)=v} q^{h(\gamma)}.$$

For this purpose, recast the construction given in the $q = 1$ case as follows. We can uniquely produce every γ with $v(\gamma) = v$ by first choosing a δ with $v(\delta) = v'$ and then choosing a way to insert v_s copies of s into δ in valid locations. The generating function for the number of ways to choose δ, where the power of q records $h(\delta)$, is by assumption

$$q^{m\sum_{i=0}^{s-1} v_i(v_i-1)/2} \prod_{i=1}^{s-1} q^{v_i \sum_{j=1}^{m}(m-j)v_{i-j}} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q .$$

To complete the proof, we need to show that the increase in the h-statistic caused by the second choice (namely, $h(\gamma) - h(\delta)$) has generating function

$$q^{mv_s(v_s-1)/2} q^{v_s \sum_{k=1}^{m}(m-k)v_{s-k}} \begin{bmatrix} v_s + v_{s-1} + \cdots + v_{s-m} - 1 \\ v_s, v_{s-1} + \cdots + v_{s-m} - 1 \end{bmatrix}_q . \tag{2.11}$$

Then the desired result will follow from the product rule for generating functions (see [3], Ch. 10).

We encode the choice of how to insert the v_s copies of s into δ as a word

$$w \in R(0^{v_s} 1^{v_{s-1}+\cdots+v_{s-m}-1}).$$

To find w, read the symbols in the completed vector γ from left to right. Write down a zero in w every time an s occurs in γ; write down a one in w every time one of the symbols $s-1, \ldots, s-m$ occurs in γ; ignore all other symbols in γ. By the conditions on γ, the first symbol in w must be a one (since some symbol in $\{s-1, \ldots, s-m\}$ must appear just before the leftmost s in γ). Erase this initial 1 to obtain the final word w.

We will prove that

$$h(\gamma) - h(\delta) = mv_s(v_s - 1)/2 + v_s \sum_{k=1}^{m} (m-k)v_{s-k} + coinv(w); \tag{2.12}$$

if this equation holds, then (2.11) immediately follows from it because of (2.7).

The proof of (2.12) proceeds by induction on the value of $coinv(w)$. Suppose $coinv(w) = 0$ first. This happens if and only if all v_s copies of s were inserted into δ

immediately following the last occurrence of any symbol in the set $\{s-1,\ldots,s-m\}$. How do these v_s newly inserted symbols affect the h-statistic? To answer this, we must compute the sum (see (2.4))

$$\sum_{i<j} sc_m(\gamma_i - \gamma_j)$$

over all pairs (i,j) such that $\gamma_i = s$ or $\gamma_j = s$.

First, consider the pairs (i,j) for which $i<j$ and $\gamma_i = s$ and $\gamma_j = s$. There are $\binom{v_s}{2}$ such pairs, and each contributes $sc_m(s-s) = sc_m(0) = m$ to the h-statistic. This gives the term $mv_s(v_s-1)/2$ in (2.12).

Second, consider the pairs (i,j) for which $i<j$ and $\gamma_i = s$ and $\gamma_j \neq s$. Since all the copies of s in γ occur in a contiguous group following all instances of the symbols $s-1,\ldots,s-m$, and since s is the largest symbol appearing in γ, $j>i$ implies that $\gamma_j < s-m$. Then $sc_m(\gamma_i-\gamma_j)=0$, since $\gamma_i-\gamma_j>m$. So these pairs contribute nothing to the h-statistic.

Third, consider the pairs (i,j) for which $i<j$ and $\gamma_i \neq s$ and $\gamma_j = s$. Since s is the largest symbol, we have $\gamma_i < s$. Write $\gamma_i = s-k$ for some $k>0$, and consider various subcases. Suppose $k \in \{1,2,\ldots,m\}$. Then $sc_m(\gamma_i-\gamma_j) = sc_m(-k) = m-k$. For how many pairs (i,j) does it happen that $i<j$, $\gamma_i = s-k$, and $\gamma_j = s$? There are v_s choices for the index j and v_{s-k} choices for the index i; the condition $i<j$ holds automatically, since all occurrences of s occur to the right of all occurrences of $s-k$. Thus, we get a total contribution to the h-statistic of $(m-k)v_s(v_{s-k})$ for this k. Adding over all k, we obtain the term

$$v_s \sum_{k=1}^{m} (m-k) v_{s-k}$$

appearing in (2.12). On the other hand, if $k>m$, then $sc_m(\gamma_i-\gamma_j) = sc_m(-k) = 0$, so there is no contribution to the h-statistic.

The three cases just considered are exhaustive, so we conclude that (2.12) is true when $coinv(w)$ is zero.

For the inductive step, consider what happens when we replace two consecutive symbols 10 in w by 01, thus increasing $coinv(w)$ by one. Let w' be the new word after the replacement, and let γ' be the associated vector obtained by inserting s's into δ according to the encoding w'. We may assume, by induction, that (2.12) is correct for γ

and w. Passing from w to w' increases the right side of (2.12) by one. Hence, (2.12) will be correct for γ' and w', provided that $h(\gamma') = h(\gamma)+1$. To obtain γ' from γ, look at the symbols in γ corresponding to the replaced string 10 in w. The symbol corresponding to the 0 is an s. This s is immediately preceded in γ by a symbol in $\{s-1,\ldots,s-m\}$ which corresponds to the 1, by the conditions on γ and the fact that $s > 0$. Say $s-k$ immediately precedes this s. The effect of replacing 10 by 01 in w is to move the s leftwards, past its predecessor $s-k$, and re-insert it in the next valid position in γ. This valid position occurs immediately to the right of the next occurrence of a symbol in $\{s, s-1, s-2, \ldots, s-m\}$ left of the symbol $s-k$. Pictorially, we have:

$$\text{original } \gamma = \ldots\ (s-j)\ z_1\ z_2\ \ldots\ z_\ell\ (s-k)\ s\ \ldots$$

where $0 \le j \le m$, $1 \le k \le m$, $\ell \ge 0$, and every $z_i < s-m$. After moving s left, we have

$$\text{new } \gamma' = \ldots\ (s-j)\ s\ z_1\ z_2\ \ldots\ z_\ell\ (s-k)\ \ldots.$$

Note that the symbol $s-j$ must exist, lest $\gamma'_0 = s > 0$.

Now, let us examine the effect of this motion on the h-statistic. When we move the s left past its predecessor $s-k$ in γ, we get a net change in the h-statistic of

$$sc_m(s-(s-k)) - sc_m((s-k)-s) = sc_m(k) - sc_m(-k) = +1,$$

since $1 \le k \le m$ (see (2.4)). As before, since $|s - z_i| > m$, moving the s past each z_i will not affect the h-statistic at all. Thus, the total change in the h-statistic is $+1$, as desired.

We can obtain an arbitrary encoding word w from the word $11\ldots100\ldots0$ with no coinversions by doing a finite sequence of interchanges of the type just described. Thus, the validity of (2.12) for all words w follows by induction on the number of such interchanges required (this number is exactly $coinv(w)$, of course). This completes the proof of the theorem. □

Corollary 2.17.

$$C_n^{(m)}(q,t) = HC_n^{(m)}(q,t).$$

We will give a bijective proof of this corollary in the next subsection.

2.2.5 Bijection Proving that $HC_n^{(m)}(q,t) = C_n^{(m)}(q,t)$

The two proofs just given to show that formula (2.8) holds for $C_n^{(m)}(q,t)$ and $HC_n^{(m)}(q,t)$ were completely combinatorial. Hence, we can combine these proofs to get a bijective proof that $HC_n^{(m)}(q,t) = C_n^{(m)}(q,t)$. Fix m and n. We describe a bijection $\phi : \mathcal{D}_n^{(m)} \to \mathcal{D}_n^{(m)}$ such that

$$h(D) = area(\phi(D)) \text{ and } area(D) = b(\phi(D)) \text{ for } D \in \mathcal{D}_n^{(m)}$$

and a bijection $\psi = \phi^{-1} : \mathcal{D}_n^{(m)} \to \mathcal{D}_n^{(m)}$ such that

$$b(D) = area(\psi(D)) \text{ and } area(D) = h(\psi(D)) \text{ for } D \in \mathcal{D}_n^{(m)}.$$

These bijections will show that the three statistics $area$, h, and b all have the same univariate distribution on $\mathcal{D}_n^{(m)}$.

Description of ϕ. Let D be an m-Dyck path of height n. To find the path $\phi(D)$:

- Represent D by the vector of row lengths $\gamma(D) = (\gamma_0(D), \ldots, \gamma_{n-1}(D))$, where $\gamma_i(D)$ is the number of area cells in the i^{th} row from the bottom.

- Define $v = (v_0, \ldots, v_s)$ by letting v_j be the number of occurrences of the value j in the vector $\gamma(D)$.

- Starting with an empty triangle, draw a bounce path from $(0,0)$ with successive vertical segments $v_0, \ldots, v_s$ and horizontal segments $h_0, h_1, \ldots$, where $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)}$ for each i.

- For $1 \leq i \leq s$, form a word w_i from $\gamma(D)$ as follows. Initially, w_i is empty. Read γ from left to right. Write down a zero every time the symbol i is seen in γ. Write down a one every time a symbol in $\{i-1, \ldots, i-m\}$ is seen in γ. Ignore all other symbols in γ. At the end, erase the first symbol in w_i (which is necessarily a 1).

- Let $R_1, \ldots, R_s$ be the empty rectangles above the bounce path. Let $R_1', \ldots, R_s'$ be these rectangles with the leftmost columns deleted (as in §2.2.3). For $1 \leq i \leq s$, use the word w_i to fill in the part of the path lying in R_i', from the southwest corner to the northeast corner, by taking a north step for each zero in w_i, and an east step for each one in w_i. Call the completed path $\phi(D)$.

The two preceding proofs have already shown that ϕ has the desired effect on the various statistics.

Example 2.18. Let D be the 2-Dyck path of height 12 depicted in Figure 2.2. We have

$$\gamma(D) = (0, 0, 1, 3, 5, 1, 2, 3, 5, 5, 4, 1); \quad area(D) = 30; \quad h(D) = 41.$$

Doing frequency counts on the entries of γ, we compute

$$v = (v_0, v_1, v_2, v_3, v_4, v_5) = (2, 3, 1, 2, 1, 3).$$

Given v, we can draw the bounce path shown in Figure 2.5 with 5 empty rectangles above it. Now, we compute the words w_i:

$$w_1 = 1000; \quad w_2 = 11101; \quad w_3 = 01101; \quad w_4 = 110; \quad w_5 = 01001.$$

Using these words to fill in the partial paths, we obtain the path D' in Figure 2.3, which has $b(D) = 30$ and $area(D) = 41$.

Here is a mild simplification of the bijection. Leave the first 1 at the beginning of each w_i instead of erasing it. Then the w_i tell us how to construct the partial paths in the full rectangles R_i (rather than the shortened rectangles R'_i). Every such partial path begins with an east step, as required by the bouncing rules.

Description of ψ. Let D be an m-Dyck path of height n. To find the path $\psi(D)$:

- Draw the bounce path derived from D according to the bouncing rules (see §2.2.2). Let $v = (v_0, \ldots, v_s)$ be the lengths of the vertical moves in this bounce path.

- Let $R_1, \ldots, R_s$ be the rectangular regions above the bounce path. These regions contain partial paths going from the southwest corner to the northeast corner. For $1 \leq i \leq s$, find the word w_i by traversing the partial path in R_i and writing a one for each east step and a zero for each north step. Note that every w_i has first symbol one.

- Build up γ as follows. Start with a string of v_0 zeroes. For $i = 1, 2, \ldots, s$, insert v_i copies of i into the current string γ according to w_i. More explicitly, read w_i left to right. When a 1 is encountered, scan γ from left to right for the next occurrence of a symbol in $\{i-1,\ldots,i-m\}$. When a 0 is encountered, place an i in the gap immediately to the right of the current symbol in γ. Continue until all symbols i have been inserted.

- Use γ to draw the picture of a new m-Dyck path D' of height n, by placing γ_i area cells in the i^{th} row of the figure. Since $\gamma \in \mathcal{G}_n^{(m)}$, the resulting picture will be a valid path.

Example 2.19. Let D be the 3-Dyck path of height 8 shown in Figure 2.4. From the bounce path drawn in that figure, we find that

$$v = (v_0,\ldots,v_9) = (1,1,1,1,2,0,0,1,1).$$

Examining the rectangles above the bounce path (several of which happen to be empty or have height zero), we get the words w_i:

$$w_1 = 10;\ w_2 = 110;\ w_3 = 1110;\ w_4 = 10011;\ w_5 = 1111;\ w_6 = 111;\ w_7 = 110;\ w_8 = 10.$$

Now, build up the vector γ as follows:

- Initially, $\gamma = 0$ (since $v_0 = 1$).
- Use $w_1 = 10$ to insert one 1 into γ to get $\gamma = 01$.
- Use $w_2 = 110$ to insert one 2 into γ to get $\gamma = 012$.
- Use $w_3 = 1110$ to insert one 3 into γ to get $\gamma = 0123$.
- Use $w_4 = 10011$ to insert two 4's into γ to get $\gamma = 014423$.
- Use $w_5 = 1111$ to insert zero 5's into γ to get $\gamma = 014423$.
- Use $w_6 = 111$ to insert zero 6's into γ to get $\gamma = 014423$.
- Use $w_7 = 110$ to insert one 7 into γ to get $\gamma = 0144723$.
- Use $w_8 = 10$ to insert one 8 into γ to get $\gamma = 01447823$.

Thus, the image path D' is the unique 3-Dyck path of height 8 such that $\gamma(D') = (0,1,4,4,7,8,2,3)$. D' is pictured in Figure 2.7.

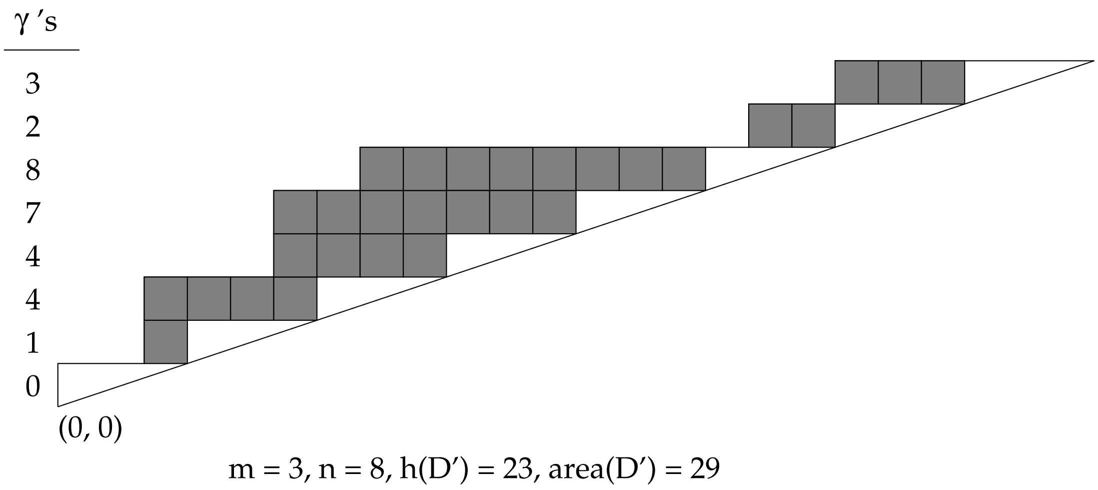


Figure 2.7: The image $\psi(D)$ for the path D from Figure 2.4.

As this example indicates, the presence of vertical moves of length zero does not alter the validity of the preceding proofs and bijections.

Remark 2.20. The main difficulty involved in the combinatorial investigation of the original q,t-Catalan sequence $OC_n(q,t)$ was discovering the two statistics *bounce* and *dinv* defined in §1.4.1 and §1.4.2. The *area* statistic, on the other hand, is quite natural to consider once one notices that $OC_n(1,1)$ counts the number of Dyck paths of height n. Similar comments apply to the higher q,t-Catalan sequences.

Having introduced the bijections ϕ and $\psi = \phi^{-1}$, we can consider the problem of finding these statistics in a new light. It is natural to count Dyck paths (or m-Dyck paths) by constructing the associated γ-sequences through successive insertion of zeroes, ones, twos, etc., as done in §2.2.4. The map ϕ arises by representing the insertion choices geometrically as paths inside rectangles and positioning these rectangles in a nice way (as in Figure 2.5). The remarkable coincidence is that the resulting picture is another m-Dyck path.

We may thus regard the area statistic and the map ϕ as the "most fundamental" concepts. Then the two new statistics h and b can be "guessed" by simply looking at what happens to the area statistic when we apply ϕ (or ϕ^{-1}). We find that ϕ sends area to the bounce statistic b, and ϕ^{-1} sends area to the generalized Haiman statistic h.

This suggests a possible approach to other problems in which there are two variables with the same univariate distribution, but a combinatorial interpretation is only known for one of the variables. Finding a combinatorial interpretation for the Kostka-Macdonald coefficients [24] provides an example of such a problem. There, the q-statistic is known (the so-called "cocharge statistic" on tableaux), but the t-statistic has not been discovered.

2.3 Recursions for $C_n^{(m)}(q,t)$

In this section, we prove several recursions for $C_n^{(m)}(q,t)$ and related sequences. Of course, the same recursions hold for $HC_n^{(m)}(q,t)$. These recursions are more convenient for some purposes than the summation formula given in §2.2.3. As an example, we use the recursion to prove a formula for $C_n^{(m)}(q,1/q)$ which shows that $C_n^{(m)}(q,1/q) = OC_n^{(m)}(q,1/q)$.

2.3.1 Recursion based on Removing the First Bounce

Our goal here is to modify the idea in the proof of Haglund's recursion (see §1.4.5) to get a recursion for $C_n^{(m)}(q,t)$. The main difficulty is that the bounce path depends on the prior bouncing history when $m > 1$, so that we cannot simply remove the first bounce and restart "from scratch." Consequently, we must add more subscripts that keep track of the lengths of the first m vertical moves in the bounce path.

Definition 2.21. Fix $m > 1$. Define $\mathcal{F}^{(m)}_{n;v_0,v_1,\ldots,v_{m-1}}$ to be the collection of m-Dyck paths of height n whose derived bounce paths start with vertical moves of lengths $v_0, v_1, \ldots, v_{m-1}$, in that order. Define $F^{(m)}_{n;v_0,\ldots,v_{m-1}}(q,t)$ to be the sum of $q^{area(D)}t^{b(D)}$ over all paths $D \in \mathcal{F}^{(m)}_{n;v_0,\ldots,v_{m-1}}$. (An empty sum is defined to be zero.) To reduce clutter in formulas, we will generally omit the superscript (m).

We make the following observations about these definitions.

Remark 2.22. (1) If $\mathcal{F}_{n;v_0,v_1,\ldots,v_{m-1}}$ is a nonempty collection of paths, then we must have $v_0 > 0$, $v_i \geq 0$ for $i > 0$, and $v_0 + \cdots + v_{m-1} \leq n$.

(2) If $v_0 = n$ and $v_i = 0$ for $i > 0$, then $\mathcal{F}_{n;n,0,\ldots,0}$ consists of the single path D that goes north n steps and then east mn steps. Hence, $F_{n;n,0,\ldots,0}(q,t) = q^{mn(n-1)/2}t^0$.

(3) Consider the collection $\mathcal{F}_{n+1;1,0,\ldots,0}$. A path D in this collection starts by going north one unit and then east m units (since $v_1 = \cdots = v_{m-1} = 0$). At this point, D has returned to the diagonal $x = my$. If we look at the rest of the path beyond this point, we get an arbitrary m-Dyck path D' of height n. Also, the bounce path for D' is the same as the latter part of the bounce path for D (starting with v_m). Note that the prior history in D is immaterial, since $v_{m-1} = \cdots = v_1 = 0$. See Figure 2.8. We conclude that

$$t^{mn}C_n^{(m)}(q,t) = F_{n+1;1,0,\ldots,0}(q,t).$$

The extra factor of t^{mn} accounts for the contribution of the first m bounces to $b(D)$, which is not present in $b(D')$.

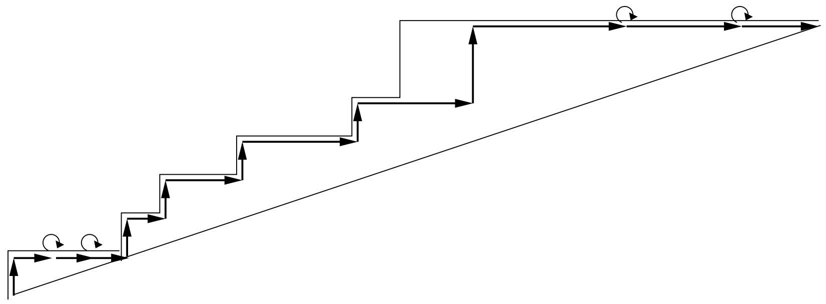


Figure 2.8: Removing a trivial bottom row of an m-Dyck path.

(4) There is a version of the formula (2.8) for $F_{n;v_0,\ldots,v_{m-1}}(q,t)$. Specifically,

$$F_{n;v_0,\ldots,v_{m-1}}(q,t) = \sum_{(v_m,v_{m+1},\ldots)} t^{\sum_{i\geq 0} iv_i} q^{m\sum_{i\geq 0}\binom{v_i}{2}} \prod_{i\geq 1} q^{v_i \sum_{j=1}^{m}(m-j)v_{i-j}} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_q.$$

This equation follows immediately from the combinatorial interpretation of the summation index $v = (v_0, v_1 \ldots)$ appearing in (2.8) as the lengths of the vertical segments in the bounce path. Since $v_0, \ldots, v_{m-1}$ are fixed in advance, we need only sum over the remaining segments $v_m, v_{m+1}, \ldots$.

To state the new recursion, it is convenient to introduce a modified version of the generating functions $F_{n;v_0,\ldots,v_{m-1}}(q,t)$. Intuitively, we need to remove the influence of v_0 on the future bouncing history to obtain a recursion. Assume that $v_0 > 0$ first.

Definition 2.23. For $v_0 > 0$, define $\mathcal{E}_{n;v_0,\ldots,v_{m-1}}$ to be the collection of all m-Dyck paths D of height n with the following properties. First, the bounce path derived from D starts with vertical moves of lengths $v_0,\ldots,v_{m-1}$. Second, the first $m-1$ rectangles $R_1,\ldots,R_{m-1}$ above the bounce path of D (see Figure 2.5) are all empty. This means that the subpath in each rectangle goes all the way east before turning north, so that there are no area cells in the rectangle. Also define

$$E^{(m)}_{n;v_0,\ldots,v_{m-1}}(q,t) = \sum_{D\in\mathcal{E}_{n;v_0,\ldots,v_{m-1}}} q^{area(D)}t^{b(D)}.$$

The case when $v_0 = 0$ is handled in the following remark.

Remark 2.24. By filling the empty rectangles $R_1,\ldots,R_{m-1}$ in a path $D \in \mathcal{E}_{n;v_0,\ldots,v_{m-1}}$ according to the bouncing rules, we deduce that

$$F_{n;v_0,\ldots,v_{m-1}}(q,t) = E_{n;v_0,\ldots,v_{m-1}}(q,t)\prod_{i=1}^{m-1}\begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i,\, v_{i-1}+\cdots+v_{i-m}-1\end{bmatrix}_q \text{ when } v_0>0. \tag{2.13}$$

This relation gives an exact formula for $E_{n;v_0,\ldots,v_{m-1}}(q,t)$ when $v_0 > 0$:

$$E_{n;v_0,\ldots,v_{m-1}}(q,t) = \sum_{(v_m,v_{m+1},\ldots)} t^{\sum_{i\ge 0} iv_i} q^{m\sum_{i\ge 0}\frac{1}{2}v_i(v_i-1)}\prod_{i\ge 1} q^{v_i\sum_{j=1}^{m\wedge i}(m-j)v_{i-j}}\prod_{i\ge m}\begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i,\, v_{i-1}+\cdots+v_{i-m}-1\end{bmatrix}_q. \tag{2.14}$$

Here, we have written $m\wedge i$ to denote the minimum of m and i. Note that the validity of equation (2.14) does not depend on the earlier convention that $v_i = 0$ for all negative i. Now, if $v_0 = 0$, we simply *define* $E_{n;v_0,\ldots,v_{m-1}}(q,t)$ by formula (2.14).

Remark 2.25. It follows from (2.13) that $E_{n+1;1,0,\ldots,0}(q,t) = F_{n+1;1,0,\ldots,0}(q,t)$. Therefore,

$$C^{(m)}_n(q,t) = t^{-mn}E_{n+1;1,0,\ldots,0}(q,t). \tag{2.15}$$

Thus, the higher-order Catalan sequence can be recovered from the E's.

Theorem 2.26. *The generating functions $E_{n;v_0,\ldots,v_{m-1}}(q,t)$ satisfy the recursion:*

$$E_{n;v_0,\ldots,v_{m-1}}(q,t) = t^{n-v_0} q^{m\binom{v_0}{2}} \prod_{i=1}^{m-1} q^{v_0 v_i (m-i)} \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} \begin{bmatrix} v_m+\cdots+v_0-1 \\ v_m,\, v_{m-1}+\cdots+v_0-1 \end{bmatrix}_q E_{n-v_0;v_1,\ldots,v_{m-1},v_m}(q,t). \tag{2.16}$$

The initial conditions are

$$E_{n;n,0,\ldots,0}(q,t) = q^{mn(n-1)/2} t^0$$

$$E_{n;0,0,\ldots,0}(q,t) = 0.$$

Remark 2.27. Observe that we recover Haglund's original recursion when $m=1$.

Proof. We obtain the recursion for $E_{n;v_0,\ldots,v_{m-1}}$ by breaking up the summation in (2.14) based on the value of v_m. Consider a fixed choice of v_m in the range $\{0,1,\ldots,n-v_0-\cdots-v_{m-1}\}$. Write down (2.14) with n replaced by $n-v_0$ and v_k replaced by v_{k+1} for all $k\geq 0$:

$$E_{n-v_0;v_1,\ldots,v_m}(q,t) = \sum_{(v_{m+1},v_{m+2},\ldots)} t^{\sum_{i\geq 0} i v_{i+1}} q^{pow_1} \prod_{i\geq m} \begin{bmatrix} v_{i+1}+v_i+\cdots+v_{i+1-m}-1 \\ v_{i+1},\, v_i+\cdots+v_{i+1-m}-1 \end{bmatrix}_q, \tag{2.17}$$

where

$$pow_1 = m\sum_{i\geq 0}\binom{v_{i+1}}{2} + \sum_{i\geq 1} v_{i+1} \sum_{j=1}^{m\wedge i}(m-j)v_{i+1-j}.$$

Replace i by $i-1$ in this formula to get

$$E_{n-v_0;v_1,\ldots,v_m}(q,t) = \sum_{(v_{m+1},v_{m+2},\ldots)} t^{\sum_{i\geq 1}(i-1)v_i} q^{pow_2} \prod_{i>m} \begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i,\, v_{i-1}+\cdots+v_{i-m}-1 \end{bmatrix}_q, \tag{2.18}$$

where

$$pow_2 = m\sum_{i\geq 1}\binom{v_i}{2} + \sum_{i\geq 2} v_i \sum_{j=1}^{m\wedge(i-1)}(m-j)v_{i-j}.$$

In the original formula for $E_{n;v_0,\ldots,v_{m-1}}$, we can sum over v_m first and then sum over the remaining v_j's. The resulting formula is:

$$E_{n;v_0,\ldots,v_{m-1}}(q,t) = \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} \sum_{(v_{m+1},v_{m+2},\ldots)} t^{\sum_{i\geq 0} i v_i} q^{pow_3} \prod_{i\geq m} \begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i,\, v_{i-1}+\cdots+v_{i-m}-1 \end{bmatrix}_q, \quad \textit{where} \tag{2.19}$$

$$pow_3 = m\sum_{i\geq 0}\binom{v_i}{2} + \sum_{i\geq 1} v_i \sum_{j=1}^{m\wedge i}(m-j)v_{i-j}.$$

To go from the formula in (2.18) to the corresponding summand in (2.19), we need to multiply the former by the expression

$$t^{0v_0+v_1+v_2+v_3+\cdots} q^{m\binom{v_0}{2}} q^{v_1 v_0(m-1)} \prod_{i=2}^{m} q^{v_i(m-i)v_0} \begin{bmatrix} v_m+\cdots+v_0-1 \\ v_m, v_{m-1}+\cdots+v_0-1 \end{bmatrix}_q.$$

Doing this multiplication and adding over all choices of v_m, we obtain the recursion stated in (2.16). The initial conditions follow immediately from the definitions. $\square$

Remark 2.28. It is hoped that (2.16) could be used to prove the conjecture $C_n^{(m)}(q,t) = SC_n^{(m)}(q,t)$. One difficulty is finding the analogues of $E_{n;v_0,\ldots,v_{m-1}}$ in the symmetric function setting. Computer experiments suggest that

$$E_{n;v_0,0,\ldots,0}(q,t) = q^{m\binom{v_0}{2}} t^{(m-1)(n-v_0)}\ \nabla^m(e_{n-v_0}[X(1+q+q^2+\cdots+q^{v_0-1})])|_{s_{1^{n-v_0}}}.$$

In other words, the conjecture is that

$$E_{n;v_0,0,\ldots,0}(q,t) = q^{m\binom{v_0}{2}} t^{(m-1)(n-v_0)}\ (\nabla^m(\phi(e_{n-v_0})))|_{s_{1^{n-v_0}}},$$

where ϕ is the unique extension of the function sending p_k to

$$p_k(1+q^k+q^{2k}+\cdots+q^{k(v_0-1)}),$$

whose existence is guaranteed by Theorem 1.20.

However, we have not found a conjectured formula for the general $E_{n;v_0,\ldots,v_{m-1}}$ in terms of the nabla operator.

Remark 2.29. It is clear that we could perform a similar manipulation of (2.8) to obtain a recursion based on removing the last nontrivial vertical bounce v_s. The inductive proof in §2.2.4 that (2.8) equals $HC_n^{(m)}(q,t)$ was based on this idea. There is a slight added complication because one must know s, not just v_s, to determine the effect of removing the last bounce on $b(D)$. On the other hand, v_s only affects the dimensions of one nontrivial rectangle in Figure 2.5.

2.3.2 Application: A Formula for the Specialization $C_n^{(m)}(q,1/q)$

We now use the recursion of the preceding subsection to derive an exact formula for the specialization

$$E^{(m)}_{n;v_0,\ldots,v_{m-1}}(q,1/q).$$

As an application of this formula, we will prove that

$$q^{mn(n-1)/2}C_n^{(m)}(q,1/q) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q.$$

Garsia and Haiman proved the same formula for $OC_n^{(m)}(q,1/q)$ in [15]. It follows that

$$C_n^{(m)}(q,1/q) = OC_n^{(m)}(q,1/q).$$

Fix m, N, and $v=(v_0,\ldots,v_{m-1})$. Our formula for $E^{(m)}_{N;v}(q,1/q)$ will involve various intermediate quantities A, B, etc., depending on N, m, and v. If the dependence on the variables needs to be made explicit, we will write $A(N,m,v)$, $B(N,m,v)$, etc.

The basic formula is

$$E^{(m)}_{N;v}(q,1/q) = A_0 - B_1 - B_2 - \cdots - B_m, \tag{2.20}$$

where A_0 and each B_j is a certain q-binomial coefficient multiplied by a certain power of q. The precise expressions appear in the next definition.

Definition 2.30. Introduce the following temporary notation, to be used only in this subsection.

$$\begin{aligned}
A = A(N,m,v_0,\ldots,v_{m-1}) &= \begin{bmatrix} (m+1)N-1-\sum_{k=0}^{m-1}(m-k)v_k \\ N-\sum_{k=0}^{m-1}v_k \end{bmatrix}_q \\
B = B(N,m,v_0,\ldots,v_{m-1}) &= \begin{bmatrix} (m+1)N-1-\sum_{k=0}^{m-1}(m-k)v_k \\ N-1-\sum_{k=0}^{m-1}v_k \end{bmatrix}_q \\
P_0 = P_0(N,m,v_0,\ldots,v_{m-1}) &= -\frac{m}{2}(N^2+N) + \left[m\left(\sum_{k=0}^{m-1}v_k\right)-(m-1)\right]N \\
&\quad + \sum_{k=0}^{m-2}(m-1-k)v_k + \sum_{0\le j<k\le m-1}(j-k)v_jv_k \\
P_j = P_j(N,m,v_0,\ldots,v_{m-1}) &= v_{m-1}+(j-1)N \\
&\quad - \sum_{\ell=0}^{m-2}\min(j-1,m-2-\ell)v_\ell \quad (1\le j\le m)
\end{aligned}$$

$$\begin{aligned} A_0 = A_0(N, m, v_0, \ldots, v_{m-1}) &= Aq^{P_0} \\ B_j = B_j(N, m, v_0, \ldots, v_{m-1}) &= Bq^{P_0+P_j} \quad (1 \le j \le m) \end{aligned}$$

Example 2.31. (1) Let $m = 1$ and $v_0 = w$. Then

$$E^{(1)}_{N;w}(q, 1/q) = q^{-(N^2+N)/2+wN} \left(\begin{bmatrix} 2N - w - 1 \\ N - w, N - 1 \end{bmatrix}_q - q^w \begin{bmatrix} 2N - w - 1 \\ N - w - 1, N \end{bmatrix}_q \right).$$

This is exactly formula (1.20) from Theorem 1.56.

(2) Let $m = 2$, $v_0 = w$, $v_1 = x$. Then

$$\begin{aligned} E^{(2)}_{N;w,x}(q, 1/q) &= q^{pow_2} \left(\begin{bmatrix} 3N - 2w - x - 1 \\ N - w - x, 2N - w - 1 \end{bmatrix}_q \right. \\ &\left. -(q^x + q^{N+x}) \begin{bmatrix} 3N - 2w - x - 1 \\ N - w - x - 1, 2N - w \end{bmatrix}_q \right), \\ \text{where } pow_2 &= -(N^2 + N) + (2w + 2x - 1)N - w(x - 1). \end{aligned}$$

(3) Let $m = 3$, $v_0 = w$, $v_1 = x$, $v_2 = y$. Then

$$\begin{aligned} E^{(3)}_{N;w,x,y}(q, 1/q) &= q^{pow_3} \left(\begin{bmatrix} 4N - 3w - 2x - y - 1 \\ N - w - x - y, 3N - 2w - x - 1 \end{bmatrix}_q \right. \\ &\left. -(q^y + q^{y+N-w} + q^{y+2N-w}) \begin{bmatrix} 4N - 3w - 2x - y - 1 \\ N-w-x-y-1, 3N-2w-x \end{bmatrix}_q \right), \end{aligned}$$

where

$$pow_3 = -3(N^2 + N)/2 + (3w + 3x + 3y - 2)N + (y - 1)(-2w - x) - wx.$$

(4) Let $m = 5$ and $(v_0, v_1, v_2, v_3, v_4) = (v, w, x, y, z)$. Then

$$\begin{aligned} P_1 &= z \\ P_2 &= z + N - v - w - x \\ P_3 &= z + 2N - 2v - 2w - x \\ P_4 &= z + 3N - 3v - 2w - x \\ P_5 &= z + 4N - 3v - 2w - x. \end{aligned}$$

Theorem 2.32. *With the notation introduced above, we have:*

$$E^{(m)}_{N;v}(q,1/q) = A_0 - B_1 - B_2 - \cdots - B_m. \tag{2.21}$$

Proof. We will need the following two identities, proved in Lemma 1.54 and Lemma 1.55 from Chapter 1.

$$\sum_{i=0}^{D-E} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \begin{bmatrix} D-i \\ E, D-i-E \end{bmatrix}_q q^{(E+1)i} = \begin{bmatrix} C+D+1 \\ D-E, C+1+E \end{bmatrix}_q. \tag{2.22}$$

$$q^{-C}\left(\begin{bmatrix} C+D \\ C, D \end{bmatrix}_q - \begin{bmatrix} C+D \\ C-1, D+1 \end{bmatrix}_q\right) = \begin{bmatrix} C+D \\ C, D \end{bmatrix}_q - q^{D-C+1}\begin{bmatrix} C+D \\ C-1, D+1 \end{bmatrix}_q. \tag{2.23}$$

To prove (2.21), we need to check that the right side satisfies the same initial conditions and recursion that the specialization $E^{(m)}_{N;v}(q,1/q)$ satisfies.

Step 1. The first step is to check that the right side of (2.21) satisfies the specialized initial conditions

$$E^{(m)}_{N;0,0,\ldots,0}(q,1/q) = 0.$$
$$E^{(m)}_{N;N,0,\ldots,0}(q,1/q) = q^{mN(N-1)/2}$$

Consider the first initial condition, where $v_0 = \cdots = v_{m-1} = 0$. For these values of the v_i's, we obtain

$$\begin{aligned} A &= \begin{bmatrix} mN+N-1 \\ N, mN-1 \end{bmatrix}_q \\ B &= \begin{bmatrix} mN+N-1 \\ N-1, mN \end{bmatrix}_q \\ P_0 &= -\frac{m}{2}(N^2+N) - mN + N \\ P_j &= (j-1)N. \end{aligned}$$

Then the right side of (2.21), namely $A_0 - B_1 - \cdots - B_m$, evaluates to

$$q^{P_0}\left(\begin{bmatrix} mN+N-1 \\ N, mN-1 \end{bmatrix}_q - (1+q^N+q^{2N}+\cdots+q^{(m-1)N})\begin{bmatrix} mN+N-1 \\ N-1, mN \end{bmatrix}_q\right).$$

After factoring, this expression can be written

$$q^{P_0}\frac{[mN+N-1]!_q}{[N-1]!_q[mN-1]!_q}\left(\frac{1}{[N]_q} - \frac{1+q^N+q^{2N}+\cdots+q^{(m-1)N}}{[mN]_q}\right).$$

The parenthesized factor can be written

$$\frac{1-q}{1-q^N} - \frac{(1-q^{mN})/(1-q^N)}{(1-q^{mN})/(1-q)} = 0,$$

and so the whole expression is zero, as desired.

The other initial condition, where $v_0 = N$ and $v_i = 0$ for $i > 0$, is easier to check. From the definitions, we have

$$\begin{aligned} A &= \begin{bmatrix} (m+1)N - 1 - mN \\ N - N \end{bmatrix}_q = 1 \\ B &= \begin{bmatrix} (m+1)N - 1 - mN \\ N - 1 - N \end{bmatrix}_q = 0 \\ P_0 &= -\frac{m}{2}(N^2 + N) + (mN - (m-1))N + (m-1)N \\ &= \frac{mN(N-1)}{2}. \end{aligned}$$

Thus, we immediately calculate

$$A_0 - B_1 - \cdots - B_m = 1q^{P_0} - 0 = q^{mN(N-1)/2}.$$

Step 2. The next step is to check that the right side of (2.21) satisfies the recursion (2.16) with t specialized to $1/q$. After setting $t = 1/q$ and simplifying, this recursion can be written

$$E^{(m)}_{N;v_0,\ldots,v_{m-1}}(q, 1/q) = q^{pow} \sum_{i=0}^{N-v_0-\ldots-v_{m-1}} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q E^{(m)}_{N-v_0;v_1,v_2\ldots,v_{m-1},i}(q, 1/q), \quad (2.24)$$

where

$$\begin{aligned} pow &= v_0 - N + m(v_0^2 - v_0)/2 + \sum_{k=1}^{m-1}(m-k)v_0 v_k \\ \text{and } C &= v_0 + v_1 + \cdots + v_{m-1} - 1. \end{aligned}$$

The proof will be finished if we can show this same relation holds with the E's replaced by the appropriate formulas from the right side of (2.21). Specifically, write A' for $A(N, m, v_0, \ldots, v_{m-1})$, write P'_j for $P_j(N, m, v_0, \ldots, v_{m-1})$, and so forth. Write A'' for $A(N - v_0, m, v_1, \ldots, v_{m-1}, i)$, write P''_j for $P_j(N - v_0, m, v_1, \ldots, v_{m-1}, i)$, and so forth. Then we must show that the quantity

$$A'_0 - B'_1 - B'_2 - \cdots - B'_m \quad (2.25)$$

is equal to the quantity

$$q^{pow} \sum_{i=0}^{N-v_0-\ldots-v_{m-1}} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \left(A_0'' - B_1'' - B_2'' - \cdots - B_m'' \right). \tag{2.26}$$

To show this, we write the latter expression as the sum of $m+1$ smaller expressions, namely

$$q^{pow} \sum_{i=0}^{N-v_0-\ldots-v_{m-1}} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q A_0''$$

and (for $1 \leq j \leq m$)

$$q^{pow} \sum_{i=0}^{N-v_0-\ldots-v_{m-1}} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q (-B_j'').$$

Each of these $m+1$ expressions can be evaluated (see below) using identity (2.22). The resulting sum is *almost* the desired quantity

$$A_0' - B_1' - B_2' - \cdots - B_m'.$$

More specifically, for $2 \leq j \leq m$, the expression involving $-B_j''$ will evaluate to $-B_{j-1}'$. On the other hand, the expression involving $-B_1''$ will evaluate to $-B_m'$ times an unwanted power of q. Similarly, the expression involving A_0'' will evaluate to A_0' times another unwanted power of q. Finally, identity (2.23) will show that these last two terms are in fact equal to $A_0' - B_m'$ without the unwanted powers of q! This will complete the proof of the formula (2.21).

Step 3. We indicate how to evaluate the expression

$$q^{pow} \sum_{i=0}^{N-v_0-\ldots-v_{m-1}} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q (A_0'') \tag{2.27}$$

from Step 2. The final answer will be $q^{-(N-v_0-\cdots-v_{m-1})} A_0'$.

We must first verify the algebraic identity

$$P_0'' + pow = P_0' - (N - v_0 - \cdots - v_{m-1}) + i \left[mN - \sum_{k=0}^{m-1} (m-k) v_k \right]. \tag{2.28}$$

To do this, recall that

$$\begin{aligned} P_0' &= P_0(N, m, v_0, v_1, \ldots, v_{m-1}) \\ &= -\frac{m}{2}(N^2+N) + \left[m \left(\sum_{k=0}^{m-1} v_k \right) - (m-1) \right] N \\ &\quad + \sum_{k=0}^{m-2} (m-1-k) v_k + \sum_{0 \le j < k \le m-1} (j-k) v_j v_k. \end{aligned}$$

Now, $P_0'' = P_0(N - v_0, m, v_1, \ldots, v_{m-1}, i)$. Replacing N by $N - v_0$, v_i by v_{i+1} for $i < m-1$, and v_{m-1} by i in the above formula, we have

$$\begin{aligned} P_0'' &= -\frac{m}{2}((N-v_0)^2 + (N-v_0)) + \left[m \left(\sum_{k=1}^{m-1} v_k + i \right) - (m-1) \right] (N - v_0) \\ &\quad + \sum_{k=0}^{m-2} (m-1-k) v_{k+1} + \sum_{0 \le j < k \le m-2} (j-k) v_{j+1} v_{k+1} \\ &\quad + \sum_{0 \le j < k = m-1} (j-k) v_{j+1} i. \end{aligned}$$

Let us first consider the terms involving i on either side of (2.28). From the above formulas, the terms divisible by i in $P_0'' + pow$ are

$$\begin{aligned} im(N-v_0) + \sum_{j=0}^{m-2} (j-(m-1)) v_{j+1} i &= im(N-v_0) + \sum_{k=1}^{m-1} (k-m) v_k i \\ &= i \left[mN - \sum_{k=0}^{m-1} (m-k) v_k \right]. \end{aligned}$$

Exactly the same terms involve i on the right side of (2.28).

Next, consider the terms involving $v_j v_k$ on either side of (2.28), where $0 \le j < k \le m-1$. If $j > 0$, comparison of the above formulas for P_0' and P_0'' shows that the term $(j-k)v_j v_k$ appears on each side. If $j = 0$, the coefficient of $v_0 v_k$ in $P_0'' + pow$ is $-m + (m-k) = -k$, while the coefficient on the other side (coming from P_0') is $0 - k = -k$. Thus, each side of (2.28) has the term $-k v_0 v_k$.

One can similarly verify that the terms involving other combinations of the parameters v_j and N are the same on either side of (2.28). These routine algebraic verifications will be left to the reader.

Using identity (2.28) and expanding the definition of A_0'', the expression (2.27) can be written

$$q^{P_0'-(N-v_0-\cdots-v_{m-1})}\sum_{i=0}^{D-E}\begin{bmatrix}C+i\\C,i\end{bmatrix}_q\begin{bmatrix}D-i\\E,D-E-i\end{bmatrix}_q q^{i(E+1)},$$

where

$$\begin{aligned}D &= (m+1)N-1-\sum_{k=0}^{m-1}(m+1-k)v_k,\\ E &= mN-1-\sum_{k=0}^{m-1}(m-k)v_k,\ \text{and}\\ D-E &= N-v_0-v_1-\cdots-v_{m-1}.\end{aligned}$$

Using the identity (2.22), this new expression becomes

$$q^{-(N-v_0-\cdots-v_{m-1})}q^{P_0'}\begin{bmatrix}C+D+1\\D-E,C+1+E\end{bmatrix}_q = q^{-(N-v_0-\cdots-v_{m-1})}A_0'.$$

Step 4. We indicate how to evaluate the expression

$$q^{pow}\sum_{i=0}^{N-v_0-\ldots-v_{m-1}}\begin{bmatrix}C+i\\C,i\end{bmatrix}_q(-B_j'') \tag{2.29}$$

from Step 2. The answer will be $-B_{j-1}'$ for $j>1$; it will be $-B'q^{P_0'-(N-v_0-\cdots-v_{m-1})}$ for $j=1$.

The calculation is similar to the one in Step 3. Using the definition of B_j'' and and the identity (2.28), we can rewrite (2.29) as

$$-q^{P_0'-(N-v_0-\cdots-v_{m-1})}\sum_{i=0}^{D-E+1}\begin{bmatrix}C+i\\C,i\end{bmatrix}_q\begin{bmatrix}D-i\\E,D-E-i\end{bmatrix}_q q^{iE}q^{P_j''}, \tag{2.30}$$

where we now set

$$\begin{aligned}D &= (m+1)N-1-\sum_{k=0}^{m-1}(m+1-k)v_k,\\ E &= mN-\sum_{k=0}^{m-1}(m-k)v_k,\ \text{and}\\ D-E &= N-v_0-v_1-\cdots-v_{m-1}-1.\end{aligned}$$

The summand where $i = D - E + 1$ is zero, so we may adjust the upper limit of the sum to be $i = D - E$ instead. To continue simplifying, one must first verify the identity

$$P'_{j-1} = P''_j - i - (N - v_0 - \cdots - v_{m-1}) \quad (j > 1).$$

The proof of this identity involves algebraic manipulations similar to those used to prove (2.28). These manipulations will be left to the reader.

Assume $j > 1$ first. Using the last identity to eliminate P''_j, the expression (2.30) becomes

$$-q^{P'_0+P'_{j-1}} \sum_{i=0}^{D-E} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \begin{bmatrix} D-i \\ E, D-E-i \end{bmatrix}_q q^{i(E+1)}.$$

Using the identity (2.22), the sum (without the outside power of q) evaluates to B'. Thus, when $j > 1$, the expression (2.29) evaluates to $-B'_{j-1}$ as claimed.

Now assume $j = 1$. Since $P''_1 = i$, the expression (2.30) becomes

$$-q^{P'_0-(N-v_0-\cdots-v_{m-1})} \sum_{i=0}^{D-E} \begin{bmatrix} C+i \\ C, i \end{bmatrix}_q \begin{bmatrix} D-i \\ E, D-E-i \end{bmatrix}_q q^{i(E+1)}.$$

Using the identity (2.22), this becomes

$$-q^{P'_0-(N-v_0-\cdots-v_{m-1})} B'$$

as claimed.

Step 5. Let us recap the preceding calculations. We have evaluated the expression (2.26), hoping to obtain the answer

$$(A'_0 - B'_m) - B'_1 - B'_2 - \cdots - B'_{m-1}$$

from (2.25). Instead, we obtained the answer

$$q^{-(N-v_0-\cdots-v_{m-1})}(A' q^{P'_0} - B' q^{P'_0}) - B'_1 - B'_2 - \cdots - B'_{m-1}.$$

Now, use the identity (2.23), setting

$$\begin{aligned} C &= N - v_0 - \cdots - v_{m-1} \text{ and} \\ D &= mN - 1 - \sum_{k=0}^{m-1} (m-1-k) v_k. \end{aligned}$$

The result is

$$q^{-(N-v_0-\cdots-v_{m-1})}(A'-B') = A' - q^{P'_m}B',$$

since one can check that $P'_m = D - C + 1$ here. Multiplying by $q^{P'_0}$, we see that

$$q^{-(N-v_0-\cdots-v_{m-1})}(A'q^{P'_0} - B'q^{P'_0}) = (A'_0 - B'_m),$$

so that (2.26) does indeed evaluate to the desired answer (2.25). This completes the proof. $\square$

Corollary 2.33. *For all $m, n \geq 1$,*

$$q^{mn(n-1)/2}C_n^{(m)}(q, 1/q) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q.$$

In particular, $C_n^{(m)}(q, 1/q) = OC_n^{(m)}(q, 1/q)$.

Proof. From (2.15) with $t = 1/q$, we have

$$C_n^{(m)}(q, 1/q) = q^{mn}E^{(m)}_{n+1;1,0,\ldots,0}(q, 1/q).$$

Now, we use the formula just proved for the E's with $N = n+1$, $v_0 = 1$, and $v_i = 0$ for $i > 0$. The reader may verify that, with these substitutions, we obtain

$$q^{mn(n-1)/2}C_n^{(m)}(q, 1/q) =$$
$$q^{n-nm}\left(\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q - \begin{bmatrix} mn+n \\ n-1, mn+1 \end{bmatrix}_q \cdot \left(\sum_{j=0}^{m-1} q^{nj+\chi(j=m-1)}\right)\right).$$

The expression in the curly braces can be written

$$\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q \cdot \left(1 - \frac{[n]_q \sum_{j=0}^{m-1} q^{nj+\chi(j=m-1)}}{[mn+1]_q}\right),$$

which in turn simplifies to

$$\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q \cdot \left(\frac{\sum_{k=0}^{mn} q^k - \sum_{k=0}^{mn} q^k\chi(k \neq mn-n)}{[mn+1]_q}\right) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q q^{mn-n}.$$

The leftover power of q is exactly what is needed to cancel the outside power q^{n-nm}. Thus, we obtain the desired result

$$q^{mn(n-1)/2}C_n^{(m)}(q, 1/q) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q.$$

The last assertion of the corollary follows from the identity

$$q^{mn(n-1)/2}OC_n^{(m)}(q,1/q) = \frac{1}{[mn+1]_q}\begin{bmatrix} mn+n \\ mn, n \end{bmatrix}_q.$$

proved by Garsia and Haiman in [15]. □

2.3.3 Recursions for $C_n^{(m)}(q,t)$ based on Removing the Last Row

We now present one more recursion that is not based directly on formula (2.8). This recursion is simpler in form than (2.16) because it has only four terms. However, one must keep track of several new statistics in this recursion.

We first introduce some temporary notation.

Definition 2.34. Let D be an m-Dyck path of height n. Let the bounce path of D have successive vertical moves $(v_0, v_1, \ldots, v_s)$ and horizontal moves $(h_0, h_1, \ldots)$ as usual. Here, v_s is the last nonzero vertical move. Define

$$\begin{aligned}
Q(D) &= area(D); \\
T(D) &= b(D); \\
Y(D) &= s; \\
Z_i(D) &= v_{s-i} \text{ for } i \geq 0; \\
K(D) &= \text{the total number of area cells in the top row of } D; \\
W(D) &= \text{the number of area cells in the top row of } D \\
&\quad \text{left of the last vertical move of the bounce path.}
\end{aligned}$$

Thus, $Y(D)$ is one less than the total number of bounces needed to reach the top rim; the statistics $Z_i(D)$ record the history of vertical moves near the end of the bounce path; and $W(D)$ counts the number of "extra" cells in the top row left of the bounce path. Define $\mathcal{D}_{n,k}^{(m)}$ to be the collection of paths $D \in \mathcal{D}_n^{(m)}$ with $K(D) = k$, for $0 \leq k \leq m(n-1)$.

Finally, define

$$C_{n,k}(q,t,y,z_0,\ldots,z_{m-1},w) = \sum_{D \in \mathcal{D}_{n,k}^{(m)}} q^{Q(D)}t^{T(D)}y^{Y(D)}w^{W(D)}\prod_{i=0}^{m-1} z_i^{Z_i(D)}. \qquad (2.31)$$

(We suppress the dependence on m from the notation.)

Example 2.35. The 2-Dyck path E in Figure 2.3 has $Q(E) = 41$, $T(E) = 30$, $Y(E) = 5$, $Z_0(E) = 3$, $Z_1(E) = 1$, $W(E) = 1$, and $K(E) = 8$. The 3-Dyck path D in Figure 2.4 has $Q(D) = 23$, $T(D) = 29$, $Y(D) = 8$, $Z_0(D) = 1$, $Z_1(D) = 1$, $Z_2(D) = 0$, $W(D) = 0$, and $K(D) = 5$.

Theorem 2.36. *For all m and n and all k with $0 \leq k < m(n-1)$, we have the recursion*

$$\begin{aligned} C_{n,k}(q,t,y,\vec{z},w) = & z_0 q^k C_{n-1,k-m}(q,t,ty,\vec{z},w) \\ & + q^{-1}w^{-1}\left(C_{n,k+1}(q,t,y,\vec{z},w) - C_{n,k+1}(q,t,y,\vec{z},0)\right) \\ & + q^{-1}tyz_0z_1^{-1}w^{-2}C_{n,k+1}(q,t,y,wz_1,wz_2,\ldots,wz_{m-1},w,0). \end{aligned} \tag{2.32}$$

The initial condition is

$$C_{n,m(n-1)}(q,t,y,z_0,\ldots,z_{m-1},w) = q^{mn(n-1)/2}z_0^n.$$

This recursion and initial condition uniquely determine the multivariable generating functions $C_{n,k}$.

Proof. Consider the initial condition first. If $k = m(n-1)$, there is only one path $D_0 \in \mathcal{D}^{(m)}_{n,m(n-1)}$, which goes north n steps and then east mn steps. We obtain

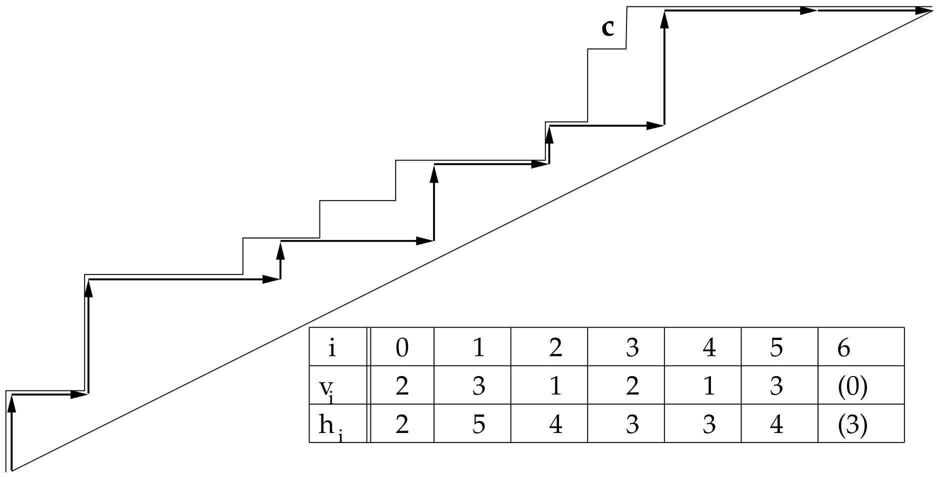
$$C_{n,m(n-1)}(q,t,y,z_0,\ldots,z_{m-1},w) = q^{mn(n-1)/2}z_0^n,$$

since, by inspection, D_0 has area $mn(n-1)/2$ and a single nontrivial bounce of height n.

Write $\vec{z}$ to denote $(z_0,\ldots,z_{m-1})$. We now give a combinatorial proof of (2.32). The idea is to classify a path $D \in \mathcal{D}^{(m)}_{n,k}$ based on what happens at the left edge of the top row of D. Exactly one of the following three cases must occur:

- **Case 1:** The path D reaches the top row by taking two consecutive north steps. See Figure 2.3 for an example.
- **Case 2:** The path D reaches the top row by taking a north step preceded by an east step, AND this east step did not block the progress of the next-to-last vertical bounce move. This means that adding one more area cell to the top row of D would not change the derived bounce path. See Figure 2.9 for an example.

- **Case 3:** The path D reaches the top row by taking a north step preceded by an east step, AND this east step did block the progress of the next-to-last vertical bounce move. This means that adding one more area cell to the top row of D would enable the next-to-last bounce to reach the top rim, so that the total number of bounces would decrease by one. See Figure 2.4 for an example.



i	0	1	2	3	4	5	6
v_i	2	3	1	2	1	3	(0)
h_i	2	5	4	3	3	4	(3)

m = 2, n = 12, area(D) = 42, b(D) = 30

Figure 2.9: A path satisfying case 2 in the recursion analysis.

The three terms on the right side of (2.32) are the respective generating functions for the paths in the three cases above.

To see this, first consider paths satisfying Case 1. We can uniquely construct each such path D by first picking a path D' of height $n-1$ with $k-m$ area cells in row $n-1$, and then placing k new area cells in row n to obtain D. See Figures 2.10 and 2.3 (where $D = E$). The generating function for the choice of D' is $C_{n-1,k-m}(q,t,y,\vec{z},w)$. Adding the new row influences the statistics as follows. The power of q increases by k since we added k new area cells. Let $(v'_0,\ldots,v'_{s'})$ be the vertical moves in the bounce path for D'. It is clear from Figure 2.10 that the bounce path of D will have vertical moves $(v_0,\ldots,v_s)$, where $s = s'$, $v_i = v'_i$ for $i < s$, and $v_s = v'_{s'}+1$. Since only the last vertical move changed, all horizontal moves before reaching the top rim are the same. Since $v_s = v'_{s'} + 1$, the power of z_0 should increase by one when we pass from D' to D. Since $v_i = v'_i$ for $i < s$, the powers of $z_1, z_2, \dots$ should not change. Similarly, since $s = s'$, the power of y does not change in the passage from D' to D. The power of w does not change either, since there are the same number of extra cells left of the last vertical move after adding the new row. Finally, we have $b(D) = \sum_{i\geq 0} iv_i = \sum_{i\geq 0} iv'_i + s = b(D') + s$, since $v_s = v'_s + 1$. We can increase the power of t in the generating function by exactly s if we replace y by ty in $C_{n-1,k-m}(q,t,y,\vec{z},w)$. To see this, recall that $Y(D') = s' = s$ and compare to definition (2.31) [with D there replaced by D']. Putting all this together, we see that the generating function for paths in Case 1 is precisely $z_0 q^k C_{n-1,k-m}(q,t,ty,\vec{z},w)$.

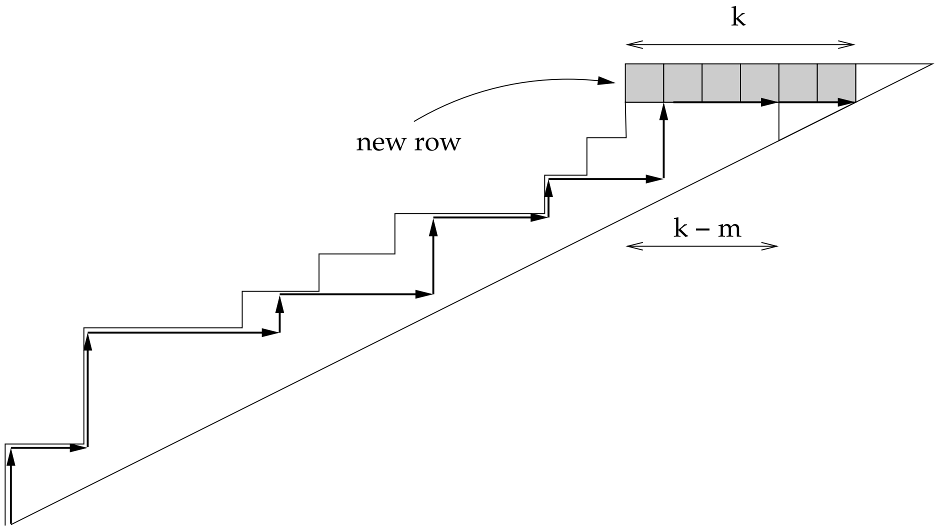


Figure 2.10: Constructing a path in Case 1 by adding a row.

We will treat the next two cases together. Note that all paths D satisfying Case 2 or Case 3 can be uniquely constructed by choosing a path $D' \in \mathcal{D}^{(m)}_{n,k+1}$ and then *removing* the leftmost area cell in the top row of D'. The generating function for the paths D' is $C_{n,k+1}(q,t,y,\vec{z},w)$. However, to determine the effect of the cell removal on the bounce statistic, we must know whether the removed cell was an "extra" cell or one that was part of the bounce path. This complication forces the introduction of two separate cases.

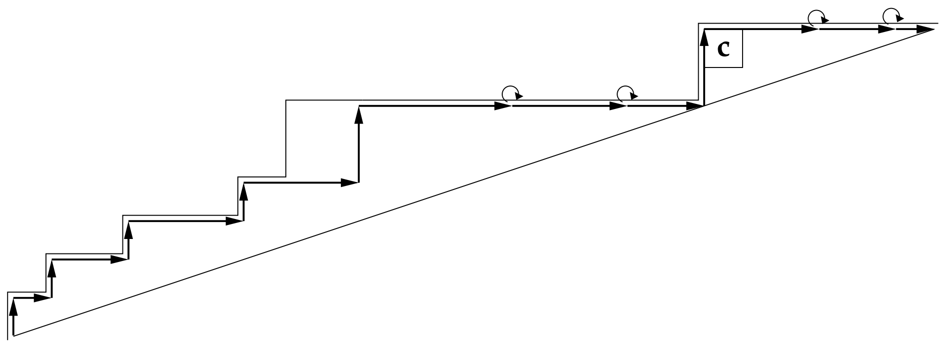
If $w(D') = 0$, then D' has no extra area cells in its top row. The path D constructed from D' therefore belongs to case 3. Consider the definition (2.31) with D

replaced by D' and k replaced by $k+1$. If we substitute $w=0$ in that definition (with the usual convention that $0^0=1$), we are left with the generating function for just those paths D' with $w(D')=0$. By the sum rule, the generating function for just those paths D' with $w(D')>0$ must be $C_{n,k+1}(q,t,y,\vec{z},w)-C_{n,k+1}(q,t,y,\vec{z},0)$.

In case 2, we start with a path D' counted by the latter generating function. For example, D' could be the path D shown in Figure 2.9 with the cell c adjoined. To go from D' to D, we remove the cell in position c. This clearly decreases $Q(D')$ and $W(D')$ by 1, but does not affect the other statistics that are determined by the bounce path. It immediately follows that the generating function for the paths D in case 2 is

$$q^{-1}w^{-1}\left(C_{n,k+1}(q,t,y,\vec{z},w)-C_{n,k+1}(q,t,y,\vec{z},0)\right)$$

To get a path D belonging to case 3, on the other hand, we must have started with a path D' such that $w(D')=0$. For example, the path D' in Figure 2.11 is used to construct the path D in Figure 2.4.



i	0	1	2	3	4	5	6	7	8	9
v_i	1	1	1	1	2	0	0	2	(0)	(0)
h_i	1	2	3	3	4	3	2	2	(2)	(2)

Figure 2.11: Constructing a path in Case 3 by deleting one cell.

The generating function for the choice of D' is $C_{n,k+1}(q,t,y,\vec{z},0)$. We obtain D from D' by removing the leftmost area cell c in the top row of D'. To see how this affects the statistics, compare Figure 2.11 to Figure 2.4. Clearly, the area $Q(D)=Q(D')-1$ because of the removed cell. Let $(v'_0,\ldots,v'_{s'})$ be the lengths of the vertical moves in the bounce path for D'; let $(v_0, \ldots, v_s)$ be the lengths of the vertical moves in the bounce path for D. In this case, removing the cell forces the last vertical move in D' to be shortened by 1 unit, so that there must be a new vertical move of length 1 afterwards in D. Thus, $v_i = v'_i$ for $i < s'$, $v_{s'} = v'_{s'} - 1$, $s = s' + 1$, and $v_s = 1$. We find that $b(D) - b(D') = (s'+1)\cdot 1 - s' \cdot 1 = 1$, so that the bounce statistic has increased by 1. We also have $Y(D) = Y(D') + 1$, $Z_0(D) = 1$, $Z_1(D) = Z_0(D') - 1$, and $Z_i(D) = Z_{i-1}(D')$ for $i \geq 2$. Finally, we must compute the new value $W(D)$. After the bounce path for D takes the vertical step of length $v_{s'} = v'_{s'} - 1$ (this step is blocked by the east step introduced by the removed cell), the bounce path moves east

$$Z_1(D) + Z_2(D) + \cdots + Z_m(D) = Z_0(D') - 1 + Z_1(D') + \cdots + Z_{m-1}(D') \text{ units.}$$

All the area cells above this horizontal move were present in D'; in D, all these cells exist except the leftmost cell c. This implies that

$$W(D) = Z_0(D') + \cdots + Z_{m-1}(D') - 2.$$

Consider the last term on the right side of (2.32):

$$q^{-1}tyz_0z_1^{-1}w^{-2}C_{n,k+1}(q,t,y,wz_1,wz_2,\ldots,wz_{m-1},w,0).$$

By the definition in (2.31) and the comments above,

$$C_{n,k+1}(q,t,y,z_0,\ldots,z_{m-1},0) = \sum_{D' \text{as in Case 3}} q^{Q(D')}t^{T(D')}y^{Y(D')}\prod_{i=0}^{m-1} z_i^{Z_i(D')}.$$

Therefore, making the indicated substitutions for the variables,

$$\begin{aligned}
&q^{-1}tyz_0z_1^{-1}w^{-2}C_{n,k+1}(q,t,y,wz_1,wz_2,\ldots,wz_{m-1},w,0)\\
&= \sum_{D'} q^{Q(D')-1}t^{T(D')+1}y^{Y(D')+1}z_0^1 z_1^{Z_0(D')-1}z_2^{Z_1(D')}\cdots z_{m-1}^{Z_{m-2}(D')}w^{pow}\\
&\quad (\text{where } pow = Z_0(D') + \cdots + Z_{m-2}(D') + Z_{m-1}(D') - 2)\\
&= \sum_{D} q^{Q(D)}t^{T(D)}y^{Y(D)}z_0^{Z_0(D)}\cdots z_{m-1}^{Z_{m-1}(D)}w^{W(D)},
\end{aligned} \tag{2.33}$$

where the sums extend over the paths D' and D appearing in the description of case 3 above. Thus, the third term in (2.32) is the correct generating function for the paths belonging to case 3. This completes the proof of the recursion. □

Remark 2.37. The given recursion (2.32) keeps track of the last m vertical bounces $Z_0(D), \dots, Z_{m-1}(D)$. This is necessary to determine what happens to the other statistics in certain cases. Though it is not necessary here, we clearly could add even more variables $z_m, \dots$ to keep track of the earlier bounce moves $Z_m(D), \dots$ if we wished. Later (§2.4), we shall consider a more general recursion in which it becomes necessary to keep track of $Z_m(D)$.

Remark 2.38. A similar recursion can be proved for a suitable generalization of $HC_n^{(m)}$. We do not give the details of the proof, which are quite messy, but merely list the appropriate reinterpretations of the statistics. In this setting, one should take

$$\begin{aligned}
Q(D) &= h(D);\\
T(D) &= area(D);\\
Y(D) &= \max_{0\le i<n} \gamma_i(D);\\
Z_i(D) &= |\{j : \gamma_j(D) = Y(D) - i\}| \text{ for } i \ge 0;\\
K(D) &= h(D) - h(D'), \text{where } D' \text{ is obtained from } D \text{ by}\\
&\quad\ \text{removing the rightmost value } Y(D);\\
W(D) &= \text{the number of symbols in } \{Y(D)-1, \dots, Y(D)-m\} \text{ appearing}\\
&\quad\ \text{in } \gamma(D) \text{ after the last occurrence of } Y(D).
\end{aligned}$$

This gives an alternate way of proving that $C_n^{(m)}(q,t) = HC_n^{(m)}(q,t)$.

2.4 Trivariate Catalan Sequences

We now introduce three-variable sequences $C_n^{(m)}(q,t,r)$ that generalize the higher q,t-Catalan sequences. Recall from Chapter 1 that

$$\begin{bmatrix} c+d \\ c \end{bmatrix}_{x,y} = \begin{bmatrix} c+d \\ c,d \end{bmatrix}_{x,y} = \sum_{w\in R(0^c1^d)} x^{inv(w)} y^{coinv(w)} = \sum_{P\in\mathcal{R}(c,d)} x^{\tilde{a}(p)} y^{a(p)}.$$

Definition 2.39. We introduce a new statistic $area'$ on m-Dyck paths D of height n as follows. Given D, draw the bounce path of D and the associated rectangles R_i as in Figure 2.5. Let R_i' denote the rectangle R_i without its leftmost column. Define $area'(D)$

to be the number of complete cells below the bounce path of D plus the number of cells inside the rectangles R_i' and *above* the path D. By contrast, $area(D)$ is the number of complete cells below the bounce path of D plus the number of cells inside the rectangles R_i' and *below* the path D. For each m and n, define

$$C_n^{(m)}(q,t,r) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{area(D)} t^{b(D)} r^{area'(D)}.$$

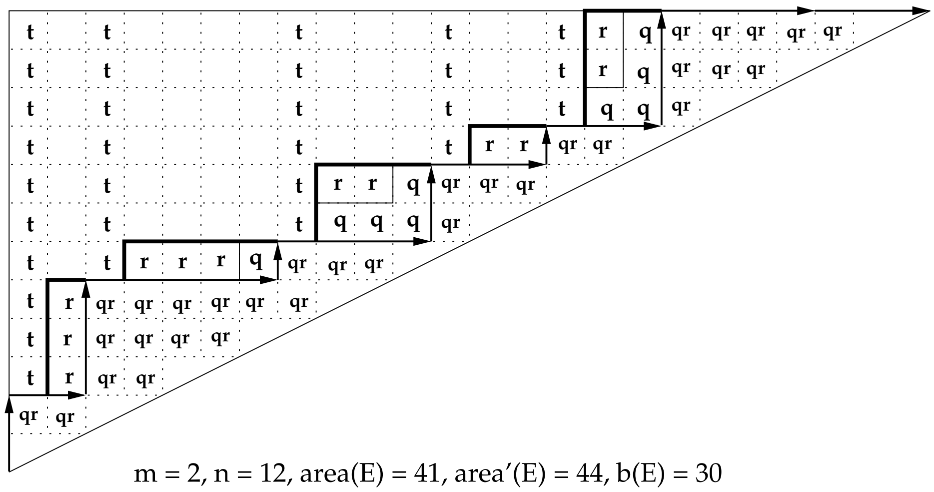


Figure 2.12: Visualizing the three statistics as counting cells.

See Figure 2.12 for an example. In this figure, cells below the bounce path contributing to both $area$ and $area'$ are labelled by their weight qr. Cells above the bounce path but below the m-Dyck path contribute only to $area$ and are labelled q. Cells inside the rectangles R_i' but above the m-Dyck path are labelled r. Finally, Figure 2.12 shows how we can interpret the bounce statistic $b(D)$ as counting certain cells in the picture as well. Specifically, we label each cell in the column above a vertical bounce move with t. Equation (2.6) shows that the number of such factors t is exactly $b(D)$.

Theorem 2.40 (Symmetry between $area$ and $area'$).

$$C_n^{(m)}(q,t,r) = C_n^{(m)}(r,t,q).$$

Proof. The proof can be read off from Figure 2.12. For, we can interchange the number of cells labelled q and the number of cells labelled r by merely rotating the contents of each shortened rectangle R'_i by $180°$. Note that this rotation will not affect the bounce path, since it does not affect the leftmost columns of the full rectangles R_i. The image of the path in Figure 2.12 under this involution is shown in Figure 2.13. $\square$

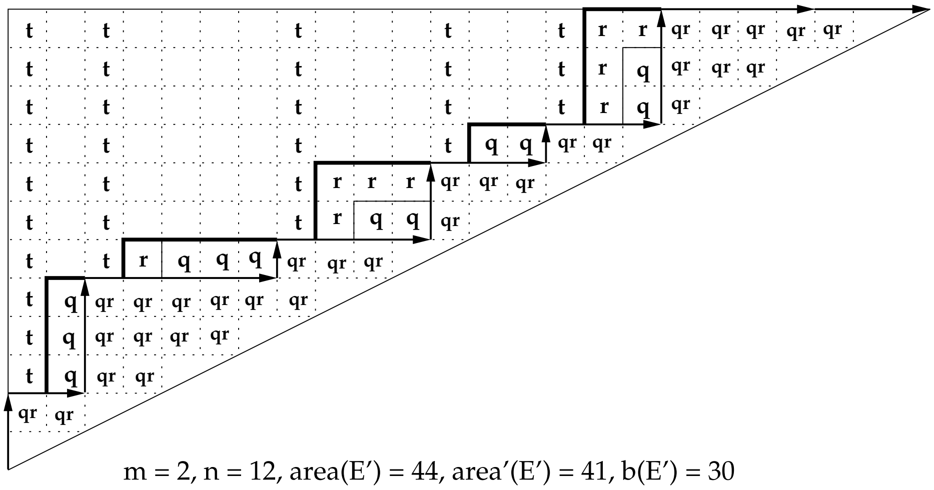


Figure 2.13: Interchanging $area$ and $area'$ by flipping rectangles.

It is easy to incorporate $area'$ into formula (2.8).

Theorem 2.41. *We have*

$$C_n^{(m)}(q,t,r) = \sum_{v\in\mathcal{V}_n^{(m)}} t^{\sum_{i\geq 0} iv_i}(qr)^{m\sum_{i\geq 0}\frac{1}{2}v_i(v_i-1)}\prod_{i\geq 1}(qr)^{v_i\sum_{j=1}^{m}(m-j)v_{i-j}}\begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i, v_{i-1}+\cdots+v_{i-m}-1\end{bmatrix}_{q,r}. \tag{2.34}$$

Proof. The new formula follows by recalling that the factors $\left[\begin{smallmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-1 \\ v_i, v_{i-1}+\cdots+v_{i-m}-1\end{smallmatrix}\right]_q$ keep track of the area cells below the path in the rectangles R'_i, whereas the remaining powers of q in (2.8) count the cells below the bounce path. Hence, to keep track of $area'$, it suffices to replace the latter occurrences of q by qr and to use q,r-binomial coefficients in place of q-binomial coefficients. $\square$

The recursion in §2.3.1 is also easily modified.

Definition 2.42. Define

$$E_{n;v_0,\dots,v_{m-1}}(q,t,r) = \sum_{(v_m,v_{m+1},\dots)} t^{\sum_{i\geq 0} iv_i}(qr)^{pow_1} r^{pow_2} \prod_{i\geq m} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_{q,r}, \tag{2.35}$$

where

$$pow_1 = m \sum_{i\geq 0} \frac{1}{2} v_i(v_i - 1) + \sum_{i\geq 1} v_i \sum_{j=1}^{m\wedge i} (m-j)v_{i-j}, \text{ and}$$

$$pow_2 = \sum_{i=1}^{m-1} v_i \left(\left(\sum_{j=0}^{i-1} v_j \right) - 1 \right).$$

The extra power r^{pow_2} accounts for the cells in the first $m-1$ rectangles R_i', which all contribute to the r-statistic.

Theorem 2.43. *We have the recursion*

$$E_{n;v_0,\dots,v_{m-1}}(q,t,r) = t^{n-v_0}(qr)^{pow_3} \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} r^{pow_4} \begin{bmatrix} v_m + \cdots + v_0 - 1 \\ v_m, v_{m-1} + \cdots + v_0 - 1 \end{bmatrix}_{q,r} E_{n-v_0;v_1,\dots,v_{m-1},v_m}(q,t,r), \tag{2.36}$$

where

$$pow_3 = m\binom{v_0}{2} + \sum_{i=1}^{m-1} v_0 v_i (m-i), \text{ and}$$

$$pow_4 = v_0(v_1 + \cdots + v_{m-1}) - v_1 - v_m(v_{m-1} + \cdots + v_0 - 1).$$

The initial condition is

$$E_{n;n,0,\dots,0}(q,t,r) = (qr)^{mn(n-1)/2} t^0.$$

Proof. The proof is the same as for the first recursion. It is easy to see that the powers of r are correct. ∎

The recursion from §2.3.3 requires a bit more work.

Definition 2.44. For an m-Dyck path D, define $R(D) = area'(D)$, and set

$$C_{n,k}(q,r,t,y,z_0,\ldots,z_m,w) = \sum_{D\in\mathcal{D}^{(m)}_{n,k}} q^{Q(D)}r^{R(D)}t^{T(D)}y^{Y(D)}w^{W(D)}\prod_{i=0}^{m} z_i^{Z_i(D)}. \qquad (2.37)$$

Observe that this generating function, unlike the original, keeps track of $Z_m(D)$ as well as $Z_i(D)$ for $i < m$. We need to make one technical adjustment in the definition of Z_m. If D_0 is the special path that goes north n steps and east mn steps, set $Z_m(D_0) = 1$; for all other paths, define $Z_m(D)$ as in §2.3.3.

Theorem 2.45. *If $0 \le k < m(n-1)$, then*

$$\begin{aligned} &C_{n,k}(q,r,t,y,\vec{z},w) = \\ &\quad z_0 q^k r^{k-1} C_{n-1,k-m}(q,r,t,ty,z_0,rz_1,\ldots,rz_m,r^{-2}w) \\ &\quad + q^{-1}w^{-1}r^{+1}\left(C_{n,k+1}(q,r,t,y,\vec{z},w) - C_{n,k+1}(q,r,t,y,\vec{z},0)\right) \\ &\quad + q^{-1}tyz_0z_1^{-1}w^{-2}r^2C_{n,k+1}(q,r,t,y,r^{-1}wz_1,r^{-2}wz_2,\ldots,r^{-2}wz_m,r^{-1},0). \end{aligned} \qquad (2.38)$$

The initial condition is

$$C_{n,k}(q,r,t,y,\vec{z},w) = (qr)^{mn(n-1)/2}z_0^n z_m^1 \text{ when } k = m(n-1).$$

Proof. The initial condition follows easily from the preceding definition. To verify the new recursion, we need only check the correctness of the powers of r and z_m. We look at three cases, as in §2.3.3. In case 1, we go from $D' \in \mathcal{D}^{(m)}_{n-1,k-m}$ to $D \in \mathcal{D}^{(m)}_{n,k}$ by adding a new top row with k area cells. By definition, $Z_m(D') = Z_m(D)$. [Note that the technical adjustment made to $Z_m(D_0)$ has no effect here, since $k < m(n-1)$ implies that $(k-m) < m((n-1)-1)$, hence $D' \neq D_0$ and $D \neq D_0$.] What happens to $area'$ when we pass from D' to D? In the new top row, $k - W(D)$ of the k new area cells are below the bounce path for D, hence contribute to $area'$. The last rectangle R_s has also gained a new top row, which contains $h_{s-1} = v_{s-1} + \cdots + v_{s-m}$ cells. Of these cells, the one in the leftmost column does not count towards $area'$, nor do the $W(D)$ new cells below the path D. These observations explain why we replace $z_1,\ldots,z_m$ by $rz_1,\ldots,rz_m$ (leaving z_0 alone) and multiply by r^{k-1} in the term

$$z_0q^kr^{k-1}C_{n-1,k-m}(q,r,t,ty,z_0,rz_1,\ldots,rz_m,r^{-2}w).$$

For, the net gain in the power of r is

$$Z_1(D') + \cdots + Z_m(D') + k - 1 - 2W(D') =$$
$$v_{s-1}(D) + \cdots + v_{s-m}(D) + (k - W(D)) - (W(D) + 1),$$

as required.

The term from Case 2, namely

$$q^{-1}w^{-1}r^{+1}\left(C_{n,k+1}(q,r,t,y,\vec{z},w) - C_{n,k+1}(q,r,t,y,\vec{z},0)\right),$$

is the easiest to derive. Recall that we go from D' to D by removing the leftmost ordinary area cell in the top row of D', which is not below the bounce path of D' or D. But "removing" this cell from D' causes the cell to contribute to $area'$ instead, since it belonges to one of the rectangles R' and is now above D. Thus, we have an extra factor r^{+1} in the generating function. As for z_m, note that $D' \neq D_0$ since $W(D') > 0 = W(D_0)$, and $D \neq D_0$ since $k < k+1 \leq m(n-1)$. Thus, $Z_m(D') = Z_m(D)$.

Finally, consider the term from Case 3, namely

$$q^{-1}tyz_0z_1^{-1}w^{-2}r^2C_{n,k+1}(q,r,t,y,r^{-1}wz_1,r^{-2}wz_2,\ldots,r^{-2}wz_m,r^{-1},0).$$

In this case, we go from D' to D by removing the leftmost ordinary area cell c in the top row of D', causing a change in the end of the bounce path. (See Figures 2.11 and 2.4.) Specifically, the bounce path of D has a new terminating vertical move v_s of length 1, and the previous vertical move v_{s-1} is one less than the corresponding move v'_{s-1} in D'. Note that the top row of the last rectangle R^*_{s-1} in D' does not belong to the rectangle R_{s-1} in D. Every cell in the top row of R^*_{s-1}, except the leftmost one, contributed to $R(D')$, because $w(D') = 0$. The number of contributing cells is one less than the horizontal dimension of R^*_{s-1}; this dimension is $Z_1(D') + \cdots + Z_m(D')$. The conclusion is that $R(D)$ drops by

$$(Z_1(D') + \cdots + Z_m(D') - 1) \tag{2.39}$$

as a result of the lost row in R_{s-1}.

On the other hand, consider cells in the top row of D' that are to the right of the bounce path in D'. After removing cell c from D', the new bounce path for D stops at the southwest corner of c, then goes east for a distance of

$$Z_0(D) + \cdots + Z_{m-1}(D) = (Z_0(D') - 1) + Z_1(D') + \cdots + Z_{m-1}(D')$$

units, then goes north one unit. The cells in the top row above this last east step used to count towards $area'(D')$, being below the bounce path of D', but will no longer count towards $area'(D)$. In more detail, cell c does not count towards $area'(D)$ because it is in the leftmost column of its rectangle. The other cells do not count towards $area'(D)$ because they count towards ordinary area instead. We conclude that $R(D)$ drops by an additional

$$Z_0(D') + \cdots + Z_{m-1}(D') - 1 \tag{2.40}$$

as a result of the change in this part of the bounce path. The total change is

$$R(D) - R(D') = -(1Z_0(D') + 2Z_1(D') + \cdots + 2Z_{m-1}(D') + 1Z_m(D')) + 2.$$

This change is modelled algebraically by the additional occurrences of r in the expression

$$q^{-1}tyz_0z_1^{-1}w^{-2}r^2C_{n,k+1}(q,r,t,y,r^{-1}wz_1,r^{-2}wz_2,\ldots,r^{-2}wz_m,r^{-1},0).$$

The argument in the last two paragraphs is correct, unless R^*_{s-1} has width zero. In this situation, there is no leftmost column in R^*_{s-1}, so we should not have subtracted 1 in (2.39). But it is easy to see that this situation occurs if and only if $D' = D_0$. Then our technical convention that $Z_m(D_0) = +1$ causes (2.39) to be correct after all, and the validity of (2.40) is not affected either. Since the new value $Z_m(D)$ comes from the old value $Z_{m-1}(D')$ (not from $Z_m(D')$), the technical convention for $Z_m(D_0)$ does not affect the correctness of the values of $Z_m(D)$ calculated using the recursion. This completes the proof of the new recursion. □

Finally, we describe an analogous way of adding a third statistic to the other combinatorial sequence $HC_n^{(m)}(q,t)$. We can "guess" what this statistic should be by seeing what happens to $area'$ when we apply the bijection ψ from §2.2.5. We are led to the following formula.

Definition 2.46. For an m-Dyck path of height n, define

$$h'(D) = \sum_{0\le i<j<n} \sum_{k=0}^{m-1} \chi\left(\gamma_i(D) - \gamma_j(D) + k \in \{-1,0,1,\ldots,m-1\}\right) - \sum_{i=0}^{n-1} \chi(\gamma_i(D) > 0).$$

The first sum is similar to the one appearing in $h(D)$. The second sum that is subtracted may look surprising, but it arises from the fact that the leftmost column of each rectangle R_i does *not* count towards $area'$. Note that the total number of cells in these columns is $n - v_0(\phi(D)) = \sum_{i=0}^{n-1} \chi(\gamma_i(D) > 0)$.

There is a formula analogous to (2.4) for $h'(D)$.

Definition 2.47. Define $sc'_m : \mathbb{Z} \to \mathbb{Z}$ by

$$sc'_m(p) = \begin{cases} m - p & \text{for } 0 \le p \le m; \\ m + 1 + p & \text{for } -m \le p \le -1; \\ 0 & \text{for other } p. \end{cases} \tag{2.41}$$

Define $adj' : \mathbb{Z} \to \mathbb{Z}$ by $adj'(p) = -1$ for $p > 0$ and $adj'(p) = 0$ for other p. We have

$$h'(D) = \sum_{0 \le i < j < n} sc'_m(\gamma_i(D) - \gamma_j(D)) + \sum_{i=0}^{n-1} adj'(\gamma_i).$$

The proof is the same as the corresponding proof of (2.4).

Finally, define

$$HC_n^{(m)}(q,t,r) = \sum_{D \in \mathcal{D}_n^{(m)}} q^{h(D)} t^{area(D)} r^{h'(D)}.$$

Theorem 2.48.

$$HC_n^{(m)}(q,t,r) =$$

$$\sum_{v \in \mathcal{V}_n^{(m)}} t^{\sum_{i \ge 0} i v_i} (qr)^{m \sum_{i \ge 0} \frac{1}{2} v_i (v_i - 1)} \prod_{i \ge 1} (qr)^{v_i \sum_{j=1}^{m} (m-j) v_{i-j}} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - 1 \\ v_i, v_{i-1} + \cdots + v_{i-m} - 1 \end{bmatrix}_{q,r}, \tag{2.42}$$

and hence $HC_n^{(m)}(q,t,r) = C_n^{(m)}(q,t,r)$. *Moreover, the bijection* ϕ *introduced in §2.2.5 maps the ordered triple of statistics* $(h, area, h')$ *to the ordered triple* $(area, b, area')$ *(similarly for* $\psi = \phi^{-1}$*).*

Proof. As in §2.2.4, we proceed by induction on the largest symbol s appearing in $\gamma(D)$. When $s = 0$, γ must consist of n zeroes, and $h'(D) = mn(n-1)/2$. This is the same as the power of r on the right side of (2.34).

For the induction step, it suffices to prove the following formula, which is the analogue of (2.12) for h':

$$h'(\gamma) - h'(\delta) = mv_s(v_s - 1)/2 + v_s \sum_{k=1}^{m} (m-k)v_{s-k} + inv(w). \tag{2.43}$$

Here, $\gamma = \gamma(D)$ has largest symbol $s > 0$; v_i is the number of occurrences of i in γ for $0 \le i \le s$; δ is obtained from γ by erasing all the symbols s; and the word w records how to insert the v_s copies of s into δ to recover γ.

We still proceed by induction on $coinv(w)$. If $coinv(w) = 0$, all v_s copies of s were inserted into δ just after the last occurrence of any symbol in the set $\{s-1, \ldots, s-m\}$. The change $h'(\gamma) - h'(\delta)$ caused by this insertion is

$$\sum_{i<j} sc'_m(\gamma_i - \gamma_j) - v_s$$

where the sum extends over all pairs (i, j) such that $\gamma_i = s$ or $\gamma_j = s$. We subtract v_s since we introduced v_s new positive entries (all equal to s) in γ.

First, consider the pairs (i, j) for which $i < j$ and $\gamma_i = s = \gamma_j$. There are $\binom{v_s}{2}$ such pairs, and each contributes $sc'_m(s - s) = sc'_m(0) = m$ to the h'-statistic. This gives the term $mv_s(v_s - 1)/2$ in (2.43).

Second, consider the pairs (i, j) for which $i < j$ and $\gamma_i = s$ and $\gamma_j \neq s$. Since all the copies of s in γ occur in a contiguous group following all instances of the symbols $s-1, \ldots, s-m$, and since s is the largest symbol appearing in γ, $j > i$ implies that $\gamma_j < s - m$. Then $sc'_m(\gamma_i - \gamma_j) = 0$, since $\gamma_i - \gamma_j > m$. So these pairs contribute nothing to the h'-statistic.

Third, consider the pairs (i, j) for which $i < j$ and $\gamma_i \neq s$ and $\gamma_j = s$. Since s is the largest symbol, we have $\gamma_i < s$. Write $\gamma_i = s - k$ for some $k > 0$, and consider various subcases. Suppose $k \in \{1, 2, \ldots, m\}$. Then $sc'_m(\gamma_i - \gamma_j) = sc'_m(-k) = m+1-k$. For how many pairs (i, j) does it happen that $i < j$, $\gamma_i = s - k$, and $\gamma_j = s$? There are v_s choices for the index j and v_{s-k} choices for the index i; the condition $i < j$ holds automatically, since all occurrences of s occur to the right of all occurrences of $s - k$. Thus, we get a total contribution to the h'-statistic of $(m + 1 - k)v_s(v_{s-k})$ for this k.

Adding over all k, we obtain

$$v_s \sum_{k=1}^{m} (m+1-k) v_{s-k} = v_s \sum_{k=1}^{m} (m-k) v_{s-k} + \sum_{k=1}^{m} v_s v_{s-k}.$$

On the other hand, if $k > m$, then $sc'_m(\gamma_i - \gamma_j) = sc_m(-k) = 0$, so there is no contribution to the h'-statistic.

Finally, recall that w is a rearrangement of v_s zeroes and $v_{s-1} + \cdots + v_{s-m} - 1$ ones. Since $coinv(w) = 0$, all zeroes in w occur at the end, and hence

$$inv(w) = v_s(v_{s-1} + \cdots + v_{s-m} - 1) = \left(\sum_{k=1}^{m} v_s v_{s-k} \right) - v_s.$$

Thus, the change $h'(\gamma) - h'(\delta)$ is precisely the expression on the right side of (2.43). So we are done when $coinv(w) = 0$.

To finish the induction step, it suffices to show that replacing 10 by 01 in w decreases h' by one (since this replacement also decreases $inv(w)$ by one). Let w' be the new word after the replacement, with corresponding vector γ' As in §2.2.4, we have

$$\text{original } \gamma = \ldots \ (s-j) \ z_1 \ z_2 \ \ldots \ z_\ell \ (s-k) \ s \ \ldots$$

where $0 \le j \le m$, $1 \le k \le m$, $\ell \ge 0$, and every $z_i < s - m$. Replacing 10 by 01 in w causes the s to move left, resulting in:

$$\text{new } \gamma' = \ldots \ (s-j) \ s \ z_1 \ z_2 \ \ldots \ z_\ell \ (s-k) \ \ldots.$$

Note that the symbol $s - j$ must exist, lest $\gamma'_0 = s > 0$.

Let us examine the effect of this motion on the h'-statistic. When we move the s left past its predecessor $s - k$ in γ, we get a net change in the h'-statistic of

$$sc'_m(s - (s-k)) - sc'_m((s-k) - s) = sc'_m(k) - sc'_m(-k) = -1,$$

since $1 \le k \le m$ (see (2.41)). As before, since $|s - z_i| > m$, moving the s past each z_i will not affect the h'-statistic at all. Thus, the total change in the h'-statistic is -1, as desired. □

Acknowledgement: This chapter is essentially a reprint, with minor modifications, of the paper "Conjectured Statistics for the Higher q,t-Catalan Sequences" by N. Loehr, which has been submitted for publication in *Electronic Journal of Combinatorics.* The dissertation author was the primary investigator and sole author of this paper.

3

Trapezoidal Lattice Paths and Multivariate Analogues

In this chapter, we extend the combinatorial work of the previous chapter to lattice paths inside trapezoids. We introduce two collections of five statistics on trapezoidal paths, one based on "bounce paths" and another based on "generalized inversion statistics." Though these two sets of statistics appear to be quite different, they have the same generating function. We give bijections to prove this fact and provide an explicit summation formula for their common generating function. We can specialize this generating function in various ways to obtain all the bivariate and trivariate sequences from Chapter 2. We also establish certain symmetry properties and recursions involving the new statistics. We will see that some proofs actually become simpler in the five-variable setting. It is an open problem to find interpretations for the new combinatorial sequences (or their specializations) in terms of representation theory or symmetric functions.

The rest of this chapter is organized as follows. In Section 3.1, we define trapezoidal lattice paths and the two families of statistics for those paths. In Section 3.2, we prove a summation formula for the common generating function of each family of statistics. The proof provides a bijection on paths that sends one family of statistics to the other family. Section 3.3 describes the symmetry properties of the five-variable generating function and its specializations. Section 3.4 uses the summation formula to prove a recursion characterizing the trapezoidal generating functions.

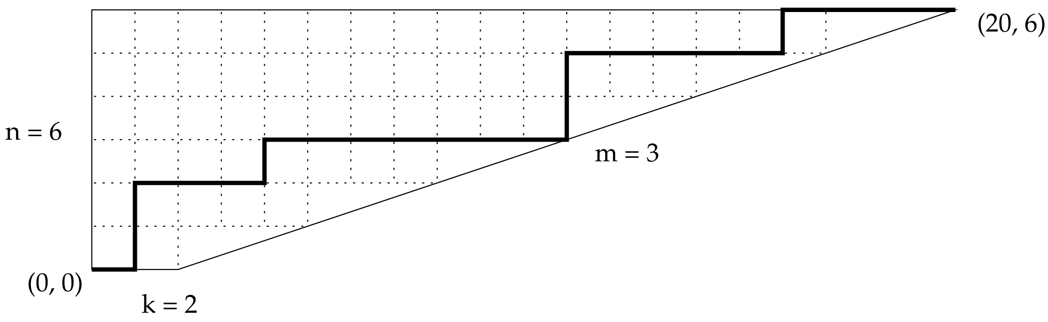


Figure 3.1: A trapezoidal lattice path.

3.1 Statistics on Trapezoidal Lattice Paths

In this section, we define statistics on lattice paths and two five-variable generating functions involving these statistics. We begin by defining trapezoidal lattice paths.

3.1.1 Trapezoidal Lattice Paths

Definition 3.1. Let n, k, and m be integers with $n > 0$, $k \geq 0$, and $m \geq 0$. Let $TZ_{n,k,m}$ denote the trapezoid with corners at $(0,0)$, $(0,n)$, $(k,0)$, and $(k+mn,n)$. (If $k=0$, then $TZ_{n,k,m}$ is really a triangle. This case was studied in Chapter 2. If $k>0$ and $m=0$, then $TZ_{n,k,m}$ is a rectangle. This degenerate case will be discussed later.) Define a *trapezoidal lattice path* of type (n,k,m) to be a path that goes from $(0,0)$ to $(k+mn,n)$ by a series of north and east steps of length one, such that no vertex of the path lies outside the trapezoid $TZ_{n,k,m}$. Define $\mathcal{T}_{n,k,m}$ to be the set of trapezoidal lattice paths of type (n,k,m).

For example, Figure 3.1 shows a trapezoidal path with $n=6$, $k=2$, and $m=3$.

Next, we define statistics for these trapezoidal lattice paths. We will use two different families of statistics. One family is based on "bounce paths," while the second family consists of "generalized inversion statistics." The theorem in the next section will show that the two families of statistics have the same generating function.

3.1.2 Statistics based on Bounce Paths

Let P be a trapezoidal path of type (n,k,m). Then P is a path that proceeds by vertical and horizontal steps from $(0,0)$ to $(k+mn,n)$ while staying inside a trapezoid. We begin with the fundamental construction of *the bounce path derived from* P. This is another lattice path $B=B(P)$ that also proceeds by vertical and horizontal steps from $(0,0)$ to $(k+mn,n)$ while staying inside the trapezoid $TZ_{n,k,m}$. The bounce path $B(P)$ always stays weakly below the original path P. It is derived from P according to the following rules.

Bouncing Algorithm. Given $P\in\mathcal{T}_{n,k,m}$, we construct its derived bounce path $B(P)$ as follows.

(1) $B(P)$ will consist of an alternating sequence of *vertical moves* and *horizontal moves*, starting at $(0,0)$ and ending at $(k+mn,n)$. A vertical move consists of zero or more vertical steps (of length one); similarly for the horizontal moves. We let $v_i=v_i(P)$ denote the length of the i^{th} vertical move, and we let $h_i=h_i(P)$ denote the length of the i^{th} horizontal move, for $i\geq 0$. Set $(x_0,y_0)=(0,0)$, which is our "initial position" on the bounce path. In general, let (x_i,y_i) denote the position on the bounce path just before the i^{th} vertical move. (All the quantities defined here and below depend on P, of course. We write $x_i(P)$, $y_i(P)$, etc., if it is necessary to make this dependence explicit. At other times it is convenient to omit the P and write B, v_i, x_i, etc., if there is no danger of confusion.)

(2) The numbers v_i and h_i are computed as follows. Set $v_i=0$ and $h_i=0$ for all $i<0$. Set $i=0$ initially. Perform the following steps repeatedly.

a. Go up from the current position (x_i,y_i) until blocked by a horizontal step of the original path P. Let the vertical distance traveled be v_i. Note that v_i may be zero. Set $y_{i+1}=y_i+v_i$.

b. Go right from the current position (x_i,y_{i+1}) by the horizontal distance

$$h_i \stackrel{def}{=} v_i+v_{i-1}+\cdots+v_{i-(m-1)}+\chi(i<k). \tag{3.1}$$

Set $x_{i+1}=x_i+h_i$, so the new position is (x_{i+1},y_{i+1}).

c. If the new position is the upper corner $(k+mn,n)$, the bounce path has been completed. Otherwise, replace i by $i+1$ and return to step a.

This algorithm can be informally summarized as follows. A "ball" starts at the southwest corner of the trapezoid. The ball moves up until it "hits" the path P. Then it moves right by a distance that is the sum of the previous m vertical moves, *plus an additional one unit for the first k bounces only.* The bouncing continues until the ball reaches the northeast corner.

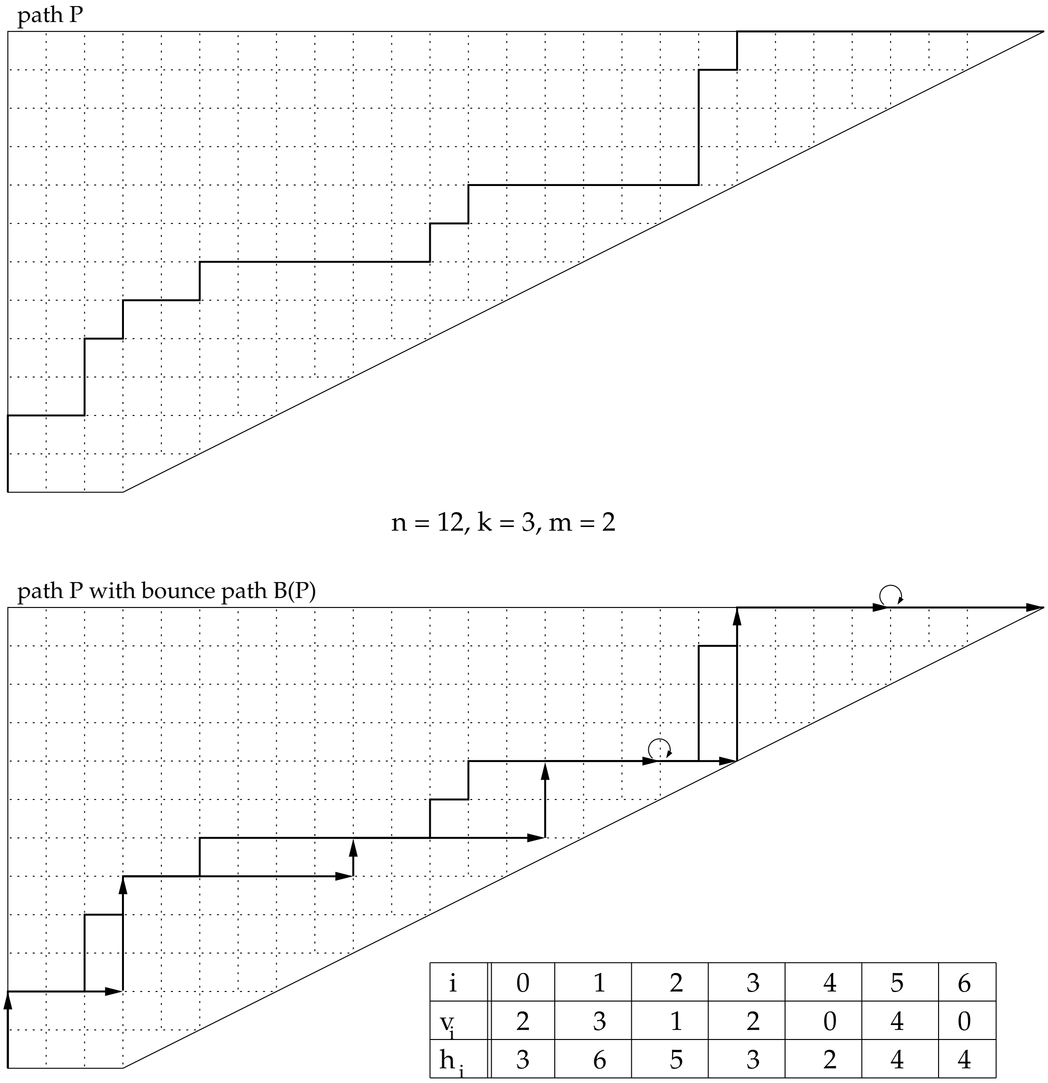
Example 3.2. Figure 3.2 shows a trapezoidal path $P \in \mathcal{T}_{12,3,2}$ and the associated bounce path $B(P)$.

Remark 3.3. For special choices of k and m, the bounce path becomes simpler. When $k=0$ and $m=1$, we obtain a bounce path that returns to the diagonal $x=y$ after each bounce. When $k=0$ and $m>1$, we obtain the bounce paths inside triangles discussed in Chapter 2. When $m=1$ and $k>0$, the first k horizontal moves are one unit longer than the immediately preceding vertical moves. After the first k bounces, the bounce path will have reached the diagonal boundary $x=y+k$. For any further bounces, the horizontal move is the same as the preceding vertical move. In the degenerate case when $m=0$ (a rectangle), the bounce path essentially coincides with the original path P. The only difference is that the bounce path takes a vertical move of height zero between consecutive horizontal steps of the original path.

We can use the bounce path $B(P)$ derived from P to dissect the figure for P into smaller geometric components. For this purpose, we introduce the following terminology.

Definition 3.4. Let $P \in \mathcal{T}_{n,k,m}$, and let B be the associated bounce path. Define v_i, h_i, x_i, and y_i as in the bouncing algorithm above.

(1) For $i \geq 1$, define the i^{th} *bounce rectangle* R_i to be the rectangle with vertices (x_{i-1},y_i), (x_{i-1},y_{i+1}), (x_i,y_i), and (x_i,y_{i+1}). This rectangle consists of the cells above the $i-1^{\text{th}}$ horizontal bounce move and left of the i^{th} vertical bounce move. The definition also makes sense for $i=0$, if we set $x_j=y_j=0$ for $j<0$. Note that we allow degenerate "rectangles" whose height or width is zero.



i	0	1	2	3	4	5	6
v_i	2	3	1	2	0	4	0
h_i	3	6	5	3	2	4	4

Figure 3.2: A trapezoidal path and its associated bounce path.

(2) For $i \geq 1$, define the i^{th} *shortened bounce rectangle* S_i to be the rectangle with vertices $(x_{i-1}+1, y_i)$, $(x_{i-1}+1, y_{i+1})$, (x_i, y_i), and (x_i, y_{i+1}). Thus S_i consists of all columns of R_i except the leftmost column. We are interested in S_i because any part of the path P that lies above the bounce bath $B(P)$ must be contained in one of the shortened rectangles S_i. This fact is immediate from the definition of the bounce path.

(3) For $i \geq 0$, define the i^{th} *bounce triangle* T_i to be the triangle with vertices $(k + my_i, y_i)$, $(k + my_i, y_{i+1})$, and $(k + my_{i+1}, y_{i+1})$. Note that the right edge of the trapezoid $TZ_{n,k,m}$ has equation $x = k+my$. Thus, each T_i is a right triangle whose hypotenuse lies on this edge of the trapezoid.

(4) For $i \geq 1$, define the i^{th} *bounce slab* Sl_i to be the rectangle with vertices (x_i, y_i), (x_i, y_{i+1}), $(k + my_i, y_i)$, and $(k + my_i, y_{i+1})$.

It is clear from these definitions that the region inside the trapezoid and below the bounce path is precisely the union of the bounce triangles T_i and the bounce slabs Sl_i. (See Figure 3.3 below.)

Example 3.5. Consider the path P from Figure 3.2. Table 3.1 gives the values of v_i, h_i, x_i, and y_i for this path. Figure 3.3 illustrates the bounce rectangles R_i, the shortened rectangles S_i, the bounce triangles T_i, and the bounce slabs Sl_i for the same path. For visual clarity, the original path P is not shown in this figure. Note that R_4, S_4, Sl_4, and T_4 have height zero, so these shapes are not visible in the figure.

At last, we are ready to introduce the five statistics in the "bounce family." We also mention two other area statistics that can be derived from these.

Definition 3.6. Let $P \in \mathcal{T}_{n,k,m}$. Let B, x_i, y_i, v_i, h_i, R_i, S_i, T_i, and Sl_i be as described above.

(1) Define a *type-1 area cell of* P to be a lattice square completely inside the trapezoid $TZ_{n,k,m}$ and below the bounce path $B(P)$. Thus, a cell is a type-1 area cell if and only if it is contained in one of the bounce triangles T_i or bounce slabs Sl_i. Define the *type-1 area of* P, denoted $a_1(P)$, to be the number of type-1 area cells of P.

Table 3.1: Values of x_i, y_i, v_i, and h_i for the path P.

i	x_i	y_i	v_i	h_i
0	0	0	2	3
1	3	2	3	6
2	9	5	1	5
3	14	6	2	3
4	17	8	0	2
5	19	8	4	4
6	23	12	0	4
7	27	12	N/A	N/A

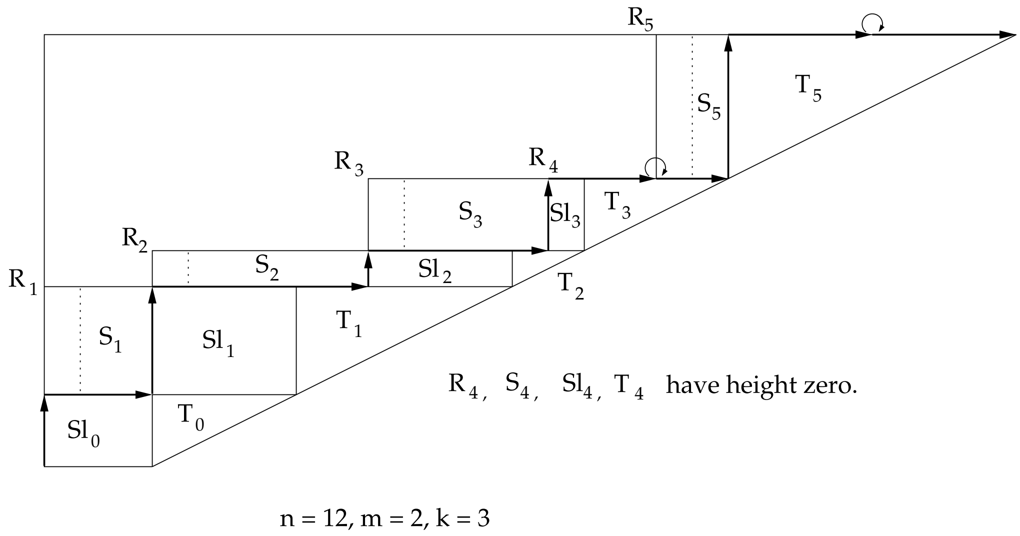


Figure 3.3: Using the bounce path to dissect the trapezoid. (Rectangles R_i are labelled at their northwest corner.)

(2) Define a *type-2 area cell of* P to be a lattice square inside the trapezoid $TZ_{n,k,m}$, below the trapezoidal path P, and above the bounce path $B(P)$. Thus, a cell is a type-2 area cell if and only if it lies in a shortened rectangle S_i and is below the original path P. Define the *type-2 area of* P, denoted $a_2(P)$, to be the number of type-2 area cells of P.

(3) Define a *type-3 area cell of* P to be a lattice square inside one of the shortened rectangles S_i and above the trapezoidal path P. Define the *type-3 area of* P, denoted $a_3(P)$, to be the number of type-3 area cells of P. Note that the cells in $R_i - S_i$ (the leftmost column of R_i) do *not* contribute to $a_3(P)$ or $a_2(P)$ or $a_1(P)$.

(4) Define the *bounce score* $b(P)$ by

$$b(P) = \sum_{i \geq 0} i v_i.$$

It is easy to see that $b(P)$ is the sum of the number of cells in the trapezoid that are directly above the first horizontal step of each horizontal move in the bounce path. These cells include, but are not limited to, the cells in the leftmost column of each R_i.

(5) Define the *bounce count* $c(P)$ to be the largest integer s such that $v_s > 0$. Roughly speaking, this statistic keeps track of how many bounces were needed to reach the top edge of the trapezoid. Note that the first vertical move is labelled v_0, not v_1.

(6) Define the *ordinary area of* P to be $a(P) = a_1(P) + a_2(P)$. This is just the number of complete lattice squares inside the trapezoid and below the path P.

(7) Define the *modified area of* P to be $a'(P) = a_1(P) + a_3(P)$. This is the number of cells either below the bounce path $B(P)$, or above the original path P and inside one of the shortened rectangles S_i.

(8) Finally, define the *bounce generating function* for the trapezoid $TZ_{n,k,m}$ by

$$B_{n,k,m}(q_1, q_2, q_3, t, z) = \sum_{P \in \mathcal{T}_{n,k,m}} q_1^{a_1(P)} q_2^{a_2(P)} q_3^{a_3(P)} t^{b(P)} z^{c(P)}.$$

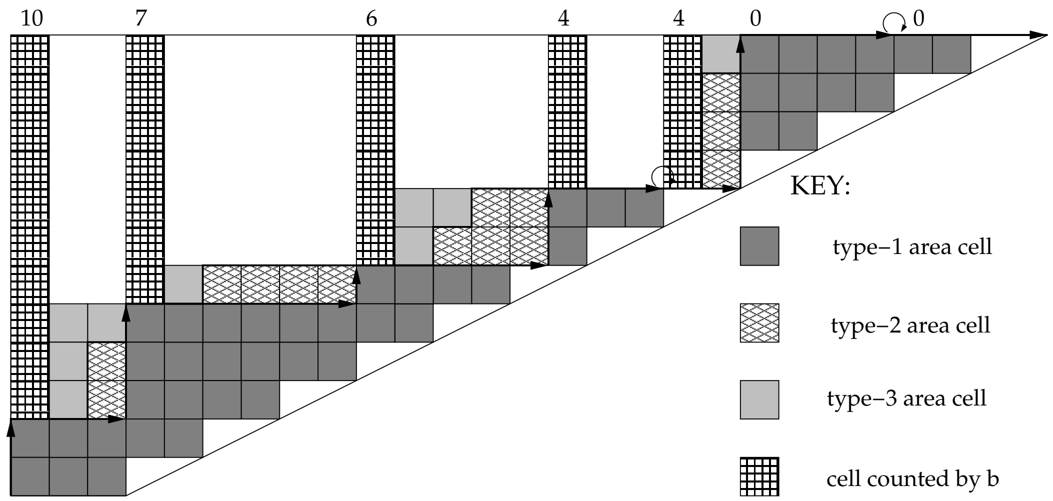


Figure 3.4: Special cells in the diagram for P.

We will also frequently be interested in the two specializations

$$B'_{n,k,m}(q,t) = B_{n,k,m}(q,q,1,t,1) = \sum_{P\in\mathcal{T}_{n,k,m}} q^{a(P)}t^{b(P)}$$

$$B''_{n,k,m}(q,r,t) = B_{n,k,m}(qr,q,r,t,1) = \sum_{P\in\mathcal{T}_{n,k,m}} q^{a(P)}r^{a'(P)}t^{b(P)}.$$

Example 3.7. For the path P from Figure 3.2, we compute the following statistics:

$$a_1(P) = 46,\ a_2(P) = 14,\ a_3(P) = 9,\ b(P) = 31,$$

$$c(P) = 5,\ a(P) = 60,\ a'(P) = 55.$$

Figure 3.4 shows how the statistics a_1, a_2, a_3, and b can be computed by counting cells in the diagram for P.

Example 3.8. In the special case of a rectangle ($m = 0$), all the rectangles R_i have width 1, so all shortened rectangles S_i are empty. It follows that, for any path P inside the rectangle $TZ_{n,k,0}$, we have $a_2(P) = a_3(P) = 0$, $a_1(P) = a(P) = a'(P) =$ the number of lattice squares below the path P, and $b(P) =$ the number of lattice squares above the path P.

The following lemma will be useful when we derive a formula for the generating function $B_{n,k,m}(q_1, q_2, q_3, t, z)$. It provides formulas for the coordinates (x_i, y_i) and the dimensions of various components of the bounce picture.

Lemma 3.9. *Let $P \in \mathcal{T}_{n,k,m}$. Let B, x_i, y_i, v_i, h_i, R_i, S_i, T_i, and Sl_i be as described above.*

(1) For $i \geq 0$, we have

$$x_i = \min(i, k) + \sum_{j=1}^{i} \min(j, m) v_{i-j}$$

and

$$y_i = \sum_{j=0}^{i-1} v_j.$$

(2) The bounce path $B(P)$ always stays inside the trapezoid $TZ_{n,k,m}$ and has $h_j > 0$ for $j \geq 0$. In particular, the bounce path always reaches the upper-right corner $(k + mn, n)$, so that the bouncing algorithm does terminate.

(3) For $i \geq 0$, the bounce rectangle R_i has height v_i and width

$$v_{i-1} + \cdots + v_{i-m} + \chi(i < k + 1).$$

(4) For $i \geq 1$, the shortened rectangle S_i has height v_i and width

$$v_{i-1} + \cdots + v_{i-m} - \chi(i > k).$$

(5) For $i \geq 0$, the bounce triangle T_i has height v_i and contains $mv_i(v_i - 1)/2$ complete lattice squares.

(6) For $i \geq 0$, the bounce slab Sl_i has height v_i and width $\max(k - i, 0) + \sum_{j=1}^{m-1}(m - j)v_{i-j}$, hence contains

$$v_i \left(\max(k - i, 0) + \sum_{j=1}^{m-1} (m - j) v_{i-j} \right)$$

complete lattice squares.

Proof. (1) Since the bounce path starts at $(0,0)$, y_i is $\sum_{j<i} v_j$, the sum of all preceding vertical moves. Similarly, $x_i = \sum_{j<i} h_j$ is the sum of all preceding horizontal moves. Substituting the definition

$$h_j = \chi(j < k) + \left(\sum_{p=0}^{m-1} v_{j-p}\right)$$

into the formula for x_i, we get

$$x_i = \sum_{j=0}^{i-1} \left(\chi(j < k) + \sum_{p=0}^{m-1} v_{j-p}\right).$$

Now, the sum of the terms $\chi(j < k)$ for $0 \le j \le i-1$ is just $\min(i,k)$. Since $v_\ell = 0$ for $\ell < 0$, we can rewrite the remaining terms as

$$\sum_{j<i} \sum_{p=0}^{m-1} v_{j-p}.$$

Let us count the occurrences of v_{i-1}, v_{i-2}, $\ldots$, v_0. The term v_{i-1} occurs only once (assuming $m > 0$), when $j = i-1$ and $p = 0$. The term v_{i-2} occurs twice (assuming $m > 1$), when $(j,p) = (i-1,1)$ and when $(j,p) = (i-2,0)$. In general, for $1 \le t \le m$, the term v_{i-t} occurs t times, by letting $(j,p) = (i-s, t-s)$ for $1 \le s \le t$. For $t \ge m$, the term v_{i-t} occurs m times, by letting $(j,p) = (i-t+s, s)$ for $0 \le s \le m-1$. Changing notation, we obtain

$$x_i = \min(i,k) + \sum_{j=1}^{i} \min(j,m) v_{i-j}$$

as claimed.

(2) To prove that $B(P)$ stays inside the trapezoid $TZ_{n,k,m}$, it suffices to show that every point (x_i, y_i) lies in this trapezoid. It is clear that $0 \le y_i \le n$ for all i, since the path P never goes above the line $y = n$. It is also clear that $x_i \ge 0$ for all i. To see that (x_i, y_i) does not lie to the right of the remaining boundary $x = k + my$, we need to check that $x_i \le k + my_i$. But this is obvious from the formulas in part (1), since

$$x_i = \min(i,k) + \sum_{j=1}^{i} \min(j,m) v_{i-j} \le k + \sum_{j=0}^{i-1} m v_j = k + m y_i.$$

Next, we show that $h_i > 0$ for all i. This is clear for $i < k$, since the term $\chi(i < k)$ in the definition of h_i shows that we move at least one step right. Now suppose $i \geq k$. To get a contradiction, assume $h_i = 0$. Then we must have $v_{i-j} = 0$ for $0 \leq j \leq m-1$. Observe that, in this case,

$$x_i = k + \sum_{j=m}^{i} m v_{i-j} \text{ and } y_i = \sum_{\ell=0}^{i-1} v_\ell = \sum_{j=m}^{i} v_{i-j}.$$

Thus, $x_i = k + my_i$, so that the point (x_i, y_i) lies on the right edge of the trapezoid $TZ_{n,k,m}$. But we also have $v_i = 0$. This can only occur if the original path P passes through (x_i, y_i) and *does not go up at the next step.* If $y_i < n$, this contradicts the requirement that the path P never go outside the trapezoid. If $y_i = n$, then the path has reached the upper-right corner of the trapezoid, so the bouncing path was completed just before step i.

(3) By definition of the vertices of R_i, the height of this rectangle is $y_{i+1} - y_i = v_i$. The width of this rectangle is

$$\begin{aligned} x_i - x_{i-1} = h_{i-1} &= v_{i-1} + \cdots + v_{i-m} + \chi(i-1 < k) \\ &= v_{i-1} + \cdots + v_{i-m} + \chi(i < k+1). \end{aligned}$$

(4) The height of the shortened rectangle S_i is the same as the height of R_i, while the width of S_i is one less than the width of R_i. The formulas in (4) thus follow from the ones in (3), together with the identity

$$\chi(i < k+1) - 1 = -\chi(i > k).$$

(5) By definition of T_i, the height of this triangle is $y_{i+1} - y_i = v_i$. Consider the rows of this triangle from bottom to top. As shown in Figure 3.3, these rows contain 0 cells, m cells, $2m$ cells, etc. The total number of complete cells in the triangle is therefore

$$0 + m + 2m + \cdots + (v_i - 1)m = m\binom{v_i}{2} = mv_i(v_i-1)/2.$$

(6) By definition of Sl_i, the height of this slab is $y_{i+1} - y_i = v_i$ and its width is $k + my_i - x_i$.

Using the formula for x_i and simplifying, the width is

$$\max(k-i,0)+\sum_{j=1}^{m-1}(m-j)v_{i-j}.$$

This completes the proof. $\square$

Table 3.2 summarizes the notation introduced in this subsection for statistics and constructs related to the bouncing algorithm.

We would like to have interpretations for the generating functions $B'_{n,k,m}(q,t)$ in terms of the nabla operator. At present, conjectured interpretations are only available for special choices of n, k, and m. The case $k=0$ was considered in Chapter 2. It follows directly from the definitions above that $B'_{n,m,m}(q,t)=B'_{n+1,0,m}(q,t)$, so the conjecture from Chapter 2 gives

$$B'_{n,m,m}(q,t)=\nabla^m(s_{1^{n+1}})|_{s_{1^{n+1}}}.$$

Next, consider the case $m=1$, $k=2$. We conjecture that

$$B'_{n,2,1}(q,t)\approx\nabla(s_{(2,1^n)})|_{s_{1^{n+2}}}.$$

(We use the symbol $\approx$ to indicate the omission of a fixed multiplier $\epsilon q^A t^B$ on one side of the equation, where $\epsilon=\pm1$ and A and B are constants.) This has been confirmed by computer for $0\le n\le 4$, which are all the values of n we can currently test. In the case $m=1$, $k=3$, we conjecture that

$$B'_{n,3,1}(q,t)\approx\nabla(s_{(2,2,1^n)})|_{s_{1^{n+4}}}.$$

This has been confirmed by computer for $0\le n\le 2$, which are all the values of n we can currently test. However, the pattern suggested here fails for $k=4$ and $n=0$, since

$$B'_{0,4,1}(q,t)\not\approx\nabla(s_{(2,2,2)})|_{s_{1^6}}.$$

3.1.3 Generalized Inversion Statistics

We now define the family of "generalized inversion statistics" for lattice paths. Let P be a trapezoidal path of type (n,k,m). Before describing the statistics, we describe a way to represent P as a list of integers.

Table 3.2: Summary of notation for bounce-related constructions.

Notation	Meaning
P	the given trapezoidal path
B	bounce path derived from P
v_i	length of i^{th} vertical move in B
h_i	length of i^{th} horizontal move in B $h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)} + \chi(i < k)$
x_i	x-coordinate of i^{th} vertical move in B
y_i	y-coordinate of $(i-1)^{\text{th}}$ horizontal move in B
(x_i, y_i)	location on B just before the i^{th} vertical move
R_i	bounce rectangle with southeast corner (x_i, y_i)
S_i	shortened bounce rectangle (R_i without its leftmost column)
T_i	bounce triangle right of the i^{th} vertical move
Sl_i	bounce slab right of the i^{th} vertical move
$a_1(P)$	number of type-1 area cells (cells below B)
$a_2(P)$	number of type-2 area cells (cells below P in some S_i)
$a_3(P)$	number of type-3 area cells (cells above P in some S_i)
$b(P)$	bounce score $\sum_{i\geq 0} iv_i$ (cells above vertical bounce moves)
$c(P)$	bounce count (largest s with $v_s > 0$)
$B_{n,k,m}(q_1, q_2, q_3, t, z)$	bounce generating function for paths inside $TZ_{n,k,m}$

Definition 3.10. Given $P \in \mathcal{T}_{n,k,m}$, the *area vector of* P, denoted $g(P)$, is the list

$$g(P) = (g_0(P), g_1(P), \ldots, g_{n-1}(P)),$$

where $g_i(P)$ is the number of complete lattice squares between the path P and the right boundary $x = k + my$ in the i^{th} row from the bottom. We let $\mathcal{G}_{n,k,m}$ denote the set of all lists $g(P)$, for $P \in \mathcal{T}_{n,k,m}$.

Example 3.11. For the path P shown in Figure 3.1, we have

$$g(P) = (1, 4, 4, 0, 3, 1).$$

It is clear that P is recoverable from its area vector $g(P)$. For, given $g = g(P)$, we shade in the appropriate number of area cells in each row of the trapezoid. Then P is the unique path obtained by following the left boundary of the shaded region from $(0, 0)$ to $(k + mn, n)$. Thus, the map γ sending P to $g(P)$ is a bijection from $\mathcal{T}_{n,k,m}$ onto $\mathcal{G}_{n,k,m}$.

Given an arbitrary list of integers $h = (h_0, h_1, \ldots)$, h may not be the area vector for any trapezoidal path. It is easy to see that $h \in \mathcal{G}_{n,k,m}$ if and only if the following conditions hold:

(G1) h is a list of integers of length n.

(G2) $h_i \geq 0$ for $0 \leq i \leq n - 1$.

(G3) $h_{i+1} \leq h_i + m$ for $0 \leq i \leq n - 2$.

(G4) $h_0 \in \{0, 1, \ldots, k\}$.

Condition (G1) says that h is the area vector for a shape consisting of n rows. Condition (G2) says that the path built from h cannot go past the right edge of the trapezoid in question. Condition (G3) says that the path built from h cannot take any west steps. In more detail, we have equality $h_{i+1} = h_i + m$ in (G3) if and only if the path takes two consecutive north steps in rows i and $i + 1$. If there are any intervening east steps between the north step in row i and the north step in row $i + 1$, then we must have $h_{i+1} < h_i + m$. Finally, condition (G4) reflects the fact that there is room for up to k area cells in the lowest row of the trapezoid $TZ_{n,k,m}$.

Now we define the five inversion statistics. It is most convenient to define these statistics on the collection of area vectors $\mathcal{G}_{n,k,m}$.

Definition 3.12. Let $g \in \mathcal{G}_{n,k,m}$, and let P be the corresponding path in $\mathcal{T}_{n,k,m}$ with $g = g(P)$.

(1) Define the *type-1 inversion statistic* h_1 for g or P by the formula

$$h_1(g) = h_1(P) = \sum_{0\le i<j\le n-1} \max(m - |g_i - g_j|, 0) + \sum_{i=0}^{n-1} \max(k - g_i, 0).$$

(2) Define the *type-2 inversion statistic* h_2 for g or P by the formula

$$h_2(g) = h_2(P) = \sum_{0\le i<j\le n-1} \chi(g_i - g_j \in \{1, 2, \ldots, m\}).$$

(3) Define the *type-3 inversion statistic* h_3 for g or P by the formula

$$\begin{aligned} h_3(g) = h_3(P) = \ & \textstyle\sum_{0\le i<j\le n-1} \chi(g_i - g_j \in \{-m, \ldots, -2, -1\}) \\ & -\textstyle\sum_{i=0}^{n-1} \chi(g_i > k). \end{aligned}$$

(4) Define the *ordinary area statistic* for g or P by

$$a(g) = a(P) = \sum_{i=0}^{n-1} g_i.$$

This is the number of complete lattice squares inside the trapezoid and below the path P. Therefore, this definition agrees with the definition of ordinary area given in the preceding subsection.

(5) Define the *height* of g or P by

$$ht(g) = ht(P) = \max_{0\le i\le n-1} g_i.$$

(6) Define the *positive inversion statistic* h^+ for g or P to be

$$h^+(P) = h_1(P) + h_2(P).$$

(7) Define the *negative inversion statistic* h^- for g or P to be

$$h^-(P) = h_1(P) + h_3(P).$$

(8) Finally, define the *inversion generating function* for the trapezoid $TZ_{n,k,m}$ by

$$H_{n,k,m}(q_1, q_2, q_3, t, z) = \sum_{P \in \mathcal{T}_{n,k,m}} q_1^{h_1(P)} q_2^{h_2(P)} q_3^{h_3(P)} t^{a(P)} z^{ht(P)}.$$

We will also frequently be interested in the two specializations

$$H'_{n,k,m}(q,t) = H_{n,k,m}(q,q,1,t,1) = \sum_{P \in \mathcal{T}_{n,k,m}} q^{h^+(P)} t^{a(P)}$$

$$H''_{n,k,m}(q,r,t) = H_{n,k,m}(qr,q,r,t,1) = \sum_{H \in \mathcal{T}_{n,k,m}} q^{h^+(P)} r^{h^-(P)} t^{a(P)}.$$

Example 3.13. To compute these statistics, it is convenient to list all the numbers $g_i(P) - g_j(P)$ for $i < j$. For the path P in Figure 3.1 with $n = 6$, $k = 2$, $m = 3$, and $g(P) = (1, 4, 4, 0, 3, 1)$, this list is:

$$-3, -3, 1, -2, 0; 0, 4, 1, 3; 4, 1, 3; -3, -1; 2.$$

Hence, we obtain the following statistics for P:

$$h_1(P) = 16 + 4 = 20,\ h_2(P) = 6,\ h_3(P) = 5 - 3 = 2,\ a(P) = 13,\ ht(P) = 4,$$

$$h^+(P) = 26,\ h^-(P) = 22.$$

3.2 Formula for the Generating Functions

In this section, we show that

$$B_{n,k,m}(q_1, q_2, q_3, t, z) = H_{n,k,m}(q_1, q_2, q_3, t, z)$$

and give an explicit formula for this generating function. Moreover, we exhibit a bijection $\phi : \mathcal{T}_{n,k,m} \to \mathcal{T}_{n,k,m}$ that maps each bounce statistic to its corresponding inversion statistic.

The formula for the common generating function is:

$$\sum_{(v_0,\ldots,v_s)} z^s t^{\sum_{i\geq 0} i v_i} q_1^{pow} \prod_{i=1}^{s} \begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \\ v_i, v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \end{bmatrix}_{q_2, q_3} \tag{3.2}$$

where we set $v_j = 0$ for all $j < 0$,

$$pow = m \sum_{i \geq 0} \binom{v_i}{2} + \sum_{i \geq 0} v_i \left(\max(k - i, 0) + \sum_{j=1}^{m} (m - j) v_{i-j} \right),$$

and we sum over all sequences $v = (v_0, \ldots, v_s)$ of nonnegative integers such that $v_s > 0$ and $v_0 + \cdots + v_s = n$. Since v may have zero entries, the collection of such sequences is infinite. However, if there exists an index $i > k$ such that $v_{i-1} = \cdots = v_{i-m} = 0$, then the summand corresponding to this choice of v has a binomial coefficient equal to zero. From this observation, it is easy to see that there is only a finite collection of v's that give nonzero summands.

3.2.1 Combinatorial Description of the Formula

We now introduce a combinatorial model of the above formula, consisting of a collection $\mathcal{I}_{n,k,m}$ of intermediate objects and five statistics on these objects. This collection will be helpful for defining the bijection ϕ and its inverse.

Definition 3.14. A typical element I of $\mathcal{I}_{n,k,m}$ consists of the following data:

(1) a nonnegative integer s.

(2) a sequence $v = (v_0, \ldots, v_s)$ of nonnegative integers such that $v_s > 0$, $v_0 + \cdots + v_s = n$, and for $1 \leq i \leq s$,

$$v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \geq 0.$$

(As usual, we set $v_j = 0$ for $j < 0$.)

(3) for $1 \leq i \leq s$, a word $w_i \in R(0^{v_i} 1^{v_{i-1} + \cdots + v_{i-m} - \chi(i>k)})$.

We write $I = (s; v; w_1, \ldots, w_s)$.

Define five statistics on these intermediate objects, as follows.

(1) The *first q-statistic* for $I = (s; v; w_1, \ldots, w_s)$ is

$$Q_1(I) = m \sum_{i \geq 0} \binom{v_i}{2} + \sum_{i \geq 0} v_i \left(\max(k - i, 0) + \sum_{j=1}^{m} (m - j) v_{i-j} \right).$$

This is exactly the expression *pow* appearing above.

(2) The *second q-statistic* for $I = (s; v; w_1, \dots, w_s)$ is

$$Q_2(I) = \sum_{i=1}^{s} coinv(w_i).$$

(3) The *third q-statistic* for $I = (s; v; w_1, \dots, w_s)$ is

$$Q_3(I) = \sum_{i=1}^{s} inv(w_i).$$

(4) The *t-statistic* for $I = (s; v; w_1, \dots, w_s)$ is

$$T(I) = \sum_{i=0}^{s} iv_i.$$

This is just the power of t in the formula above.

(5) The *z-statistic* for $I = (s; v; w_1, \dots, w_s)$ is

$$Z(I) = s.$$

This is just the power of z in the formula above.

(6) Finally, define the *intermediate generating function* of type (n, k, m) by

$$F_{n,k,m}(q_1, q_2, q_3, t, z) = \sum_{I \in \mathcal{I}_{n,k,m}} q_1^{Q_1(I)} q_2^{Q_2(I)} q_3^{Q_3(I)} t^{T(I)} z^{Z(I)}.$$

Theorem 3.15. *$F_{n,k,m}(q_1, q_2, q_3, t, z)$ is given by formula (3.2).*

Proof. This assertion follows easily from the above definitions. The formula (3.2) classifies objects $I = (s; v; w_1, \dots, w_s)$ in $\mathcal{I}_{n,k,m}$ based on the values of s and v. For fixed s and $v = (v_0, \dots, v_s)$, the powers of q_1, t, and z in formula (3.2) are exactly the values of $Q_1(I)$, $T(I)$, and $Z(I)$, respectively. We still must choose the words w_i with the appropriate number of zeroes and ones. The generating function for the choice of w_i, where q_2 counts coinversions and q_3 counts inversions, is exactly

$$\begin{bmatrix} v_i + v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \\ v_i, v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \end{bmatrix}_{q_2,q_3}.$$

Multiplying these factors together, for $1 \le i \le s$, gives us a generating function where the power of q_2 records $\sum_{i=1}^{s} coinv(w_i) = Q_2(I)$ and the power of q_3 records $\sum_{i=1}^{s} inv(w_i) = Q_3(I)$. Thus, (3.2) is the generating function for $\mathcal{I}_{n,k,m}$ relative to the five given statistics.

□

To show that the bounce generating function $B_{n,k,m}$ and the inversion generating function $H_{n,k,m}$ are also given by formula (3.2), it suffices to exhibit bijections $\alpha : \mathcal{T}_{n,k,m} \to \mathcal{I}_{n,k,m}$ and $\beta : \mathcal{G}_{n,k,m} \to \mathcal{I}_{n,k,m}$ that preserve the appropriate statistics. Recall there is a trivial statistic-preserving bijection $\gamma : \mathcal{T}_{n,k,m} \to \mathcal{G}_{n,k,m}$ sending P to its area vector $g(P)$. By looking at the composites $\phi = \gamma^{-1} \circ \beta^{-1} \circ \alpha$ and $\phi^{-1} = \alpha^{-1} \circ \beta \circ \gamma$, we obtain a bijective proof that $B_{n,k,m} = H_{n,k,m}$.

3.2.2 Mapping Bounce Statistics to Intermediate Statistics

This subsection describes the bijection $\alpha : \mathcal{T}_{n,k,m} \to \mathcal{I}_{n,k,m}$, which sends statistics in the bounce family to their counterparts in the intermediate setting. We also describe α^{-1}.

Let P be a given path in $\mathcal{T}_{n,k,m}$. To find $\alpha(P)$, we first draw the bounce path of P and compute the quantities listed in Table 3.2. Define $\alpha(P)$ to be the intermediate object $I = (s; v; w_1, \ldots, w_s) \in \mathcal{I}_{n,k,m}$ constructed as follows. Set $s = c(P)$ and $v = (v_0(P), v_1(P), \ldots, v_s(P))$, the sequence of vertical moves in the bounce path for P. Clearly, v does satisfy the necessary requirements from the definition of $\mathcal{I}_{n,k,m}$.

We now describe the construction of the words w_i. For each i with $1 \le i \le s$, consider the portion of the path P contained in the shortened rectangle S_i. Call this partial path P_i. The path P_i goes from the southwest corner $(x_{i-1}+1, y_i)$ of S_i to the northeast corner (x_i, y_{i+1}) of S_i. To obtain the word w_i, replace each vertical step in P_i by a zero and each horizontal step by a one. By part (4) of the lemma from §3.1.2, the resulting word w_i does have the appropriate number of zeroes and ones (namely, v_i and $v_{i-1} + \cdots + v_{i-m} - \chi(i > k)$, respectively).

We now describe α^{-1}. Let $I = (s; v; w_1, \ldots, w_s) \in \mathcal{I}_{n,k,m}$. Construct the path $P = \alpha^{-1}(I) \in \mathcal{T}_{n,k,m}$ as follows. Start with an empty trapezoid $TZ_{n,k,m}$. Draw a bounce path with successive vertical steps $v_0, \dots, v_s$ and horizontal steps given by the usual rule

$$h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)} + \chi(i < k).$$

Define (x_i, y_i) to be the point on the bounce path just prior to the vertical step v_i (as usual). Next, use each word w_i to draw a subpath in the shortened rectangle of height v_i and width $h_i - 1$ whose southeast corner is (x_i, y_i). The subpath is obtained by drawing a vertical step for each zero in w_i, and a horizontal step for each one in w_i. This procedure is obviously inverse to the one described in the last paragraph.

We must still check that α has the desired effect on the five statistics. Fix the path P and $I = \alpha(P)$. Consider each of the five statistics in turn.

First, recall that $a_1(P)$ is the sum of the number of complete cells in all bounce triangles T_i and bounce slabs Sl_i. By parts (5) and (6) of the lemma from §3.1.2,

$$a_1(P) = \sum_{i=0}^{s} m v_i(v_i - 1)/2 + \sum_{i=0}^{s} v_i \left(\max(k - i, 0) + \sum_{j=1}^{m-1} (m - j) v_{i-j} \right) = Q_1(I).$$

Second, recall that $a_2(P)$ is the total number of cells in the shortened rectangles S_i below the partial paths P_i. Using the notation from Chapter 1, we can write

$$a_2(P) = \sum_{i=1}^{s} a(P_i) = \sum_{i=1}^{s} coinv(w_i) = Q_2(I).$$

Third, recall that $a_3(P)$ is the total number of cells in the shortened rectangles S_i above the partial paths P_i. Using the notation from Chapter 1, we can write

$$a_3(P) = \sum_{i=1}^{s} \tilde{a}(P_i) = \sum_{i=1}^{s} inv(w_i) = Q_3(I).$$

Fourth, it is immediate from the definitions that

$$b(P) = \sum_{i=0}^{s} i v_i = T(I).$$

Fifth, we trivially have

$$c(P) = s = Z(I).$$

Theorem 3.16. *$B_{n,k,m}(q_1, q_2, q_3, t, z)$ is given by formula (3.2).*

Proof. This follows immediately from Theorem 3.15 and the existence of the weight-preserving map α just described. $\square$

Example 3.17. Let $n = 12$, $m = 2$, $k = 3$, and let P be the path in $\mathcal{T}_{n,k,m}$ shown in Figure 3.2. We have $s(P) = 5$ and $v(P) = (2, 3, 1, 2, 0, 4)$. Encoding the partial paths P_i in each shortened rectangle S_i, we get words

$$\begin{aligned} w_1(P) &= 10010 \\ w_2(P) &= 101111 \\ w_3(P) &= 101011 \\ w_4(P) &= 11 \\ w_5(P) &= 00010 \end{aligned}$$

Therefore,

$$\alpha(P) = I = (5; (2, 3, 1, 2, 0, 4); 10010, 101111, 101011, 11, 00010).$$

3.2.3 Mapping Inversion Statistics to Intermediate Statistics

This subsection describes the bijection

$$\beta : \mathcal{G}_{n,k,m} \to \mathcal{I}_{n,k,m},$$

which sends inversion-type statistics to intermediate statistics. We also describe β^{-1}.

Fix $g = (g_0, g_1, \ldots, g_{n-1}) \in \mathcal{G}_{n,k,m}$. Let $\beta(g)$ be the intermediate object $I = (s; v; w_1, \ldots, w_s)$ constructed as follows. Set $s = \max_{0 \leq i \leq n-1} g_i$. For $0 \leq j \leq s$, let v_j be the number of occurrences of j in the list g. Clearly, $v_j \geq 0$, $v_s > 0$, and $v_0 + \cdots + v_s = n$. We will see below that $v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \geq 0$ for $1 \leq i \leq s$.

We now describe the construction of the words w_i. Fix i with $1 \leq i \leq s$. Form a word w_i' from g as follows. Initially, w_i' is empty. Read the entries of g from left to right. Write down a zero in w_i' every time the symbol i is seen in g. Write down a one in w_i' every time a symbol in $\{i-1, \ldots, i-m\}$ is seen in g. Ignore all other symbols in g. By definition of v_i and w_i', there are v_i zeroes in w_i' and $v_{i-1} + \cdots + v_{i-m}$ ones in w_i'. (As usual, we set $v_j = 0$ for $j < 0$.)

If $i \leq k$, define $w_i = w_i'$. If $i > k$, define w_i to be w_i' with its first symbol erased. We assert that this erased symbol must be a 1. This is clear if g does not contain any occurrences of the symbol i. Suppose, instead, that g has at least one i. Consider the smallest index j such that $g_j \geq i$. Since $g_0 \leq k < i$, we have $j > 0$. Next, recall that $g_j \leq g_{j-1} + m$. Thus, $g_{j-1} \geq g_j - m \geq i - m$. Also, by minimality of j, $g_{j-1} \leq i - 1$. Finally, the first occurrence of i in g occurs at position j or later, by definition of j. It follows that the first occurrence of i in g is preceded (at position $j-1$) by an occurrence of some symbol in $\{i-1, \ldots, i-m\}$. This forces the word w_i' to start with 1. (On the other hand, for $i \leq k$, the first symbol of $w_i' = w_i$ could be 0, thanks to the condition $g_0 \leq k$.) Note that the word w_i does consist of v_i zeroes and $v_{i-1} + \cdots + v_{i-m} - \chi(i > k)$ ones, as required in the definition of $\mathcal{I}_{n,k,m}$. In particular, the requirement $v_{i-1} + \cdots + v_{i-m} - \chi(i > k) \geq 0$ also holds.

Example 3.18. Let $n = 6$, $k = 2$, $m = 3$, and $g = (1, 4, 4, 0, 3, 1) = g(P)$, where P is the path from Figure 3.1. Let $\beta(g) = I = (s; v; w_1, \ldots, w_s)$. We have $s = 4$, the maximum value appearing in g. We have $v_0 = 1$, $v_1 = 2$, $v_2 = 0$, $v_3 = 1$, and $v_4 = 2$. The words w_i' and w_i are:

$$\begin{aligned} w_1' &= 010, & w_1 &= 010 \\ w_2' &= 111, & w_2 &= 111 \\ w_3' &= 1101, & w_3 &= 101 \\ w_4' &= 10011, & w_4 &= 0011. \end{aligned}$$

Therefore,

$$I = (4; (1, 2, 0, 1, 2); 010, 111, 101, 0011).$$

If we compute $\alpha^{-1}(I)$, we obtain the path shown in Figure 3.5.

Before describing the inverse of β, we introduce the following terminology. Given $g \in \mathcal{G}_{n,k,m}$, define u_i to be the subword of g consisting of all symbols $g_j \leq i$ (for $0 \leq i \leq s = \max g_j$). Each word u_i clearly satisfies conditions (G2), (G3), and (G4), since g does. Also, we could use the word u_i instead of g to obtain the words w_i' and w_i, and the result would be the same. Note that $u_s = g$. Finally, observe that any occurrence of the symbol i in u_i is either the first symbol of u_i (which can only happen if $i \leq k$), or is immediately preceded by one of the symbols $i, i-1, \ldots, i-m$.

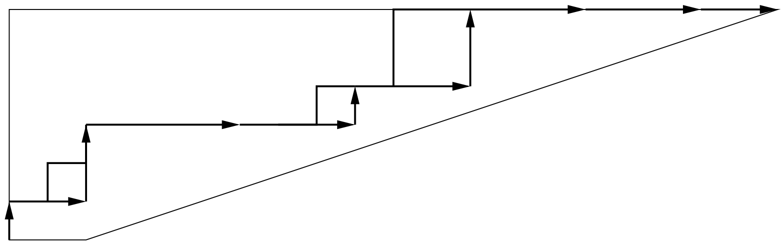


Figure 3.5: Path obtained by applying bijections to the path from Figure 3.1.

We now describe the inverse of β. Let $I = (s; v; w_1, \ldots, w_s)$ be an intermediate object. For $i \le k$, let $w'_i = w_i$. For $i > k$, let w'_i be w_i with an additional symbol 1 placed at the beginning. Let u_0 be the word consisting of v_0 zeroes (u_0 might be the empty word). We shall use the words $w'_1, \ldots, w'_s$ and u_0 to construct words $u_1, \ldots, u_s$, each of which satisfies (G2), (G3), and (G4). Finally, we will define $g = \beta^{-1}(I)$ to be u_s, which also satisfies (G1). In particular, u_s is an element of the target of β^{-1}, namely $\mathcal{G}_{n,k,m}$.

Assume that $1 \le i \le s$, and that the word u_{i-1} has already been constructed. We may also assume that u_{i-1} consists of v_0 zeroes, v_1 ones, …, and v_{i-1} copies of $i-1$. We further assume that u_{i-1} satisfies (G2), (G3), and (G4). To construct the next word u_i, we use the word w'_i to insert v_i copies of i into the current word u_{i-1}, as follows. Place a "marker" at the far left end of u_{i-1}, pointing to the gap just before the leftmost symbol of u_{i-1}. Scan the word w'_i from left to right, and perform the following actions.

- If the next symbol of w'_i is a zero, insert one copy of the symbol i in the gap pointed to by the marker. After the insertion, the marker points to the gap just after the newly inserted symbol.

- If the next symbol of w'_i is a one, scan right in u_{i-1} from the position of the marker, looking for the next occurrence of a symbol in the set $\{i-1, \ldots, i-m\}$. Set the marker to point to the gap immediately to the right of the first such symbol found.

This procedure produces the new word u_i, which clearly still satisfies all the inductive assumptions. In particular, condition (G4) holds for the following reason. If $i > k$, then w'_i starts with 1 by definition. Hence, the marker will advance past the far left end of

u_{i-1} before the first symbol i is inserted. We should also point out that the number of 1's in w_i' is always equal to the number of symbols $i-1, \ldots, i-m$ appearing in u_{i-1}. Hence, the algorithm that moves the marker right will never run out of symbols in u_{i-1} before reaching the end of w_i'. Finally, it is evident that this method of producing u_i from w_i' is the inverse of the method used in the definition of β to go from u_i (or equivalently g) to w_i'. It follows that the process just described is the inverse map to β.

Example 3.19. In the example at the end of §3.2.2, we saw that the path P from Figure 3.2 mapped to

$$\alpha(P) = I = (5; (2, 3, 1, 2, 0, 4); 10010, 101111, 101011, 11, 00010).$$

Let us compute $\beta^{-1}(I)$. Recall that $n = 12$, $m = 2$, $k = 3$ here. First, we write down the words

$$\begin{aligned} w_1' &= 10010 &&= w_1 \\ w_2' &= 101111 &&= w_2 \\ w_3' &= 101011 &&= w_3 \\ w_4' &= 111 &&= 1w_4 \\ w_5' &= 100010 &&= 1w_5 \end{aligned}$$

Next, we use the insertion algorithm above to compute

$$\begin{aligned} u_0 &= 00 \\ u_1 &= 01101 \\ u_2 &= 021101 \\ u_3 &= 02313101 \\ u_4 &= 02313101 \\ u_5 &= 023555135101. \end{aligned}$$

So, $\beta^{-1}(I) = (0, 2, 3, 5, 5, 5, 1, 3, 5, 1, 0, 1)$. If we apply the trivial map γ^{-1} to this last vector, we get the path shown in Figure 3.6.

We must still check that β sends each inversion statistic to the corresponding intermediate statistic. Fix $g = (g_0, \ldots, g_{n-1})$ and $I = \beta(g) = (s; v; w_1, \ldots, w_s)$. Consider each of the five statistics in turn.

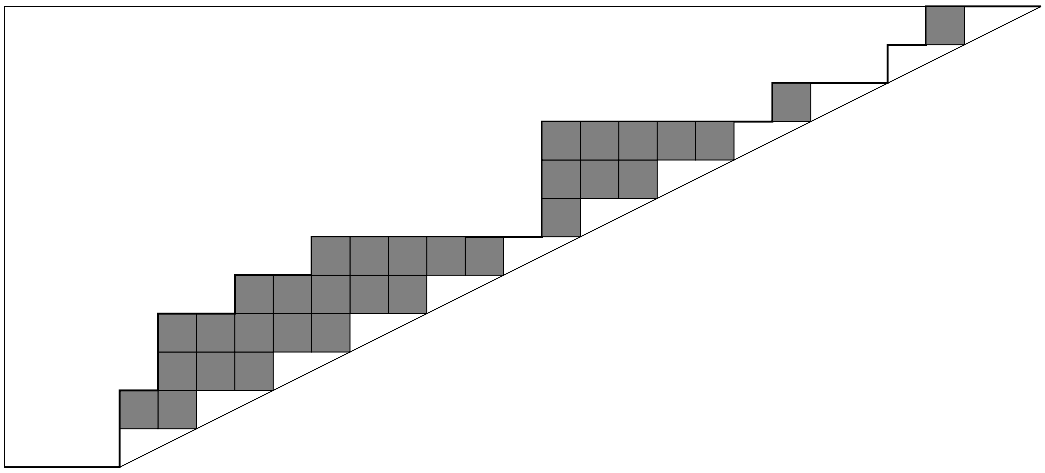


Figure 3.6: Path obtained by applying bijections to the path from Figure 3.2.

First, recall that $h_1(g)$ is given by the formula

$$h_1(g) = \sum_{0 \le p < t \le n-1} \max(0, m - |g_p - g_t|) + \sum_{p=0}^{n-1} \max(k - g_p, 0).$$

Of course, this formula makes sense for any integer vector g, not just for elements of $\mathcal{G}_{n,k,m}$. We will first show that rearranging the entries of g does not change the value of h_1. It clearly suffices to show that interchanging two adjacent entries of g, say g_i and g_{i+1}, does not change the value of h_1. Let g' denote g with g_i and g_{i+1} interchanged. We certainly have

$$\sum_{p=0}^{n-1} \max(k - g_p, 0) = \sum_{p=0}^{n-1} \max(k - g'_p, 0).$$

Consider terms in the two summations

$$\sum_{0 \le p < t \le n-1} \max(0, m - |g_p - g_t|) \text{ and } \sum_{0 \le p < t \le n-1} \max(0, m - |g'_p - g'_t|).$$

The terms in these two summations are identical (up to rearrangement), except that the term $\max(0, m - |g_i - g_{i+1}|)$ in the first sum is replaced by the term $\max(0, m - |g'_i - g'_{i+1}|)$ in the second sum. But the latter term is just $\max(0, m - |g_{i+1} - g_i|) = \max(0, m - |g_i - g_{i+1}|)$. Thus, the value of the h_1 statistic is the same for g and g'.

Now, g consists of v_0 zeroes, v_1 ones, ..., and v_s copies of s, in *some* order. For the purposes of computing h_1, we can replace g by the word

$$g'' = 0^{v_0}1^{v_1}\cdots s^{v_s}$$

consisting of v_0 zeroes, v_1 ones, ..., and v_s copies of s in *increasing* order. Using the definition of h_1, it is easy to show that

$$h_1(g) = h_1(g'') =$$
$$m\sum_{i\geq 0}\binom{v_i}{2} + \sum_{i\geq 0} v_i\left(\max(k-i,0) + \sum_{j=1}^{m}(m-j)v_{i-j}\right) = Q_1(I).$$

To see this, we group terms in the formula

$$h_1(g'') = \sum_{0\leq p<t\leq n-1}\max(0, m-|g''_p - g''_t|) + \sum_{p=0}^{n-1}\max(k-g''_p, 0).$$

The terms here with $g''_t = i$ and $g''_p - g''_t = 0$ group together to give the term $m\binom{v_i}{2}$ above. The terms here with $g''_t = i$ and $0 < |g''_p - g''_t| = j \leq m$ group together to give the term $(m-j)v_iv_{i-j}$ above. Finally, the terms in the last sum here with $g''_p = i$ group together to give the term $v_i\max(k-i,0)$ above.

Second, recall that $h_2(g)$ is given by the formula

$$h_2(g) = \sum_{u<v}\chi(g_u - g_v \in \{1, 2, \ldots, m\}).$$

We can rewrite this formula by classifying contributing terms based on the value of g_u. If $g_u = i$, then a pair $u < v$ contributes to the sum if and only if $g_v \in \{i-1, i-2, \ldots, i-m\}$. This occurs if and only if g_u is encoded as a 0 in w'_i and g_v is encoded as a 1 in w'_i and the 0 precedes the 1 in w'_i. In other words, the contributions to $h_2(g)$ coming from pairs $u < v$ with $g_u = i$ correspond exactly to the *coinversions* in w'_i. Adding over all possible i, we get

$$h_2(g) = \sum_{u<v}\chi(g_u - g_v \in \{1, 2, \ldots, m\}) = \sum_{i=1}^{s} coinv(w'_i).$$

Next, recall that w_i is either equal to w'_i (when $i \leq k$) or is w'_i with the initial 1 deleted (when $i > k$). But deleting the initial 1 from w'_i (which consists of zeroes and ones only) has no effect on the number of coinversions. Hence,

$$h_2(g) = \sum_{i=1}^{s} coinv(w_i') = \sum_{i=1}^{s} coinv(w_i) = Q_2(I).$$

Third, recall that $h_3(g)$ is given by the formula

$$h_3(g) = \sum_{u<v} \chi(g_u - g_v \in \{-m, \ldots, -2, -1\}) - \sum_{u=0}^{n-1} \chi(g_u > k).$$

Rewrite the first summation by classifying contributing terms based on the value of g_v. If $g_v = i$, then a pair $u < v$ contributes to the sum if and only if $g_u \in \{i-m, \ldots, i-2, i-1\}$. This occurs if and only if g_u is encoded as a 1 in w_i' and g_v is encoded as a 0 in w_i' and the 1 precedes the 0 in w_i'. In other words, the contributions to the first summation coming from pairs $u < v$ with $g_v = i$ correspond exactly to the *inversions* in w_i'. Adding over all possible i, we get

$$\sum_{u<v} \chi(g_u - g_v \in \{-m, \ldots, -2, -1\}) = \sum_{i=1}^{s} inv(w_i').$$

Thus, we have now shown that

$$h_3(g) = \sum_{i=1}^{s} inv(w_i') - \sum_{u=0}^{n-1} \chi(g_u > k).$$

Let us compare $\sum_{i=1}^{s} inv(w_i')$ to $Q_3(I) = \sum_{i=1}^{s} inv(w_i)$. Recall that w_i is either equal to w_i' (when $i \leq k$) or is w_i' with the initial 1 deleted (when $i > k$). Fix a value $i > k$. Deleting the initial 1 from w_i' will decrease the number of inversions by the total number of zeroes in w_i', which all come after the initial 1. By definition, the number of such zeroes is v_i, which is the number of i's in the list g, which is the number of positions u such that $g_u = i$. If we add up these losses over all choices of $i > k$, we see that

$$Q_3(I) = \sum_{i=1}^{s} inv(w_i) = \sum_{i=1}^{s} inv(w_i') - \sum_{u=0}^{n-1} \chi(g_u > k) = h_3(g).$$

Fourth, note that there are v_i copies of i in the list g, by definition of β. Therefore,

$$a(g) = \sum_{j=0}^{n-1} g_j = \sum_{i=0}^{s} iv_i = T(I).$$

Fifth, by definition of β,

$$ht(g) = \max_j g_j = s = Z(I).$$

Theorem 3.20. *$H_{n,k,m}(q_1, q_2, q_3, t, z)$ is given by formula (3.2).*

Proof. This follows immediately from Theorem 3.15 and the existence of the weight-preserving map β just described. □

3.3 Symmetry Properties

The five-variable generating function from the previous section has various symmetry properties, which are discussed in this section.

Theorem 3.21.

$$B_{n,k,m}(q_1, q_2, q_3, t, z) = B_{n,k,m}(q_1, q_3, q_2, t, z).$$

Proof. This is obvious from the formula for the generating function, since the q_2, q_3-binomial coefficients there are symmetric in q_2 and q_3. There is also an easy bijective proof of this symmetry. Given a path P, we simply rotate the partial paths inside each shortened rectangle S_i by 180°. This action does not affect the bounce path, so that the type-1 area, bounce score, and bounce count are unaffected. However, the rotation interchanges the number of type-2 area cells and type-3 area cells. □

For example, Figure 3.7 shows the result of applying this operation to the path P from Figure 3.4.

The corresponding bijection acting on the intermediate objects $I = (s; v; w_1, \dots, w_s)$ simply reverses each word w_i. This reversal obviously interchanges $inv(w_i)$ and $coinv(w_i)$, but the other three intermediate statistics are unaffected.

On the other hand, the symmetry of the statistics h_2 and h_3 on $\mathcal{G}_{n,k,m}$ is not immediately evident from their definition. To interchange these statistics, one must first use β to obtain an intermediate object I, then reverse each word w_i, then apply β^{-1}.

The second symmetry property involves the trivariate generating function

$$B''_{n,k,m}(q, r, t) = H''_{n,k,m}(q, r, t).$$

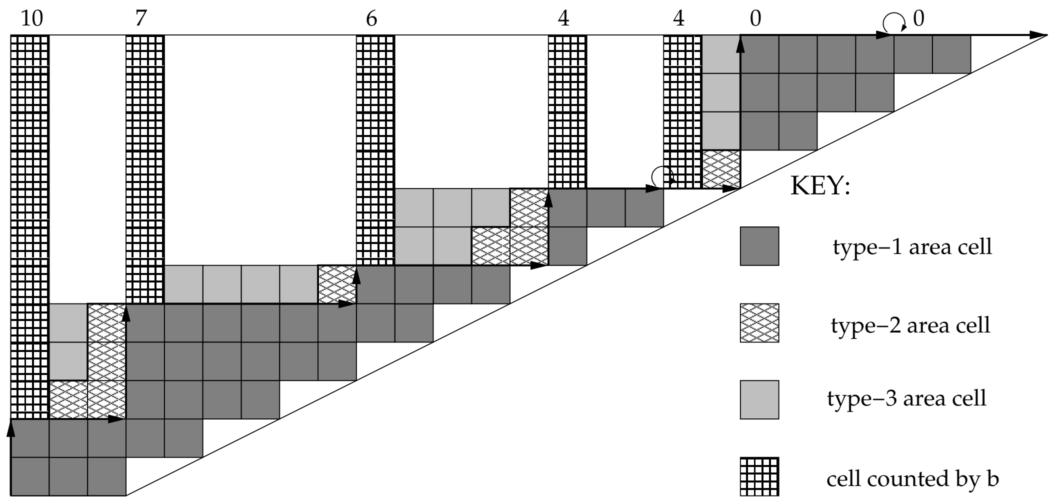


Figure 3.7: Interchanging type-2 cells and type-3 cells in a trapezoidal path.

Corollary 3.22.

$$B''_{n,k,m}(q,r,t) = B''_{n,k,m}(r,q,t).$$

Proof. This is obvious from the previous theorem, since

$$B''_{n,k,m}(q,r,t) = B_{n,k,m}(qr,q,r,t,1) = B_{n,k,m}(rq,r,q,t,1) = B''_{n,k,m}(r,q,t).$$

$\square$

Theorem 3.23. *The five statistics a, a', b, h^+ and h^- all have the same* univariate *distribution on $\mathcal{T}_{n,k,m}$, i.e.,*

$$\sum_{P\in\mathcal{T}_{n,k,m}} q^{a(P)} = \sum_{P\in\mathcal{T}_{n,k,m}} q^{a'(P)} = \sum_{P\in\mathcal{T}_{n,k,m}} q^{b(P)} = \sum_{P\in\mathcal{T}_{n,k,m}} q^{h^+(P)} = \sum_{P\in\mathcal{T}_{n,k,m}} q^{h^-(P)}.$$

Proof. Here is a bijective proof. Consider a path $P \in \mathcal{T}_{n,k,m}$. Let $Q = \phi^{-1}(P)$. Then $a(P) = b(Q)$, so a and b have the same distribution. Also, $a(Q) = a_1(Q) + a_2(Q) = h_1(P) + h_2(P) = h^+(P)$, so a and h^+ have the same distribution. Finally, the rotation maps discussed above show that a and a' have the same distribution, as do h^+ and h^-. $\square$

We can write this result in terms of generating functions:

$$B''_{n,k,m}(q,1,1) = B''_{n,k,m}(1,q,1) = B''_{n,k,m}(1,1,q)$$

$$= H''_{n,k,m}(q,1,1) = H''_{n,k,m}(1,q,1) = H''_{n,k,m}(1,1,q).$$

For the bivariate generating functions, we have

$$B'_{n,k,m}(q,1) = B'_{n,k,m}(1,q) = H'_{n,k,m}(q,1) = H'_{n,k,m}(1,q).$$

Conjecture 3.24 (Joint Symmetry).

$$B'_{n,k,m}(q,t) = B'_{n,k,m}(t,q).$$

This has been confirmed by computer experiments for many small values of n, k, and m. The formula has been proved true for all n only in the case $k = 0$ and $m = 1$. The proof, due to Garsia and Haglund, is long and non-combinatorial (see [14]).

3.4 Recursions

In this section, we derive a recursion that can be used to compute $B_{n,k,m}(q_1, q_2, q_3, t, z)$ and related generating functions. The basic idea is to reduce a path D to a smaller path D' by "removing the first bounce" from the bounce path $B(D)$. Some complications arise when $m > 1$ or $k > 0$ because the first bounce affects the dimensions of subsequent bounces according to the rule (3.1).

To deal with this problem, we must introduce more intermediate generating functions that will be used in the recursion.

Definition 3.25. Let $(v_0, \ldots, v_{m-1})$ be a fixed sequence of m nonnegative integers whose sum is at most n. Consider the subset $\mathcal{I}^{v_0,\ldots,v_{m-1}}_{n,k,m}$ of the collection $\mathcal{I}_{n,k,m}$ consisting of all intermediate objects $I = (s; y; w_1, \ldots, w_s)$ such that:

- $y_i = v_i$ for $0 \leq i \leq m-1$.
- w_i consists of a string of ones followed by a string of zeroes, for $1 \leq i \leq m-1$.

In terms of bounce paths, the first condition says that the first m vertical moves in the bounce path have been fixed to be $(v_0,\ldots,v_{m-1})$. The second condition says that the first $m-1$ bounce rectangles contain no type-2 area cells. We refer to such a rectangle as an *empty* rectangle.

Assume temporarily that the subset $\mathcal{I}_{n,k,m}^{v_0,\ldots,v_{m-1}}$ is *non-empty*. Define the generating function

$$F_{n,k,m}^{v_0,\ldots,v_{m-1}} = \sum_{I\in\mathcal{I}_{n,k,m}^{v_0,\ldots,v_{m-1}}} q_1^{Q_1(I)}q_2^{Q_2(I)}q_3^{Q_3(I)}t^{T(I)}z^{Z(I)}.$$

There is a summation formula for this generating function similar to formula (3.2) from §3.2. If $v_0+\cdots+v_{m-1}<n$, then we have

$$F_{n,k,m}^{v_0,\ldots,v_{m-1}} = \sum_{(v_m,\ldots,v_s)} z^s t^{\sum_{i=0}^s iv_i} q_1^{pow_1} q_3^{pow_3} \prod_{i=m}^{s} \begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-\chi(i>k) \\ v_i, v_{i-1}+\cdots+v_{i-m}-\chi(i>k) \end{bmatrix}_{q_2,q_3}, \tag{3.3}$$

where

$$pow_1 = m\sum_{i=0}^{s}\binom{v_i}{2} + \sum_{i=0}^{s} v_i \left(\max(k-i,0) + \sum_{j=1}^{\min(m,i)} (m-j)v_{i-j}\right),$$

and

$$pow_3 = \sum_{i=1}^{m-1} v_i \left((\sum_{j=0}^{i-1} v_j) - \chi(i>k) \right),$$

and we sum over all sequences $(v_m,\ldots,v_s)$ of nonnegative integers such that $v_s>0$ and $v_m+\cdots+v_s = n-(v_0+\cdots+v_{m-1})$. The proof is virtually the same as the proof of (3.2). Since v_0 through v_{m-1} have been fixed in advance, we only sum over the remaining indices v_m through v_s. Furthermore, since the first $m-1$ words w_i have been specified in advance, we omit the first $m-1$ q_2,q_3-binomial coefficients in (3.2). They are replaced by fixed powers of q_2 and q_3 reflecting the contributions of the fixed words w_i to Q_2 and Q_3, namely

$$\prod_{i=1}^{m-1} q_2^0 q_3^{v_i(v_{i-1}+\cdots+v_0-\chi(i>k))}.$$

On the other hand, if $v_0+\cdots+v_{m-1}=n$, let s be the largest index i such that $v_i>0$. Then we have

$$F_{n,k,m}^{v_0,\ldots,v_{m-1}} = z^s t^{\sum_{i=0}^s iv_i} q_1^{pow_1} q_3^{pow_3}. \tag{3.4}$$

The preceding discussion assumed that the subset $\mathcal{I}_{n,k,m}^{v_0,\ldots,v_{m-1}}$ was not empty. If it is empty, we use formulas (3.3) and (3.4) as the *definition* of $F_{n,k,m}^{v_0,\ldots,v_{m-1}}$ in the cases $v_0+\cdots+v_{m-1}<n$ and $v_0+\cdots+v_{m-1}=n$, respectively.

We now derive a recursion for $F_{n,k,m}^{v_0,\ldots,v_{m-1}}$ in the case where $v_0+\cdots+v_{m-1}<n$. The initial conditions for this recursion are given by the formulas in (3.4), which cover the case where $v_0+\cdots+v_{m-1}=n$. To obtain the recursion, we break up formula (3.3) based on the value of the first summation index v_m. Assume that $k>0$ first. Consider a fixed choice of v_m in the range from 0 to $n-v_0-\cdots-v_{m-1}$. Write down (3.3) with n replaced by $n-v_0$, k replaced by $k-1$, v_i replaced by v_{i+1} for all i, and s replaced by $s-1$:

$$F_{n-v_0,k-1,m}^{v_1,\ldots,v_m} = \sum_{(v_{m+1},\ldots,v_s)} z^{s-1} t^{\sum_{i=0}^{s-1} i v_{i+1}} q_1^{pow'_1} q_3^{pow'_3} \prod_{i=m}^{s-1} \begin{bmatrix} v_{i+1}+\cdots+v_{i+1-m}-\chi(i>k-1) \\ v_{i+1}, v_i+\cdots+v_{i+1-m}-\chi(i>k-1) \end{bmatrix}_{q_2,q_3}, \tag{3.5}$$

where

$$pow'_1 = m\sum_{i=0}^{s-1}\binom{v_{i+1}}{2} + \sum_{i=0}^{s-1} v_{i+1}\left(\max(k-1-i,0) + \sum_{j=1}^{\min(m,i)} (m-j)v_{i+1-j}\right),$$

and

$$pow'_3 = \sum_{i=1}^{m-1} v_{i+1}\left((\sum_{j=0}^{i-1} v_{j+1}) - \chi(i>k-1) \right).$$

Replace each summation index i by $i-1$ and simplify. The result is:

$$F_{n-v_0,k-1,m}^{v_1,\ldots,v_m} = \sum_{(v_{m+1},\ldots,v_s)} z^{s-1} t^{\sum_{i=1}^{s} (i-1) v_i} q_1^{pow'_1} q_3^{pow'_3} \prod_{i=m+1}^{s} \begin{bmatrix} v_i+\cdots+v_{i-m}-\chi(i-1>k-1) \\ v_i, v_{i-1}+\cdots+v_{i-m}-\chi(i-1>k-1) \end{bmatrix}_{q_2,q_3}, \tag{3.6}$$

where

$$pow'_1 = m\sum_{i=1}^{s}\binom{v_i}{2} + \sum_{i=1}^{s} v_i\left(\max(k-i,0) + \sum_{j=1}^{\min(m,i-1)} (m-j)v_{i-j}\right),$$

and

$$pow'_3 = \sum_{i=2}^{m} v_i\left((\sum_{j=0}^{i-2} v_{j+1}) - \chi(i-1>k-1) \right) = \sum_{i=2}^{m} v_i\left((\sum_{j=1}^{i-1} v_j) - \chi(i>k) \right).$$

Compare this to the corresponding summand in the expression (3.3):

$$F^{v_0,\dots,v_{m-1}}_{n,k,m} = \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} \sum_{(v_{m+1},\dots,v_s)} z^s t^{\sum_{i=0}^s iv_i} q_1^{pow_1} q_3^{pow_3} \prod_{i=m}^{s} \begin{bmatrix} v_i+v_{i-1}+\cdots+v_{i-m}-\chi(i>k) \\ v_i,v_{i-1}+\cdots+v_{i-m}-\chi(i>k) \end{bmatrix}_{q_2,q_3}, \tag{3.7}$$

where

$$pow_1 = m\sum_{i=0}^{s}\binom{v_i}{2} + \sum_{i=0}^{s} v_i \left(\max(k-i,0) + \sum_{j=1}^{\min(m,i)} (m-j)v_{i-j}\right),$$

and

$$pow_3 = \sum_{i=1}^{m-1} v_i \left((\sum_{j=0}^{i-1} v_j) - \chi(i>k)\right).$$

To go from formula (3.6) to the corresponding summand in (3.7), we need to multiply the former by

$$z^1 t^{v_1+\cdots+v_s} q_1^{pow_1-pow_1'} q_3^{pow_3-pow_3'} \begin{bmatrix} v_m+\cdots+v_0-\chi(m>k) \\ v_m, v_{m-1}+\cdots+v_0-\chi(m>k)\end{bmatrix}_{q_2,q_3}.$$

Doing this multiplication, noting that $n - v_0 = v_1 + \cdots + v_s$, and adding over all choices of v_m, we obtain the recursion

$$F^{v_0,\dots,v_{m-1}}_{n,k,m} = zt^{n-v_0} q_1^{pow_1''} \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} q_3^{pow_3''} \begin{bmatrix} v_m+\cdots+v_0-\chi(m>k) \\ v_m,v_{m-1}+\cdots+v_0-\chi(m>k)\end{bmatrix}_{q_2,q_3} F^{v_1,\dots,v_m}_{n-v_0,k-1,m}, \tag{3.8}$$

where

$$pow_1'' = pow_1 - pow_1' = m\binom{v_0}{2} + kv_0 + \sum_{i=1}^{m-1} v_0 v_i (m-i),$$

and

$$pow_3'' = pow_3 - pow_3' = \sum_{i=1}^{m-1} v_0 v_i - v_m \left(\sum_{j=1}^{m-1} v_j - \chi(m>k)\right).$$

If $k = 0$, an entirely analogous computation (left to the reader) shows that

$$F^{v_0,\dots,v_{m-1}}_{n,0,m} = zt^{n-v_0} q_1^{pow_1''} \sum_{v_m=0}^{n-v_0-\cdots-v_{m-1}} q_3^{pow_3''} \begin{bmatrix} v_m+\cdots+v_0-1 \\ v_m, v_{m-1}+\cdots+v_0-1\end{bmatrix}_{q_2,q_3} F^{v_1,\dots,v_m}_{n-v_0,0,m}, \tag{3.9}$$

where now

$$pow_1'' = m\binom{v_0}{2} + \sum_{i=1}^{m-1} v_0 v_i (m-i),$$

and

$$pow_3'' = \sum_{i=1}^{m-1} v_0 v_i - v_1 - v_m \left(\sum_{j=1}^{m-1} v_j - 1 \right).$$

This concludes the derivation of the recursion.

The original generating functions $B_{n,k,m}$ can easily be recovered from the expressions $F_{n,k,m}^{v_0,\ldots,v_{m-1}}$. For, given a path $D \in \mathcal{T}_{n,k,m}$, we can append m empty columns of height n at the far left of the figure. This produces a path D' counted by the generating function $F_{n,k+m,m}^{0,0,\ldots,0}$; moreover, every such path D' arises in this way. The bounce statistics for D' are the same as those for D, except that $b(D') = b(D) + mn$ and $c(D') = c(D) + m$ because of the m extra bounces at the left side of the diagram for D'. We conclude that

$$B_{n,k,m}(q_1, q_2, q_3, t, z) = F_{n,k+m,m}^{0,0,\ldots,0}(q_1, q_2, q_3, t, z)/(t^{mn} z^m).$$

Remark 3.26. We derived the recursion above using messy, though straightforward, algebraic manipulations. Here is the combinatorial intuition behind those manipulations. Assume $k > 0$ for simplicity. Start with an object D' counted by $F_{n-v_0,k-1,m}^{v_1,\ldots,v_m}$. In terms of bounce statistics, D' is a path of height $n - v_0$ whose bounce path starts with vertical moves $v_1, \ldots, v_m$ and whose first $m-1$ bounce rectangles are empty. Assume we are trying to construct an object counted by $F_{n,k,m}^{v_0,\ldots,v_{m-1}}$. Then we get to choose v_m (accounting for the summation over v_m), but $v_0, \ldots, v_{m-1}$ are determined for us. If we add a new initial bounce of height v_0 to the beginning of D', we obtain a larger path D of height n that now has m empty bounce rectangles. The binomial coefficient in the recursion reflects the fact that we can now "fill in" the m^{th} empty rectangle to obtain an object counted by $F_{n,k,m}^{v_0,\ldots,v_{m-1}}$. The other fixed powers in the recursion appear because certain bounce rectangles and bounce slabs at the bottom of the figure for D' get "stretched" by the addition of the new initial bounce of height v_0. This stretching also changes the horizontal dimension of certain bounce rectangles, which explains why we insist that the first $m-1$ bounce rectangles be empty. Without this restriction, we would need to multiply and divide by the appropriate q_2, q_3-binomial coefficients, leading to an even messier recursion.

Remark 3.27. As an application of the recursion, one can compute simple explicit formulas for the specialization

$$F_{n,k,m}^{v_0,\ldots,v_{m-1}}(q,q,1,1/q,1)$$

for certain choices of the parameters. The case $k=0$, $m=1$ has already been discussed in Chapter 1. The more complicated case $k=0$, $m>1$ was treated in Chapter 2. We now briefly consider the case $k>0$, $m=1$. Define $G_{n,k}^s(q) = F_{n,k,1}^s(q,q,1,1/q,1)$; observe that $B'_{n,k,1}(q,1/q) = G_{n,k+1}^0(q)\cdot q^n$. Recall from Chapter 1 that

$$G_{n,0}^s(q) = q^{-(n^2+n)/2+ns}\left(\begin{bmatrix}2n-s-1\\ n-s,n-1\end{bmatrix}_q - q^s\begin{bmatrix}2n-s-1\\ n-s-1,n\end{bmatrix}_q\right).$$

An easy combinatorial argument shows that $G_{n,1}^s(q) = G_{n+1,0}^{s+1}(q)$, so the last formula also gives

$$G_{n,1}^s(q) = q^{\frac{-(n+1)^2-(n+1)}{2}+(n+1)(s+1)}\left(\begin{bmatrix}2n-s\\ n-s,n\end{bmatrix}_q - q^{s+1}\begin{bmatrix}2n-s\\ n-s-1,n+1\end{bmatrix}_q\right).$$

Assume that $k\geq 1$ now. Making the appropriate substitutions in the five-variable recursion, we find that the specialized generating functions satisfy the recursion

$$G_{n,k}^s(q) = q^{s(s-1)/2+ks+s-n}\sum_{r=0}^{n-s}\begin{bmatrix}r+s\\ r,s\end{bmatrix}_q G_{n-s,k-1}^r(q),$$

subject to the initial condition

$$G_{n,k}^n(q) = q^{n(n-1)/2+kn}.$$

Substituting the formula for $G_{n-s,1}^r(q)$ into this recursion, one can obtain the formula

$$G_{n,2}^s(q) = q^{(n-n^2)/2-2n+s(n+3)}\left(\begin{bmatrix}2n-s+1\\ n-s,n+1\end{bmatrix}_q - q^1\begin{bmatrix}2n-s+1\\ n-s-1,n+2\end{bmatrix}_q\right).$$

The proof of this formula is exactly like the corresponding proof for $G_{n,0}^s(q)$ in Chapter 1, so we omit it.

One would expect this process to continue, leading to formulas for $G_{n,k}^s(q)$ as differences of two q-binomial coefficients multiplied by appropriate powers of q. Unfortunately, this appears *not* to be true for $k\geq 3$. For example, when $(n,k,s)=(2,3,1)$, we have

$$G_{n,k}^s(q) = q^C(1+q^2+q^4+q^5),$$

and this polynomial is not of the expected form

$$q^A \begin{bmatrix} 5 \\ 1,4 \end{bmatrix}_q - q^B \begin{bmatrix} 5 \\ 0,5 \end{bmatrix}_q$$

for any choice of A and B. The earlier proofs using the recursion break down at $k = 3$ because the powers of q occurring in the formula for $G^s_{n,2}(q)$ are "bad," leading to a q-series that is not easily evaluated.

Acknowledgement: This chapter is essentially a reprint, with minor modifications, of the paper "Trapezoidal Lattice Paths and Multivariate Analogues" by N. Loehr, which has been accepted for publication in *Advances in Applied Mathematics.* The dissertation author was the primary investigator and sole author of this paper.

4

Labelled Lattice Paths and Generalized Hilbert Series

In Chapter 1, we discussed the Hilbert series of the diagonal harmonics modules and gave conjectured combinatorial interpretations of these Hilbert series in terms of labelled Dyck paths. We briefly recall the relevant definitions.

Definition 4.1. Let n be a positive integer.

(1) Consider the diagonal harmonics module DH_n, as discussed in §1.3.5. Let $d(h,k,n)$ denote the dimension of the (h,k)-component of DH_n. Explicitly, $d(h,k,n)$ is the dimension of the subspace of polynomials $f \in DH_n$ such that f has degree h in the x-variables and f has degree k in the y-variables. The *Hilbert series of DH_n* is

$$H_n(q,t) = \sum_{h\geq 0}\sum_{k\geq 0} d(h,k,n)q^h t^k.$$

(2) The *symmetric-function formula* for the Hilbert series is

$$SH_n(q,t) = \nabla(s_{1^n})|_{s_\lambda = f_\lambda},$$

where ∇ denotes the nabla operator from §1.3.3.

(3) The *first combinatorial formula* for the Hilbert series is

$$CH_n(q,t) = \sum_{P\in\mathcal{P}_n} q^{area(P)} t^{dinv(P)},$$

where $\mathcal{P}_n$ is the collection of labelled Dyck paths of order n, and *dinv* is the statistic defined in §1.5.1.

(4) The *second combinatorial formula* for the Hilbert series is

$$CH_n'(q,t) = \sum_{P \in \mathcal{Q}_n} q^{dmaj(P)} t^{area'(P)},$$

where $\mathcal{Q}_n$ is the second collection of labelled Dyck paths of order n, and $dmaj$ and $area'$ are the statistics defined in §1.5.2.

It follows from the work of Haiman [18, 21] that $H_n(q,t) = SH_n(q,t)$ for all n. We showed in Chapter 1 that $CH_n(q,t) = CH_n'(q,t)$ for all n. It is a conjecture of Haglund, Haiman, and the present author [17] that $H_n(q,t) = CH_n(q,t)$ for all n.

This chapter begins by describing certain generalizations of the diagonal harmonics modules, which were first studied by Garsia and Haiman. We then introduce combinatorial statistics on labelled trapezoidal lattice paths that generalize the *area* and *dinv* statistics appearing in $CH_n(q,t)$. In the case of labelled lattice paths inside triangles, these statistics give a conjectured combinatorial model for the Hilbert series of the generalized diagonal harmonics modules.

We prove an explicit formula for the generating function of these new statistics on labelled trapezoidal paths. The formula (1.24) mentioned in Chapter 1 follows as a special case of this new formula. Remarkably, the new formula can be derived in a completely different way, leading to yet another pair of statistics on labelled paths with the same generating function. As a corollary, it follows that all statistics being discussed have the same univariate distribution. In fact, there are explicit bijections on labelled paths that map any given statistic to any other statistic. These results, when specialized to labelled Dyck paths, settle one of the open questions from [17]. We also conjecture that the pairs of statistics introduced here are *jointly* symmetric. This conjecture has been confirmed by computer for small values of the parameters, but remains unproved (as of this writing) even in the case of labelled Dyck paths.

4.1 Generalizations of Diagonal Harmonics Modules

This section describes generalizations of the diagonal harmonics modules, which were introduced by Garsia and Haiman in [15].

Definition 4.2. Fix integers $m, n \geq 1$. We define the *generalized diagonal harmonics module* $DH_n^{(m)}$ *of order* m *in* n *variables* as follows. As in §1.3.5, let S_n act on the polynomial ring $R_n = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ via the diagonal action. Let A_n denote the ideal in R_n generated by all polynomials $P \in R_n$ for which

$$\sigma \cdot P = sgn(\sigma)P \text{ for all } \sigma \in S_n.$$

Let A_n^m denote the ideal in R_n generated by all products $P_1 P_2 \cdots P_m$, where each $P_i \in A_n$. Let J_n denote the ideal in R_n generated by all polarized power sums

$$\sum_{i=1}^{n} x_i^h y_i^k \quad (h + k \geq 1).$$

Finally, define

$$R_n^{(m)}[X;Y] = A_n^{m-1}/JA_n^{m-1}.$$

If $\sigma \in S_n$ and $f \in R_n^{(m)}[X;Y]$, the diagonal action induces an action of S_n on this module, which we denote by $\sigma \cdot f$. Define a new action of S_n by setting

$$\sigma \star f = (sgn(\sigma))^{m-1} \sigma \cdot f.$$

$DH_n^{(m)}$ is defined to be the doubly-graded module $R_n^{(m)}[X;Y]$ with this new action.

As with the original diagonal harmonics module, we would like to understand the Frobenius series $F_n^{(m)}(q,t)$, the Hilbert series $H_n^{(m)}(q,t)$, and the generating function for the sign character $RC_n^{(m)}(q,t)$ of $DH_n^{(m)}$. We have the following results, analogous to those in §1.3.5.

First, Haiman's work implies that the Frobenius series of $DH_n^{(m)}$ is given by

$$F_n^{(m)}(q,t) = \nabla^m(s_{1^n}).$$

By Theorem 1.29 and the definition of nabla, we have

$$\nabla^m(s_{1^n}) = \sum_{\mu \vdash n} \frac{\tilde{H}_\mu t^{mn(\mu)} q^{mn(\mu')} (1-t)(1-q) \Pi_\mu(q,t) B_\mu(q,t)}{h_\mu(q,t) h'_\mu(q,t)}. \tag{4.1}$$

As in the case $m = 1$, there are nice formulas for the specializations at $t = 1$ and $t = 1/q$. Recall from Definition 2.6 that $\mathcal{D}_n^{(m)}$ denotes the collection of m-Dyck paths of order n.

Theorem 4.3. *(1) For an m-Dyck path D of order n, define $a_i(D)$ to be the number of vertical steps taken by the path along the line $x = i$. Then*

$$\nabla^m(s_{1^n})|_{t=1} = \sum_{D \in \mathcal{D}_n^{(m)}} q^{area(D)} \prod_{i=0}^{mn-1} e_{a_i(D)},$$

where e_j denotes an elementary symmetric function, as usual.

(2)

$$q^{mn(n-1)/2} \nabla^m(s_{1^n})\Big|_{t=1/q} = \sum_{\mu \vdash n} s_\mu \frac{s_{\mu'}(1, q, q^2, \ldots, q^{mn})}{[mn+1]_q}.$$

Proof. See Theorem 4.3 and Corollary 4.1 in [15]. ☐

Formula (4.1) gives the Frobenius series of $DH_n^{(m)}$ in terms of the symmetric functions $\tilde{H}_\mu$. To get the Hilbert series of $DH_n^{(m)}$, we can expand $\tilde{H}_\mu$ in terms of Schur functions and replace each s_λ by f_λ.

4.2 Statistics for Labelled Trapezoidal Lattice Paths

In Chapter 2, we discussed combinatorial statistics on m-Dyck paths whose generating functions are conjectured to give $OC_n^{(m)}(q,t)$. This section generalizes some of those statistics to labelled trapezoidal paths. The new statistics for triangular paths give a conjectured combinatorial interpretation for the Hilbert series of the modules $DH_n^{(m)}$.

Definition 4.4. Fix integers $n, k, m \geq 0$.

(1) A *labelled lattice path of height n* consists of a lattice path having n vertical steps labelled $1, 2, \ldots, n$ and an unspecified number of unlabelled horizontal steps. When drawing a labelled path, our convention is to place the label for each vertical step in the lattice square directly right of that vertical step. We call a labelled lattice path *valid* if and only if the labels in each column increase from bottom to top.

(2) A *labelled trapezoidal path* of type (n, k, m) is a valid labelled lattice path whose underlying unlabelled path P lies in $\mathcal{T}_{n,k,m}$. Let $\mathcal{P}_{n,k,m}$ denote the collection of all such labelled paths.

As in the case of labelled Dyck paths, we can specify a labelled trapezoidal path P by giving a pair of vectors

$$\vec{g}(P) = (g_0, g_1, \ldots, g_{n-1}), \quad \vec{p}(P) = (p_0, p_1, \ldots, p_{n-1}),$$

where $g_i(P)$ is the number of area cells in the i^{th} row from the bottom, and p_i is the label of the vertical step in the i^{th} row from the bottom. It is easy to see that a vector of n integers $(g_0, \ldots, g_{n-1})$ corresponds to a legal path in $\mathcal{T}_{n,k,m}$ if and only if the following conditions hold:

(A) $g_0 \in \{0, 1, \ldots, k\}$.

(B) $g_i \geq 0$ for all i.

(C) $g_{i+1} \leq g_i + m$ for all i.

Moreover, the associated vector of integers $\vec{p}(P)$ represents a valid labelling if and only if:

(D) $p_0, \ldots, p_{n-1}$ is a permutation of $1, 2, \ldots, n$.

(E) For all i, if $g_{i+1} = g_i + m$, then $p_i < p_{i+1}$.

Thus, when convenient, we may regard $\mathcal{P}_{n,k,m}$ as the set of all pairs of vectors $(\vec{g}, \vec{p})$ satisfying (A)—(E).

Example 4.5. Figure 4.1 shows a typical labelled path in $\mathcal{P}_{6,2,3}$. This object corresponds to the vector pair

$$((1, 4, 4, 0, 3, 1), (3, 5, 4, 1, 6, 2)).$$

We have the following analogues of the *area* and *dinv* statistics.

Definition 4.6. (1) The *area* of $P = (\vec{g}, \vec{p}) \in \mathcal{P}_{n,k,m}$ is defined by

$$area(P) = \sum_{i=0}^{n-1} g_i.$$

This is the number of area cells in the diagram of P, as usual.

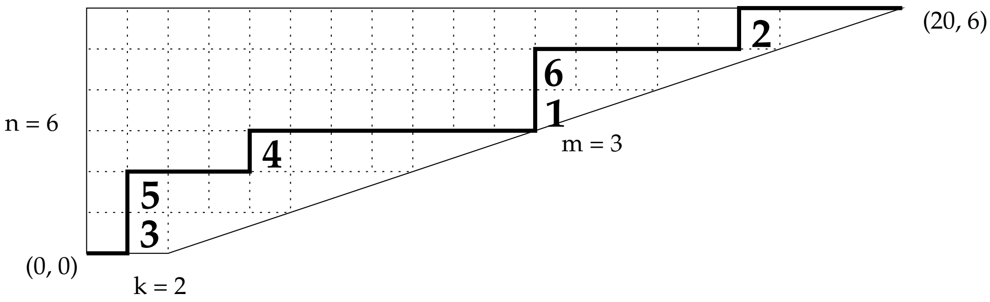


Figure 4.1: A labelled trapezoidal path.

(2) As above, set $r^+ = \max(r, 0)$ for any integer r. The *inversion statistic* of P is defined by letting

$$\begin{aligned}
h(P) &= h_1(P) + h_2(P) + h_3(P) - h_4(P), \text{ where:} \\
h_1(P) &= \sum_{i<j} (m - |g_i - g_j|)^+ \\
h_2(P) &= \sum_{i=0}^{n-1} (k - g_i)^+ \\
h_3(P) &= \sum_{i<j} \chi(g_i - g_j \in \{1, 2, \ldots, m\} \text{ and } p_i > p_j) \\
h_4(P) &= \sum_{i<j} \chi(g_i - g_j \in \{0, -1, -2, \ldots, -(m-1)\} \text{ and } p_i > p_j)
\end{aligned}$$

Equivalently, we can define

$$h(P) = h_2(P) + \sum_{i<j} \sum_{d=0}^{m-1} \chi(A_{i,j,d}),$$

where $A_{i,j,d}$ is the logical statement

$$\begin{aligned}
&(g_i - g_j + d = 0 \text{ and } p_i < p_j) \text{ or} \\
&(g_i - g_j + d \in \{1, 2, \ldots, m-1\}) \text{ or} \\
&(g_i - g_j + d = m \text{ and } p_i > p_j).
\end{aligned}$$

The verification of this equivalence involves checking that the summands corresponding to a fixed choice of i and j in $h_1(P) + h_3(P) - h_4(P)$ always add up to the corresponding summand $\sum_{d=0}^{m-1} \chi(A_{i,j,d})$. This is done by considering cases

Table 4.1: Checking the equivalence of the two formulas for $h(P)$.

Value of $z = g_i - g_j$	Order of labels $p_i,\ p_j$	Contribution to $h_1(P) + h_3(P) - h_4(P)$	Value of $\sum_{d=0}^{m-1} A_{i,j,d}$
$z > m$	$p_i > p_j$	0	0
$z > m$	$p_i < p_j$	0	0
$1 \leq z \leq m$	$p_i > p_j$	$m - z + 1$	$m - z + 1$
$1 \leq z \leq m$	$p_i < p_j$	$m - z$	$m - z$
$-m < z \leq 0$	$p_i > p_j$	$m - \lvert z\rvert - 1$	$m - \lvert z\rvert - 1$
$-m < z \leq 0$	$p_i < p_j$	$m - \lvert z\rvert$	$m - \lvert z\rvert$
$z \leq -m$	$p_i > p_j$	0	0
$z \leq -m$	$p_i < p_j$	0	0

based on the value of $g_i - g_j$ and whether $p_i > p_j$ or $p_i < p_j$ holds. These cases are checked in Table 4.1.

(3) Define

$$CH_{n,k,m}(q,t) = \sum_{P \in \mathcal{P}_{n,k,m}} q^{area(P)} t^{h(P)}.$$

It is easy to check that $CH_{n,0,1}(q,t) = CH_n(q,t)$.

Example 4.7. For the path

$$P = ((1,4,4,0,3,1),(3,5,4,1,6,2)).$$

shown in Figure 4.1, where $n = 6$, $k = 2$, $m = 3$, the values of $g_i - g_j$ for $i < j$ are:

$$\begin{aligned} i = 1 : &\quad -3, -3, 1, -2, 0; \\ i = 2 : &\quad 0, 4, 1, 3; \\ i = 3 : &\quad 4, 1, 3; \\ i = 4 : &\quad -3, -1; \\ i = 5 : &\quad 2. \end{aligned}$$

Hence, we compute:

$$area(P) = 13,\ h_1(P) = 16,\ h_2(P) = 4,\ h_3(P) = 4,\ h_4(P) = 2,\ h(P) = 22.$$

Conjecture 4.8. *For all $n, m \geq 1$, we have*

$$CH_{n,0,m}(q,t) = H_n^{(m)}(q,t).$$

In other words, the statistics for labelled paths inside the triangle with vertices $(0,0)$, $(0,n)$, and (mn,n) give a combinatorial interpretation for the Hilbert series of the generalized diagonal harmonics module $DH_n^{(m)}$.

This conjecture has been confirmed for small values of n and m by computer, using the formula

$$H_n^{(m)}(q,t) = \nabla^m(s_{1^n})|_{s_\lambda = f_\lambda}.$$

Conjecture 4.9. *For all $n, m \geq 1$, we have the specializations*

$$q^{mn(n-1)/2} CH_{n,0,m}(q, 1/q) = [mn+1]_q^{n-1};$$

$$q^{n+mn(n-1)/2} CH_{n,1,m}(q, 1/q) = (1+q^{n+1}) \cdot [mn+2]_q^{n-1}.$$

The author has recently proved the first conjecture when $m = 1$; this proof will appear in a later work. At present, there are no conjectures for the corresponding specializations when $k > 1$.

Conjecture 4.10. *For all n, k, m, we have the joint symmetry*

$$CH_{n,k,m}(q,t) = CH_{n,k,m}(t,q).$$

As evidence for this conjecture, we will prove the univariate symmetry

$$CH_{n,k,m}(q,1) = CH_{n,k,m}(1,q).$$

The proof will use a generalization of the *bounce* statistic to labelled paths, which is defined later. First, we need to establish an explicit summation formula for $CH_{n,k,m}(q,t)$.

4.3 Summation Formula for $CH_{n,k,m}(q,t)$

In this section, we will derive a formula for the generating function $CH_{n,k,m}(q,t)$ as a summation over a collection of functions (equation (4.2) below). One application of this formula is the proof of the univariate symmetry $CH_{n,k,m}(q,1) = CH_{n,k,m}(1,q)$ mentioned above.

Here are some remarks that explain how the new formula was discovered. Examining the proof of the corresponding formula (1.24), which appears in [17], suggests that we should look at subcollections of $\mathcal{P}_{n,k,m}$ where the labels appearing on each "diagonal" are fixed in advance. More precisely, suppose we are given an ordered partition $S_0, S_1, \ldots, S_{k+mn}$ of the set of labels $\{1, 2, \ldots, n\}$ into pairwise disjoint subsets, some of which may be empty. Then we can consider only those labelled paths $P = (\vec{g}, \vec{p})$ in $\mathcal{P}_{n,k,m}$ such that $p_i \in S_j$ implies $g_i = j$. In other words, the set of labels in S_j must appear in rows of P that contain exactly j area cells.

In the original formula (1.24), where $k = 0$ and $m = 1$, it was convenient to represent the set partition $S_0, S_1, \ldots,$ as a permutation σ as follows. First, write down the word

$$w = |\ S_n\ |\ S_{n-1}\ | \cdots S_3\ |\ S_2\ |\ S_1\ |\ S_0$$

in which the elements of each S_j (read from left to right) appear in increasing order, and a bar symbol is drawn between consecutive sets S_j. Now, it is easy to see that conditions (A)—(E) imply the following properties of w when $k = 0$ and $m = 1$:

- $S_j = \emptyset$ implies $S_k = \emptyset$ for all $k > j$.
- The largest element of S_j is greater than the smallest element of S_{j-1} whenever both sets are nonempty.

Let σ denote w with all bar symbols erased; clearly, σ is a permutation of $\{1, 2, \ldots, n\}$. The first property says that there are never two or more consecutive bar symbols, except possibly at the beginning of the word w. The second property says that the *descents* of w occur precisely at the locations of the erased bars (occurring after the beginning of the word). Therefore, w is recoverable from σ: given σ, we simply draw bars wherever descents occur, and then draw extra bars at the beginning of w until there are n bars total. Of course, the sets $S_0, S_1, \ldots$ are recoverable from w.

Unfortunately, the two properties above are no longer guaranteed in the case where $k > 0$ or $m > 1$. Hence, we are led to seek another representation for the set partition $S_0, S_1, \ldots$. It is convenient to introduce *functions* for this purpose. Let

$f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k + mn\}$ be a function. Then we obtain a set partition of $\{1, 2, \ldots, n\}$ by setting $S_j = f^{-1}(\{j\})$ for $0 \leq j \leq k + mn$. In this notation, we wish to consider the subcollection of paths $P = (\vec{g}, \vec{p})$ in $\mathcal{P}_{n,k,m}$ such that $f(p_i) = g_i$ for $1 \leq i \leq n$. It is also convenient to set up further notation to describe these functions.

Definition 4.11. Fix n, k, m. Let $f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k+mn\}$ be any function.

(1) Define the subcollection of *labelled paths of type* (n, k, m) *associated to* f by

$$\mathcal{P}_{n,k,m}(f) = \{P = (\vec{g}, \vec{p}) \in \mathcal{P}_{n,k,m} : f(p_i) = g_i \text{ for } 1 \leq i \leq n\}.$$

Note that, for certain choices of f, this subcollection may be empty.

(2) For any set T, define the usual *inverse image of* T *under* f by

$$f^{-1}(T) = \{x \in \{1, 2, \ldots, n\} : f(x) \in T\}.$$

Also, for any integer i, define

$$f^{-1}_{<i}(T) = \{x : x < i \text{ and } f(x) \in T\},$$

$$f^{-1}_{>i}(T) = \{x : x > i \text{ and } f(x) \in T\}.$$

For brevity, we may write $f^{-1}(j)$ instead of $f^{-1}(\{j\})$, etc.

(3) Define the *set partition associated to* f to be the list $(S_0, \ldots, S_{k+mn})$, where $S_j = f^{-1}(j)$ for each j.

(4) Define the *word of* f by

$$w(f) = S_0 \mid S_1 \mid S_2 \mid \cdots \mid S_{k+mn} \mid$$

where the elements of each S_j appear in *decreasing* order from left to right, followed by a bar symbol. Consecutive bar symbols appear in the word if and only if some S_j is empty. Note that this is the reversal of the word w described in the special case $k = 0$, $m = 1$ above. Also note that f is recoverable from $w(f)$, thanks to the bar symbols. We may safely omit bar symbols that occur together at the far right of the word of f.

(5) Let $\mathcal{F}_{n,k,m}$ denote the set of all functions $f : \{1, 2, \ldots, n\} \to \{0, 1, \ldots, k+m(n-1)\}$. For $f \in \mathcal{F}_{n,k,m}$, define

$$\begin{aligned}
maj(f) &= \sum_{j=1}^{n} f(j) \\
count(f,j) &= \chi(f(j) \le k) + |f_{<j}^{-1}(f(j)-m)| + |f_{>j}^{-1}(f(j))| \\
&\quad + |f^{-1}(\{f(j)-1, \ldots, f(j)-(m-1)\})|. \\
x_0(f) &= \sum_{j_1<j_2} (m - |f(j_1) - f(j_2)|)^+ + \sum_{j=1}^{n} (k - f(j))^+ \\
x_j(f) &= -|f_{>j}^{-1}(\{f(j), f(j)-1, \ldots, f(j)-(m-1)\})|
\end{aligned}$$

Example 4.12. Let $n = 7$, $k = 2$, $m = 2$. Let the function f be given by

$$f(1) = 2,\ f(2) = 0,\ f(3) = 0,\ f(4) = 3,\ f(5) = 0,\ f(6) = 2,\ f(7) = 2.$$

The word of f is

$$w(f) = 5\ 3\ 2 \mid \mid 7\ 6\ 1 \mid 4 \mid\mid\mid\mid\mid\mid\mid\mid\mid\mid\mid\mid,$$

where there are 12 trailing bar symbols. Also

$$maj(f) = 9,\ count(f,1) = 3,\ count(f,2) = 3,\ count(f,3) = 2,$$

$$count(f,4) = 3,\ count(f,5) = 1,\ count(f,6) = 5,\ count(f,7) = 4,$$

$$x_0(f) = 15 + 6 = 21,\ x_1(f) = -2,\ x_2(f) = -2,\ x_3(f) = -1,$$

$$x_4(f) = -2,\ x_5(f) = 0,\ x_6(f) = -1,\ x_7(f) = 0.$$

The goal of the rest of this section is to establish the following formula.

Theorem 4.13.

$$CH_{n,k,m}(q,t) = \sum_{f \in \mathcal{F}_{n,k,m}} q^{maj(f)} t^{x_0(f)} \prod_{j=1}^{n} t^{x_j(f)} [count(f,j)]_t. \tag{4.2}$$

In the coming proofs, it will be convenient to use the following notation. Given a labelled path $P = (\vec{g}, \vec{p})$, we can think of P as a single list of n "tiles"

$$P = \boxed{\begin{matrix} g_1 \\ p_1 \end{matrix}}\ \boxed{\begin{matrix} g_2 \\ p_2 \end{matrix}} \cdots \boxed{\begin{matrix} g_n \\ p_n \end{matrix}}. \tag{4.3}$$

In this notation, specifying a function $f \in \mathcal{F}_{n,k,m}$ is equivalent to specifying a collection of n tiles, namely $\begin{array}{|c|}\hline f(i) \\ i \\ \hline\end{array}$ for $1 \leq i \leq n$. The subcollection $\mathcal{P}_{n,k,m}(f)$ consists precisely of all rearrangements of these n tiles that satisfy the restrictions (A)—(E) above. Note that (B) and (D) are guaranteed to hold, by definition of f.

We will be interested in building the object P by putting down tiles one at a time. Thus, it is of interest to consider "partial" objects Q satisfying (A)—(C) and (E) but not necessarily (D).

Lemma 4.14. *Suppose $P \in \mathcal{P}_{n,k,m}(f)$, so P satisfies conditions (A)—(E). Let $i_1, \ldots, i_n$ be the word of f with all bar symbols erased. Let $S = \{i_s, \ldots, i_n\}$ be any suffix of this word, where $2 \leq s \leq n$. Let Q be obtained from P by removing all tiles of the form $\begin{array}{|c|}\hline f(i) \\ i \\ \hline\end{array}$ for $i \in S$. Then Q satisfies conditions (A)—(C) and (E).*

Proof. We prove the contrapositive in each case. If Q does not satisfy condition (A), then it begins with a tile of the form $\begin{array}{|c|}\hline x \\ y \\ \hline\end{array}$, where $x > k$. By definition of the word of f and S, all the removed tiles must have had top entries $x' \geq x$. Thus, the first tile of P must have had top entry at least x, and so P does not satisfy condition (A).

If Q does not satisfy condition (B), then P does not satisfy (B) either, since every tile in Q is a tile in P.

Suppose Q does not satisfy condition (C), so that there are consecutive tiles $\begin{array}{|c|c|}\hline x & x' \\ y & y' \\ \hline\end{array}$ in Q with $x' > x + m$. As before, the definitions of $w(f)$ and S show that any tiles in P that were between these two tiles of Q before being removed must have had top entry $x'' \geq x'$. Hence, the tile immediately following $\begin{array}{|c|}\hline x \\ y \\ \hline\end{array}$ in P still has top entry larger than $x + m$. So P does not satisfy condition (C).

Finally, suppose Q does not satisfy condition (E), so that are consecutive tiles $\begin{array}{|c|c|}\hline x & x' \\ y & y' \\ \hline\end{array}$ in Q with $x' = x + m$ and $y > y'$. Let $\begin{array}{|c|}\hline x'' \\ y'' \\ \hline\end{array}$ be the tile immediately following $\begin{array}{|c|}\hline x \\ y \\ \hline\end{array}$ in P. If $x'' = x'$ and $y'' = y'$, then P fails condition (E) already. Otherwise, by definition of $w(f)$ and S, we must have $x'' \geq x' = x + m$. Since P satisfies condition (C), we in fact have $x'' = x + m = x'$. Now, since the elements of S_{x+m} appear in *decreasing* order in the word of f, the presence of the tile $\boxed{\begin{array}{c} x+m \\ y' \end{array}}$ in Q is only possible if the value y'' in the removed tile is less than y'. But then $y > y''$, so that P fails condition (E) in this case too. $\square$

Lemma 4.15. *Given n, k, m and $f \in \mathcal{F}_{n,k,m}$, we have*

$$|\mathcal{P}_{n,k,m}(f)| = \prod_{j=1}^{n} count(f, j).$$

Proof. We can uniquely construct every object $P \in \mathcal{P}_{n,k,m}(f)$ as follows. Start with a pool of n available tiles $\boxed{\begin{array}{c} f(i) \\ i \end{array}}$, for $1 \leq i \leq n$. Let $i_1, \ldots, i_n$ be the word of f with all bar symbols erased. Starting with an empty list of tiles, form the object P by successively inserting each tile

$$\boxed{\begin{array}{c} f(i_1) \\ i_1 \end{array}}, \boxed{\begin{array}{c} f(i_2) \\ i_2 \end{array}}, \ldots, \boxed{\begin{array}{c} f(i_n) \\ i_n \end{array}}$$

into the list of previously inserted tiles. At each step, the new tile may be inserted anywhere in the existing list, provided that conditions (A), (C), and (E) hold. The previous lemma guarantees that all objects in $\mathcal{P}_{n,k,m}(f)$ can be constructed under these restrictions on tile insertions. Since the tiles are distinct, it is clear that there is a unique insertion order that will produce any given object P.

Thus, we need only count how many legal positions are available when each tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ is inserted. Fix j. First, observe that the insertion order ensures that $f(j) \geq f(i)$ for all previously inserted tiles $\boxed{\begin{array}{c} f(i) \\ i \end{array}}$. This means that conditions (C) and (E) automatically hold for the tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ and the tile immediately following it (if any). Thus, to check that conditions (A), (C), and (E) continue to hold after the insertion of tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$, we need only check that: (i) $f(j) \leq k$ if $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ is inserted in the leftmost position; or (ii) conditions (C) and (E) hold for the pair of tiles $\begin{array}{|c|c|}\hline f(\ell) & f(j) \\ \ell & j \\ \hline\end{array}$, where $\boxed{\begin{array}{c} f(\ell) \\ \ell \end{array}}$ is the tile immediately preceding $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$. By condition (C), we must have $f(\ell) \in \{f(j), f(j)-1, \ldots, f(j)-m\}$. By condition (E), if $f(\ell) = f(j) - m$, then we must also have $\ell < j$.

Now, consider the various places where the new tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ may be inserted.

- The tile may be inserted at the far left position, becoming the new first tile in the list. By condition (A), this is allowable if and only if $f(j) \le k$. So, we get a contribution of $\chi(f(j) \le k)$ to the position count.

- The tile may be inserted immediately after a tile of the form $\boxed{\begin{array}{c} f(j)-m \\ \ell \end{array}}$, where we need $\ell < j$ by condition (E). By definition of $w(f)$ and the tile insertion order, all such tiles have already been placed when tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ is being inserted. Therefore, the number of such tiles is
$$|f^{-1}_{<j}(f(j) - m)|.$$

- The tile may be inserted immediately after a tile of the form $\boxed{\begin{array}{c} f(j)-u \\ \ell \end{array}}$, where $1 \le u < m$ and ℓ is arbitrary. By definition of $w(f)$ and the tile insertion order, all such tiles have already been placed when tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ is being inserted. Therefore, the number of such tiles is
$$|f^{-1}(\{f(j) - 1, \ldots, f(j) - (m-1)\})|.$$

- The tile may be inserted immediately after a tile of the form $\boxed{\begin{array}{c} f(j) \\ \ell \end{array}}$, where ℓ is arbitrary. However, by definition of $w(f)$ and the tile insertion order, only those tiles with $\ell > j$ have been inserted prior to the insertion of tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$. Therefore, the number of such tiles is only

$$|f_{>j}^{-1}(f(j))|.$$

- The new tile can *only* be inserted in positions of the type described in the last four cases, thanks to condition (C).

In summary, for each j between 1 and n, the number of ways to place tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ is precisely

$$\chi(f(j) \leq k) + |f_{<j}^{-1}(f(j)-m)| + |f^{-1}(\{f(j)-1, \ldots, f(j)-(m-1)\})| + |f_{>j}^{-1}(f(j))|,$$

which is just $count(f, j)$. The formula in the statement of the lemma now follows from the product rule. ◻

Corollary 4.16.

$$CH_{n,k,m}(q,1) = \sum_{f \in \mathcal{F}_{n,k,m}} q^{maj(f)} \prod_{j=1}^{n} count(f,j). \tag{4.4}$$

Proof. Note that $\mathcal{P}_{n,k,m}$ is the disjoint union of the sets $\mathcal{P}_{n,k,m}(f)$ over all $f \in \mathcal{F}_{n,k,m}$. Fix f, and consider any $P \in \mathcal{P}_{n,k,m}$. We have

$$area(P) = \sum_{i=0}^{n-1} g_i = \sum_{i=0}^{n-1} f(p_i) = \sum_{i=1}^{n} f(i) = maj(f),$$

since the labels p_i are a permutation of $1, 2, \ldots, n$. Thus, all paths in $\mathcal{P}_{n,k,m}(f)$ contribute a summand $q^{maj(f)}$ to the generating function $CH_{n,k,m}(q,1)$. The stated formula then follows immediately from the previous lemma. ◻

Example 4.17. Let n,k,m, and f be as in the previous example. To construct an object $P \in \mathcal{P}_{n,k,m}(f)$, we should insert tiles in the following order:

$$\boxed{\begin{array}{c} 0 \\ 5 \end{array}} \; \boxed{\begin{array}{c} 0 \\ 3 \end{array}} \; \boxed{\begin{array}{c} 0 \\ 2 \end{array}} \; \boxed{\begin{array}{c} 2 \\ 7 \end{array}} \; \boxed{\begin{array}{c} 2 \\ 6 \end{array}} \; \boxed{\begin{array}{c} 2 \\ 1 \end{array}} \; \boxed{\begin{array}{c} 3 \\ 4 \end{array}}$$

An example of an object created in this way is

$$P = \boxed{\begin{array}{c} 2 \\ 6 \end{array}} \; \boxed{\begin{array}{c} 0 \\ 3 \end{array}} \; \boxed{\begin{array}{c} 2 \\ 7 \end{array}} \; \boxed{\begin{array}{c} 3 \\ 4 \end{array}} \; \boxed{\begin{array}{c} 2 \\ 1 \end{array}} \; \boxed{\begin{array}{c} 0 \\ 5 \end{array}} \; \boxed{\begin{array}{c} 0 \\ 2 \end{array}}$$

Note that $area(P) = 9 = maj(f)$.

Our next goal is to extend formula (4.4) to keep track of the statistic $h(P) = h_1(P)+h_2(P)+h_3(P)-h_4(P)$. The final formula, equation (4.2), will be proved in two steps. The first (easier) step involves analyzing the contribution of $h_1(P)+h_2(P)$. The next result shows that this quantity is constant for all objects P in a given subcollection $\mathcal{P}_{n,k,m}(f)$.

Lemma 4.18.

$$\sum_{P\in\mathcal{P}_{n,k,m}} q^{area(P)}t^{h_1(P)+h_2(P)} = \sum_{f\in\mathcal{F}_{n,k,m}} q^{maj(f)}t^{x_0(f)}\prod_{j=1}^{n} count(f,j). \tag{4.5}$$

Proof. In light of formula (4.4) and its proof, we need only show that

$$h_1(P)+h_2(P) = x_0(f) \text{ for all } P\in\mathcal{P}_{n,k,m}(f).$$

Recall that for $P\in\mathcal{P}_{n,k,m}(f)$, we have $g_i = f(p_i)$ for all i. Also, $p_0,\ldots,p_{n-1}$ is a rearrangement of $1,2,\ldots,n$, so we have

$$h_2(P) = \sum_{i=0}^{n-1}(k-g_i)^+ = \sum_{i=0}^{n-1}(k-f(p_i))^+ = \sum_{j=1}^{n}(k-f(j))^+.$$

Next, recall that

$$h_1(P) = \sum_{i<j}(m-|g_i-g_j|)^+.$$

This sum extends over all ordered pairs (i,j) with $0\le i<j\le n-1$. However, since $|g_i-g_j| = |g_j-g_i|$, we could equally well sum over all *unordered* pairs $\{i,j\}$ with $0\le i,j\le n-1$ and $i\ne j$. Hence,

$$\begin{aligned}
h_1(P) &= \sum_{\{i_1,i_2\}:i_1\ne i_2}(m-|g_{i_1}-g_{i_2}|)^+\\
&= \sum_{\{i_1,i_2\}:i_1\ne i_2}(m-|f(p_{i_1})-f(p_{i_2})|)^+\\
&= \sum_{\{j_1,j_2\}:j_1\ne j_2}(m-|f(j_1)-f(j_2)|)^+\\
&= \sum_{j_1<j_2}(m-|f(j_1)-f(j_2)|)^+.
\end{aligned}$$

Combining these calculations and comparing to the definition of $x_0(f)$, we get $h_1(P)+h_2(P) = x_0(f)$ as desired. $\square$

The next step is to analyze the value of $h_3(P) - h_4(P)$ for $P \in \mathcal{P}_{n,k,m}(f)$. Consider the partial objects

$$P_0, P_1, \ldots, P_n = P$$

that are constructed in Lemma 4.15 by inserting tiles in the order given by the word of f. We think of each newly inserted tile as contributing a certain increment to the statistic $h_3(P) - h_4(P)$. More specifically, let $y_0 = 0$ and, for $1 \le i \le n$, let

$$y_i = [h_3(P_i) - h_4(P_i)] \; - \; [h_3(P_{i-1}) - h_4(P_{i-1})] \, .$$

Then $h_3(P) - h_4(P) = \sum_{i=1}^{n} y_i$; note that y_i is the change in the statistic $h_3 - h_4$ (which may be positive or negative) resulting from the insertion of the i^{th} tile.

It will be convenient to alter the indexing scheme slightly, as follows. Suppose the i^{th} tile in the insertion order is $\boxed{\begin{matrix} f(j) \\ j \end{matrix}}$. Then define $z_j = y_i$. In words, z_j is the change in the statistic $h_3 - h_4$ due to the insertion of tile $\boxed{\begin{matrix} f(j) \\ j \end{matrix}}$. Note that $h_3(P) - h_4(P) = \sum_{j=1}^{n} z_j$.

We have shown, in the proof of Lemma 4.15, that there are exactly $c_j = count(f, j)$ valid positions in which tile $\boxed{\begin{matrix} f(j) \\ j \end{matrix}}$ may be inserted. Temporarily number these valid positions $0, 1, \ldots, c_j - 1$ reading from right to left. We will show later that, if the tile is placed in the valid position numbered p, then

$$z_j = x_j(f) + p \quad (0 \le p < c_j). \tag{4.6}$$

Thus, the contribution to $h_3 - h_4$ due to this particular tile insertion can be accounted for by the polynomial

$$t^{x_j(f)} \sum_{p=0}^{c_j - 1} t^p = t^{x_j(f)} \cdot [count(f, j)]_t,$$

which is a t-analogue of the number $count(f, j)$ in (4.4). By the product rule for generating functions, we conclude that

$$\sum_{P \in \mathcal{P}_{n,k,m}} t^{h_3(P) - h_4(P)} = \sum_{f \in \mathcal{F}_{n,k,m}} \prod_{j=1}^{n} x_j(f) \cdot [count(f, j)]_t.$$

Combining this with the previous analysis for $area$ and h_1+h_2, the desired formula (4.2) will follow immediately.

To prove the claims in the last paragraph, we need to consider the effect of inserting tile $\begin{array}{|c|}\hline f(j)\\ j\\ \hline\end{array}$ in *arbitrary* positions in the current tile configuration, not just the *valid* positions. We will now label each position with the change in h_3-h_4 caused by inserting tile $\begin{array}{|c|}\hline f(j)\\ j\\ \hline\end{array}$ in this position, regardless of the validity of the resulting partial object.

Example 4.19. Continuing Example 4.17, consider the partial object

$$Q = \begin{array}{|c|}\hline 2\\ 6\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 3\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ 7\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 5\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 2\\ \hline\end{array},$$

which has $h_3(Q)-h_4(Q)=3$. The next tile to be inserted is $\begin{array}{|c|}\hline 2\\ 1\\ \hline\end{array}$. The following diagram shows the change in h_3-h_4 when we insert this tile in all possible positions. We have also labelled which positions are valid.

$$\begin{array}{rccccccccccc}
 & \text{—} & \begin{array}{|c|}\hline 2\\ 6\\ \hline\end{array} & \text{—} & \begin{array}{|c|}\hline 0\\ 3\\ \hline\end{array} & \text{—} & \begin{array}{|c|}\hline 2\\ 7\\ \hline\end{array} & \text{—} & \begin{array}{|c|}\hline 0\\ 5\\ \hline\end{array} & \text{—} & \begin{array}{|c|}\hline 0\\ 2\\ \hline\end{array} & \text{—} \\
\text{validity:} & \text{yes} & & \text{yes} & & \text{no} & & \text{yes} & & \text{no} & & \text{no} \\
\text{change in } h_3-h_4\text{:} & 0 & & -1 & & -1 & & -2 & & -2 & & -2
\end{array}$$

Note that if we look at only the *valid* positions, from right to left, the changes we get are -2, -1, and 0, which are exactly the numbers $x_1(f)+p$ for $0 \le p < 3 = count(f,1)$.

Now, as in the previous example, assume that we choose to insert tile $\begin{array}{|c|}\hline 2\\ 1\\ \hline\end{array}$ immediately after $\begin{array}{|c|}\hline 2\\ 7\\ \hline\end{array}$, producing the partial object

$$Q' = \begin{array}{|c|}\hline 2\\ 6\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 3\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ 7\\ \hline\end{array}\ \begin{array}{|c|}\hline 2\\ 1\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 5\\ \hline\end{array}\ \begin{array}{|c|}\hline 0\\ 2\\ \hline\end{array},$$

with $h_3(Q') - h_4(Q') = 1$. The next tile to be inserted is $\boxed{\begin{matrix}3\\4\end{matrix}}$. The following diagram shows the change in $h_3 - h_4$ when we insert this tile in all possible positions. We have also labelled which positions are valid.

	—	$\boxed{\begin{matrix}2\\6\end{matrix}}$	—	$\boxed{\begin{matrix}0\\3\end{matrix}}$	—	$\boxed{\begin{matrix}2\\7\end{matrix}}$	—	$\boxed{\begin{matrix}2\\1\end{matrix}}$	—	$\boxed{\begin{matrix}0\\5\end{matrix}}$	—	$\boxed{\begin{matrix}0\\2\end{matrix}}$	—
validity:	no		yes		no		yes		yes		no		no
change in $h_3 - h_4$:	1		0		0		-1		-2		-2		-2

Note that if we look at only the *valid* positions, from right to left, the changes we get are -2, -1, and 0, which are exactly the numbers $x_4(f) + p$ for $0 \leq p < 3 = count(f, 4)$.

Let us make some observations about these examples. First, note that there are *blocks* of consecutive insertion positions for which the change in $h_3 - h_4$ is the same. Each such block (except possibly the leftmost block) consists of zero or more invalid positions terminated by one valid position, scanning from right to left. The leftmost block may or may not end with a valid position, depending on the value of k. Next, note that the change in $h_3 - h_4$ for positions in the rightmost block is $x_j(f)$. As we pass from one block to the next, scanning from right to left as always, the change in $h_3 - h_4$ increases by 1 each time. Finally, we have already proved (in Lemma 4.15) that the number of valid positions is exactly $count(f, j)$. Combining all these observations, we deduce that the claim (4.6) does hold in these two examples. The next lemma shows that these observations are true in general, and hence claim (4.6) always holds.

Lemma 4.20. *Fix n, k, m, and $f \in \mathcal{F}_{n,k,m}$. Let Q be a partial object, constructed as in the proof of Lemma 4.15 by inserting tiles in the order given by the word of f. Suppose $T_j = \boxed{\begin{matrix}f(j)\\j\end{matrix}}$ is the next tile to be inserted into Q. Label each insertion position with the change in $h_3 - h_4$ caused by inserting the new tile in that position. Then we have the following properties:*

(1) The rightmost position is labelled $x_j(f)$.

(2) Suppose two consecutive insertion positions in Q are separated by a tile $T_p =$

$\boxed{\begin{array}{c} f(p) \\ p \end{array}}$.

(a) *If the position just right of T_p is an invalid position for T_j, then the position just left of T_p has the same label as the position just right of T_p.*

(b) *If the position just right of T_p is a valid position for T_j, then the label of the position just left of T_p is one more than the label of the position just right of T_p.*

(3) *Hence, when scanning the count(f,j)* **valid** *insertion positions from right to left, their labels are precisely the numbers*

$$x_j(f) + p \quad (0 \le p < count(f,j)).$$

Proof. We begin by observing that, for any tile $T_p = \boxed{\begin{array}{c} f(p) \\ p \end{array}}$ in the partial object Q, we must have $f(j) \ge f(p)$; moreover, if $f(j) = f(p)$, then $p > j$. This follows directly from the definition of the word of f and the tile insertion order.

To prove (1), recall the definitions of h_3 and h_4:

$$\begin{aligned} h_3(Q) &= \sum_{i_1<i_2} \chi(g_{i_1} - g_{i_2} \in \{1,2,\ldots,m\} \text{ and } p_{i_1} > p_{i_2}) \\ h_4(Q) &= \sum_{i_1<i_2} \chi(g_{i_1} - g_{i_2} \in \{0,-1,-2,\ldots,-(m-1)\} \text{ and } p_{i_1} > p_{i_2}) \end{aligned}$$

Suppose we insert tile T_j in the far right position, after all the tiles $T_p = \boxed{\begin{array}{c} f(p) \\ p \end{array}}$ in Q. The change in h_3 caused by this insertion is

$$\sum_{T_p \in Q} \chi(f(p) - f(j) \in \{1,2,\ldots,m\} \text{ and } p > j) = 0,$$

since the observation above gives $f(p) - f(j) \le 0$. On the other hand, the change in h_4 caused by the insertion is

$$\sum_{T_p \in Q} \chi(f(p) - f(j) \in \{0,-1,\ldots,-(m-1)\} \text{ and } p > j)$$

$$= |f_{>j}^{-1}(\{f(j), f(j)-1, \ldots, f(j)-(m-1)\})|.$$

The last equality uses the fact that all tiles with lower entries in the set

$$f_{>j}^{-1}(\{f(j), \ldots, f(j)-(m-1)\})$$

have already been inserted in Q before tile T_p is inserted; this again follows from the definition of the insertion order. To summarize, the change in $h_3 - h_4$ caused by inserting T_p at the far right is

$$0 - |f_{>j}^{-1}(\{f(j), \ldots, f(j)-(m-1)\})| = x_j(f),$$

which proves (1).

In the proof of (2), we will consider configurations where the new tile T_j is inserted immediately right or left of an existing tile T_p. Note that passing from the configuration $\cdots T_p T_j \cdots$ to $\cdots T_j T_p \cdots$ simply amounts to interchanging the two adjacent tiles T_p and T_j. This interchange will only affect a single term in the formulas for h_3 and h_4. Specifically, in h_3, the term

$$t_1 = \chi(f(p) - f(j) \in \{1, 2, \ldots, m\} \text{ and } p > j)$$

will be replaced by the term

$$t_2 = \chi(f(j) - f(p) \in \{1, 2, \ldots, m\} \text{ and } j > p).$$

In h_4, the term

$$t_3 = \chi(f(p) - f(j) \in \{0, -1, \ldots, -(m-1)\} \text{ and } p > j)$$

will be replaced by the term

$$t_4 = \chi(f(j) - f(p) \in \{0, -1, \ldots, -(m-1)\} \text{ and } j > p).$$

The net change in the statistic $h_3 - h_4$ due to the interchange is therefore $(t_2 - t_4) - (t_1 - t_3) = t_2 + t_3 - t_1 - t_4$.

To prove (2a), assume that p and j are such that the position just right of T_p is an invalid position for T_j. This situation occurs in the following two cases.

(i) We have $f(p) - f(j) < -m$, so that the position right of T_p is invalid because condition (C) fails. We have $t_1 = t_2 = t_3 = t_4 = 0$, so the change in $h_3 - h_4$ when we move T_j to the left of T_p is zero.

(ii) We have $f(p) - f(j) = -m$ and $p > j$, so that the position right of T_p is invalid because condition (E) fails. We have $t_1 = t_2 = t_3 = t_4 = 0$, so the change in $h_3 - h_4$ when we move T_j to the left of T_p is zero.

To prove (2b), assume that p and j are such that the position just right of T_p is a valid position for T_j. This situation occurs in the following four cases.

(iii) We have $f(p) - f(j) = -m$ and $p < j$. Then $t_1 = t_4 = 0$, while $t_2 = 1$ and $t_3 = 0$. Hence, the change in $h_3 - h_4$ when we move T_j to the left of T_p is $+1$.

(iv) We have $-(m-1) \leq f(p) - f(j) \leq -1$ and $p > j$. Then $t_1 = t_4 = 0$, while $t_2 = 0$ and $t_3 = 1$. Hence, the change in $h_3 - h_4$ when we move T_j to the left of T_p is $+1$.

(v) We have $-(m-1) \leq f(p) - f(j) \leq -1$ and $p < j$. Then $t_1 = t_4 = 0$, while $t_2 = 1$ and $t_3 = 0$. Hence, the change in $h_3 - h_4$ when we move T_j to the left of T_p is $+1$.

(vi) We have $f(p) - f(j) = 0$, which forces $p > j$ by definition of $w(f)$ and the tile insertion order. Then $t_1 = t_2 = 0$, while $t_3 = 1$ and $t_4 = 0$. Hence, the change in $h_3 - h_4$ when we move T_j to the left of T_p is $+1$.

Note that the cases (i)—(vi) are exhaustive, since the tile insertion order rules out the possibility that $f(p) - f(j) > 0$. This completes the proof of (2).

To prove (3), note that (1) shows the rightmost position has label $x_j(f)$. Reading the positions from right to left, (2a) implies that there will be a block of positions with label $x_j(f)$, consisting of zero or more invalid positions followed by one valid position. By (2b), the next position to the left will have label $x_j(f) + 1$. Then (2a) implies that there is another block of positions labelled $x_j(f) + 1$, consisting of zero or more invalid positions followed by one valid position. This process continues until all valid positions have been encountered. We saw in the proof of Lemma 4.15 that the number of valid positions is exactly $count(f, j)$. Note that the leftmost block of positions may or may not end with a valid position, depending on k. This ambiguity does not affect the correctness of the present argument, since we stop as soon as the last (leftmost) valid position has been scanned. (This is illustrated by the two preceding examples, where the leftmost position is valid in one case and invalid in the other.) □

This lemma, together with the discussion preceding it, completes the proof of formula (4.2). We leave to the reader the task of showing that this formula reduces to formula (1.24) from Chapter 1 in the case $m = 1$, $k = 0$. This is merely a matter of notation translation, keeping in mind that the permutation σ corresponds to the reversal of the word of f.

4.4 Statistics based on Parking Policies

Recall from Chapter 1 that there are two pairs of statistics $(area, dinv)$ and $(dmaj, area')$ on parking functions that give conjectured combinatorial interpretations for the Hilbert series $H_n(q,t)$ of DH_n. This section introduces a third pair of statistics $(pmaj, area)$ on parking functions that has the same generating function as the previous two. In symbols, we have

$$\sum_{Q\in\mathcal{Q}_n} q^{dmaj(Q)}t^{area'(Q)} = \sum_{P\in\mathcal{P}_n} q^{area(P)}t^{dinv(P)} = \sum_{P\in\mathcal{P}_n} q^{pmaj(P)}t^{area(P)}.$$

Letting $q = 1$ here shows that $area$, $dinv$ and $area'$ have the same univariate distribution, while letting $t = 1$ shows that $pmaj$, $area$, and $dmaj$ have the same univariate distribution. Hence, all five individual statistics have the same univariate distribution. This result settles one of the open questions from [17]. We will prove the analogous result for labelled trapezoidal paths in the next section.

Our starting point is the formula

$$CH_n(q,t) = \sum_{P\in\mathcal{P}_n} q^{area(P)}t^{dinv(P)} = \sum_{\sigma\in S_n} q^{maj(\sigma)}\prod_{i=1}^{n}\sum_{p=0}^{w_i(\sigma)-1} t^p. \tag{4.7}$$

It is convenient to represent this formula combinatorially. To do this, consider objects $I = (\sigma; u_1, \ldots, u_n)$, where $\sigma \in S_n$ and u_i are integers satisfying $0 \leq u_i < w_i(\sigma)$. Let $\mathcal{I}_n$ denote the collection of such objects. Define $qstat(I) = maj(\sigma)$ and $tstat(I) = \sum_{i=1}^{n} u_i$.

It is obvious from these definitions and formula (4.7) that

$$CH_n(q,t) = \sum_{I \in \mathcal{I}_n} q^{qstat(I)} t^{tstat(I)}. \tag{4.8}$$

In particular, letting $q = t = 1$ here, we obtain

$$|\mathcal{I}_n| = |\mathcal{P}_n| = (n+1)^{n-1}. \tag{4.9}$$

We will define a statistic $pmaj$ on $\mathcal{P}_n$ and give a bijection $G : \mathcal{I}_n \to \mathcal{P}_n$ such that

$$qstat(I) = pmaj(G(I)) \text{ and } tstat(I) = area(G(I)).$$

It will then follow that

$$CH_n(q,t) = \sum_{P \in \mathcal{P}_n} q^{pmaj(P)} t^{area(P)}.$$

The simplest way to define $pmaj$ involves parking functions, which were discussed in §1.5.5. Let $P \in \mathcal{P}_n$, and let f be the associated parking function. Recall that $f(x) = j$ is interpreted to mean that car x prefers spot j. Let $S_j = f^{-1}(j)$ be the set of cars that want to park in spot j. Let $T_j = \bigcup_{k=1}^{j} S_k$ be the set of cars that want to park at or before spot j. The definition of a parking function states that $|T_j| \geq j$ for $1 \leq j \leq n$.

We introduce the following new *parking policy*. Consider parking spots $1, \ldots, n$ in this order. These spots will be filled with cars $\tau_1, \ldots, \tau_n$ according to certain rules. The car τ_1 that gets spot 1 is the largest car x in the set $S_1 = T_1$. The car τ_2 that gets spot 2 is the largest car x in $T_2 - \{\tau_1\}$ such that $x < \tau_1$; if there is no such car, then x is the largest car in $T_2 - \{\tau_1\}$. In general, the car τ_i that gets spot i is the largest car x in $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$ such that $x < \tau_{i-1}$; if there is no such car, then x is the largest car in $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$. Since $|T_i| \geq i$, the set $T_i - \{\tau_1, \ldots, \tau_{i-1}\}$ is never empty. So this selection process makes sense. At the end of this process, we obtain a *parking order* $\tau = \tau_1, \ldots, \tau_n$, which is a permutation of $1, \ldots, n$. We let $\sigma = \sigma(P)$ be the reversal of τ, so that $\sigma_j = \tau_{n+1-j}$ and $\tau_j = \sigma_{n+1-j}$ for $1 \leq j \leq n$. Finally, we define $pmaj(f) = pmaj(P) = maj(\sigma(P))$. Recall that $maj(\sigma_1 \cdots \sigma_n) = \sum_{i=1}^{n-1} i\chi(\sigma_i > \sigma_{i+1})$.

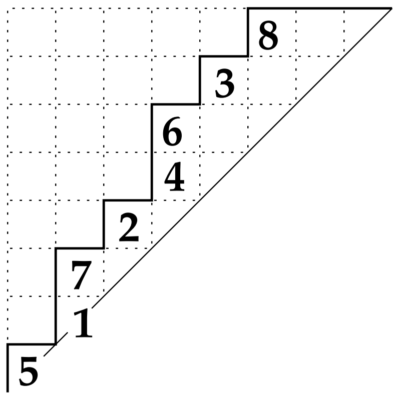


Figure 4.2: The labelled path used in Example 4.21.

Example 4.21. For the parking function f corresponding to the labelled path P in Figure 4.2, the new parking policy gives

$$\tau = 5, 1, 7, 6, 4, 3, 2, 8.$$

Hence, $\sigma = 8 > 2, 3, 4, 6, 7 > 1, 5$, and so $pmaj(P) = maj(\sigma) = 1 + 6 = 7$.

Example 4.22. Consider the labelled path P in Figure 4.3, in which the labels 1 to n appear in order from bottom to top.

The new parking policy gives

$$\tau = 1, 3, 2, 6, 5, 4, 8, 7.$$

Hence, $\sigma = 7, 8 > 4, 5, 6 > 2, 3 > 1$, and so $pmaj(P) = maj(\sigma) = 14$. On the other hand, drawing the bounce path for the corresponding unlabelled path (starting at $(0, 0)$, as in Chapter 2) gives bounces of lengths $1, 2, 3, 2$. Thus, the bounce statistic for this path is also 14.

Remark 4.23. As in the previous example, it is easy to see that the $pmaj$ statistic always reduces to the bounce statistic in the case where the labels 1 to n increase from bottom to top. The proof, which is by induction on the number of bounces, is left to the reader.

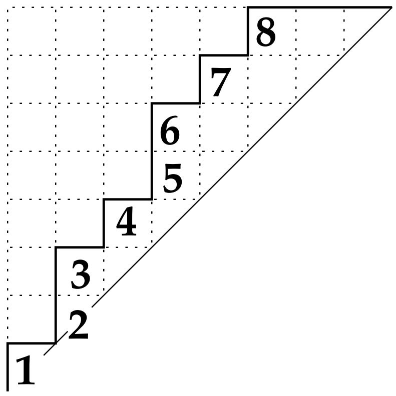


Figure 4.3: A labelled path with labels in increasing order.

We now define a map $G : \mathcal{I}_n \to \mathcal{P}_n$. Let $I = (\sigma; u_1, \dots, u_n) \in \mathcal{I}_n$. We define $G(I)$ to be the function $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that

$$f(\sigma_i) = (n + 1 - i) - u_i \text{ for } 1 \le i \le n. \tag{4.10}$$

Lemma 4.24. *The function G does map into the set $\mathcal{P}_n$.*

Proof. By definition, $w_i(\sigma)$ is no greater than the length of the list $\sigma_i, \sigma_{i+1}, \dots, \sigma_n$. Hence,

$$0 \le u_i < w_i(\sigma) \le n + 1 - i,$$

which shows that

$$1 \le f(\sigma_i) \le n + 1 - i \le n.$$

In particular, the image of f is contained in the codomain $\{1, 2, \dots, n\}$. This inequality also shows that the set $f^{-1}(\{1, 2, \dots, i\})$ contains at least the i elements $\sigma_n, \dots, \sigma_{n+1-i}$, so that f is a parking function. This shows that the image of G is contained in the set $\mathcal{P}_n$. $\square$

We will see shortly that G is a weight-preserving bijection.

Example 4.25. Let $n = 8$ and let $I = (\sigma; u_1, \dots, u_n)$, where

$$\sigma = 8 > 2, 3, 4, 6, 7 > 1, 5;$$

$$w_1 = 5,\ w_2 = 5,\ w_3 = 4,\ w_4 = 3,$$

$$w_5 = 3,\ w_6 = 2,\ w_7 = 2,\ w_8 = 1;$$

$$u_1 = 2,\ u_2 = 4,\ u_3 = 1,\ u_4 = 1,$$

$$u_5 = 0,\ u_6 = 1,\ u_7 = 0,\ u_8 = 0.$$

Using the formula above, we have $G(I) = f$, where

$$f(1) = f(\sigma_7) = 2,\ f(2) = f(\sigma_2) = 3,\ f(3) = f(\sigma_3) = 5,\ f(4) = f(\sigma_4) = 4,$$

$$f(5) = f(\sigma_8) = 1,\ f(6) = f(\sigma_5) = 4,\ f(7) = f(\sigma_6) = 2,\ f(8) = f(\sigma_1) = 6.$$

The labelled path P corresponding to this f appears in Figure 1.15. Note that

$$qstat(I) = 6 = pmaj(f) \text{ and } tstat(I) = 9 = area(f).$$

We now define a map $H : \mathcal{P}_n \to \mathcal{I}_n$ that will turn out to be the inverse of G. Let $P \in \mathcal{P}_n$, and let f be the associated parking function. Construct a permutation σ, as in the definition of $pmaj$, by reversing the parking permutation τ. Define

$$u_i = n + 1 - i - f(\sigma_i) \text{ for } 1 \leq i \leq n. \tag{4.11}$$

Finally, set $H(P) = H(f) = (\sigma; u_1, \ldots, u_n)$.

Lemma 4.26. *H does map $\mathcal{P}_n$ into the set $\mathcal{I}_n$. Moreover,*

$$G \circ H = Id_{\mathcal{P}_n}, \tag{4.12}$$

$$pmaj(P) = qstat(H(P)) \text{ and } area(P) = tstat(H(P)).$$

Proof. Let $f \in \mathcal{P}_n$. As usual, we set $S_j = f^{-1}(j)$ and $T_j = f^{-1}(\{1, 2, \ldots, j\})$. To see that H maps into $\mathcal{I}_n$, we need only show that $0 \leq u_i < w_i(\sigma)$. Observe that $\sigma_i = \tau_{n+1-i}$ is an element of T_{n+1-i}, and so $1 \leq f(\sigma_i) \leq n + 1 - i$. Hence, $u_i = n + 1 - i - f(\sigma_i)$ always satisfies the inequalities

$$0 \leq u_i \leq n - i < n + 1 - i. \tag{4.13}$$

We now consider several cases.

(I) σ_i occurs in the rightmost ascending run of σ. By definition of w_i, this implies $w_i(\sigma) = n + 1 - i$. In this case, inequality (4.13) immediately gives the desired conclusion $0 \leq u_i < w_i(\sigma)$.

(II) σ_i is not in the rightmost ascending run of σ, and σ can be written

$$\sigma = \cdots \sigma_i \cdots \sigma_k > \sigma_{k+1} \cdots \sigma_j \cdots ,$$

where: σ_k is the last entry in the ascending run containing σ_i (so $k \geq i$); σ_j and σ_{k+1} are in the same ascending run; $\sigma_j < \sigma_i$; and either: (a) $j = n$, or (b) $\sigma_j > \sigma_{j+1}$, or (c) $\sigma_j < \sigma_{j+1}$ and $\sigma_{j+1} > \sigma_i$. By definition, $w_i(\sigma) = j - i$. It suffices to check that $u_i < w_i(\sigma)$. Substituting $u_i = n + 1 - i - f(\sigma_i)$ and $w_i(\sigma) = j - i$, it suffices to check that $f(\sigma_i) > n + 1 - j$. If this inequality did not hold, we would have $f(\sigma_i) \leq n+1-j$, hence $\sigma_i \in T_{n+1-j}$. This will contradict the definition of the parking policy used to create τ, as follows. Consider $\sigma_j = \tau_{n+1-j}$. In subcase (a), $\sigma_j = \sigma_n = \tau_1 = \max T_1$. But our assumption gives $\sigma_i \in T_1$ and $\sigma_i > \sigma_j$, a contradiction. In subcase (b), $\sigma_j > \sigma_{j+1}$ means that $\tau_{n+1-j} > \tau_{n-j}$, which implies that all elements of the set

$$T_{n+1-j} - \{\tau_1, \ldots, \tau_{n-j}\} = T_{n+1-j} - \{\sigma_{j+1}, \ldots, \sigma_n\}$$

are larger than $\tau_{n-j} = \sigma_{j+1}$, and σ_j is the largest element in this set. But σ_i is also an element of this set, and it is larger than σ_j, a contradiction. In subcase (c), $\sigma_j < \sigma_{j+1}$ implies that σ_j is the largest element in the set

$$T_{n+1-j} - \{\tau_1, \ldots, \tau_{n-j}\} = T_{n+1-j} - \{\sigma_{j+1}, \ldots, \sigma_n\}$$

that is smaller than σ_{j+1}. But our assumption gives that σ_i is in this set and satisfies $\sigma_j < \sigma_i < \sigma_{j+1}$, a contradiction. Thus, the desired inequality must hold in all subcases.

(III) σ_i is not in the rightmost ascending run of σ, and σ can be written

$$\sigma = \cdots \sigma_i \cdots \sigma_j > \sigma_{j+1} \cdots ,$$

where: σ_j is the last entry in the ascending run containing σ_i (so $j \geq i$); and $\sigma_i < \sigma_{j+1}$. These inequalities force $\sigma_i < \sigma_j$. By definition, $w_i(\sigma) = j - i$. As in case (II), the desired inequality $u_i < w_i(\sigma)$ is equivalent to the inequality $f(\sigma_i) > n+1-j$. If the latter inequality fails, then $\sigma_i \in T_{n+1-j}$. As in case (II) subcase (b), $\sigma_j > \sigma_{j+1}$ means that $\tau_{n+1-j} > \tau_{n-j}$, which implies that all elements of the set

$$T_{n+1-j} - \{\tau_1, \ldots, \tau_{n-j}\} = T_{n+1-j} - \{\sigma_{j+1}, \ldots, \sigma_n\}$$

are larger than $\tau_{n-j} = \sigma_{j+1}$, and σ_j is the largest element in this set. But σ_i is an element of this set that is smaller than σ_{j+1}, which is a contradiction. So the desired inequality must hold.

This completes the proof that H maps into $\mathcal{I}_n$.

Next, the definitions of u_i and G in (4.11) and (4.10) make it clear that

$$G \circ H = Id_{\mathcal{P}_n}.$$

It is also obvious from the definition of H that

$$pmaj(P) = qstat(H(P)).$$

On the other hand, note that

$$\begin{aligned} tstat(H(P)) &= \textstyle\sum_{i=1}^n u_i = \sum_{i=1}^n (n+1-i) - \sum_{i=1}^n f(i) \\ &= n(n+1)/2 - \textstyle\sum_{i=1}^n f(i) = area(P), \end{aligned}$$

where the last equality is formula (1.25). □

Example 4.27. Let $n = 8$ and let $f \in \mathcal{P}_8$ be given by

$$f(1) = 2,\ f(2) = 3,\ f(3) = 5,\ f(4) = 4,$$

$$f(5) = 1,\ f(6) = 4,\ f(7) = 2,\ f(8) = 6.$$

As in Example 4.21, we compute $\sigma = 8 > 2, 3, 4, 6, 7 > 1, 5$. We then compute

$$u_1 = 2,\ u_2 = 4,\ u_3 = 1,\ u_4 = 1,$$

$$u_5 = 0,\ u_6 = 1,\ u_7 = 0,\ u_8 = 0.$$

Note that $H(f) = I$, where I is the object in $\mathcal{I}_n$ from Example 4.25. We have $G(H(f)) = f$ and $H(G(I)) = I$.

Theorem 4.28. *The maps $G : \mathcal{I}_n \to \mathcal{P}_n$ and $H : \mathcal{P}_n \to \mathcal{I}_n$ are bijections with $H = G^{-1}$. G and H are weight-preserving in the sense that*

$$pmaj(P) = qstat(H(P)) \text{ and } area(P) = tstat(H(P)); \tag{4.14}$$

$$qstat(I) = pmaj(G(I)) \text{ and } tstat(I) = area(G(I)). \tag{4.15}$$

Consequently,

$$\sum_{P\in\mathcal{P}_n} q^{pmaj(P)} t^{area(P)} = CH_n(q,t) = \sum_{P\in\mathcal{P}_n} q^{area(P)} t^{dinv(P)} = \sum_{Q\in\mathcal{Q}_n} q^{dmaj(Q)} t^{area'(Q)}, \tag{4.16}$$

and so all these statistics have the same univariate distribution.

Proof. We have already shown that G maps into $\mathcal{P}_n$, H maps into $\mathcal{I}_n$, and $G \circ H = Id_{\mathcal{P}_n}$. The last equation implies that H is an injection and G is a surjection. But we have seen in (4.9) that

$$|\mathcal{I}_n| = |\mathcal{P}_n| = (n+1)^{n-1} < \infty.$$

Since the sets are finite, H is automatically a surjection, G is automatically an injection, and $H = G^{-1}$. The properties in (4.14) were proved in the previous lemma, and (4.15) follows by replacing P by $G(I)$ and simplifying. Finally, the first equality in (4.16) follows from (4.8) and the existence of the weight-preserving map bijection G. The other formulas for $CH_n(q,t)$ have already been discussed. Letting $q = 1$ or $t = 1$ in (4.16) gives the final assertion of the theorem. □

Remark 4.29. It can be shown directly from the definitions of H and G that $H \circ G = Id_{\mathcal{I}_n}$, without using the identity $|\mathcal{P}_n| = |\mathcal{I}_n|$. Given a labelled path of the form $G(I)$, where $I = (\sigma; u_1, \ldots, u_n)$, one shows by backwards induction that the algorithm defining $H(G(I))$ correctly recovers $\sigma_n, \sigma_{n-1}, \ldots, \sigma_1$. The argument is similar to the case analysis in the proof of Lemma 4.26, and is left to the interested reader.

4.5 Univariate Symmetry of $CH_{n,k,m}(q,t)$

This section generalizes the constructions of §4.4 to labelled trapezoidal paths of type (n,k,m). We obtain another combinatorial interpretation of the right side of formula (4.2) in which t keeps track of area and q keeps track of a new statistic $pmaj$. As in §4.4, we can conclude that the ordered pairs of statistics $(area, h)$ and $(pmaj, area)$ have the same bivariate distribution on labelled paths. Therefore, all three statistics have the same univariate distribution. Unfortunately, the arguments given here are not strong enough to prove the conjectured joint symmetry of $CH_{n,k,m}(q,t)$.

4.5.1 Combinatorial Model of the Generating Function

We begin by introducing a simple combinatorial model for the formula (4.2).

Definition 4.30. (1) Given n, k, m, and $f \in \mathcal{F}_{n,k,m}$, define the *right limit* of j relative to f by

$$R_j(f) = |f_{>j}^{-1}(\{f(j), \dots, f(j) - (m-1)\})| = |x_j(f)|,$$

and define the *left limit* of j relative to f by

$$L_j(f) = x_j(f) + count(f,j) - 1.$$

Formula (4.2) can then be rewritten

$$CH_{n,k,m}(q,t) = \sum_{f \in \mathcal{F}_{n,k,m}} q^{maj(f)} t^{x_0(f)} \prod_{j=1}^{n} \sum_{p=-R_j(f)}^{p=L_j(f)} t^p.$$

(2) Fix n, k, and m. Define an *intermediate object* of type (n,k,m) to be a pair

$$I = (f;\ u_1, u_2, \dots, u_n),$$

where $f \in \mathcal{F}_{n,k,m}$ and where u_j are integers such that $-R_j(f) \le u_j \le L_j(f)$ for all j. Denote the collection of such intermediate objects by $\mathcal{I}_{n,k,m}$.

(3) Define the *intermediate q-statistic* for I to be

$$qstat(I) = maj(f) = \sum_{j=1}^{n} f(j).$$

Define the *intermediate t-statistic* for I to be

$$tstat(I) = x_0(f) + \sum_{j=1}^{n} u_j.$$

It is obvious from the definition of the intermediate objects and statistics that

$$\sum_{I\in\mathcal{I}_{n,k,m}} q^{qstat(I)}t^{tstat(I)} = \sum_{f\in\mathcal{F}_{n,k,m}} q^{maj(f)}t^{x_0(f)}\prod_{j=1}^{n}\sum_{p=-R_j(f)}^{p=L_j(f)} t^p = CH_{n,k,m}(q,t).$$

Theorem 4.31. *There exists a bijection $F:\mathcal{P}_{n,k,m}\to\mathcal{I}_{n,k,m}$ such that*

$$area(P) = qstat(F(P)) \text{ and } h(P) = tstat(F(P)) \text{ for all } P\in\mathcal{P}_{n,k,m}.$$

Proof. The bijection F is based on the tile insertion process from the last section (see Lemma 4.15). If $P=(\vec{g},\vec{p})$ is a labelled path, we define $f\in\mathcal{F}_{n,k,m}$ by setting $f(p_i)=g_i$, and we define u_j to be the change in the statistic h_3-h_4 caused by the insertion of the tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$. We then set $F(P)=(f;u_1,\ldots,u_n)$. Lemma 4.20 shows that each u_j satisfies the required inequalities

$$-R_j(f)\le u_j\le L_j(f).$$

The discussion in the last section shows that $area(P) = qstat(F(P))$ and $h(P) = tstat(F(P))$. The map F^{-1} is defined similarly: given $I=(f;u_1,\ldots,u_n)$, the function f tells us which tiles to use, and the numbers u_j tell us where to insert each tile to reconstruct P. Lemma 4.20 shows that there exists a unique valid insertion position for tile $\boxed{\begin{array}{c} f(j) \\ j \end{array}}$ that causes a change of u_j in the statistic h_3-h_4, so that F^{-1} is well-defined. Thus F is a bijection. □

Corollary 4.32.

$$|\mathcal{I}_{n,k,m}| = |\mathcal{P}_{n,k,m}| \text{ for all } n,k,m. \tag{4.17}$$

Proof. This is immediate from the existence of the bijection $F:\mathcal{P}_{n,k,m}\to\mathcal{I}_{n,k,m}$. □

Example 4.33. Let us compute $F(P)$, where P is the path given in tile notation by

$$P = \boxed{\begin{array}{c}2\\6\end{array}}\,\boxed{\begin{array}{c}0\\3\end{array}}\,\boxed{\begin{array}{c}2\\7\end{array}}\,\boxed{\begin{array}{c}3\\4\end{array}}\,\boxed{\begin{array}{c}2\\1\end{array}}\,\boxed{\begin{array}{c}0\\5\end{array}}\,\boxed{\begin{array}{c}0\\2\end{array}}$$

and $(n,k,m)=(7,2,2)$. First, examination of the tiles in P gives

$$f(1)=2,\ f(2)=0,\ f(3)=0,\ f(4)=3,\ f(5)=0,\ f(6)=2,\ f(7)=2.$$

Recall that $w(f)$, $count(f,j)$, etc., were computed before in Example 4.12. To find the numbers u_j, we build up P by inserting tiles in the order given in Example 4.17. For instance, Example 4.19 discusses the last two steps of the tile insertion. From that example, we see that $u_1 = -2$ and $u_4 = -1$. Similar analysis of the earlier tile insertions shows that

$$u_5 = 0, \quad u_3 = 0, \quad u_2 = -2, \quad u_7 = 2, \quad u_6 = 3.$$

We conclude that

$$F(P) = (f; -2, -2, 0, -1, 0, 3, 2).$$

The reader should consult Example 4.12 to confirm that $-R_j(f) \leq u_j \leq L_j(f)$ for $1 \leq j \leq 7$.

Our goal in the rest of this section is to describe another bijection $G : \mathcal{I}_{n,k,m} \to \mathcal{P}_{n,k,m}$ such that $tstat$ maps to $area$. The definition of the new statistic $pmaj$ on labelled paths is engineered so that $qstat$ maps to $pmaj$ under G. Indeed, we will use the equation $pmaj(P) = qstat(G^{-1}(P))$ as the definition of $pmaj$. Except in the case $m = 1$ and $k = 0$ considered earlier, the $pmaj$ statistic does not seem to have a particularly simple direct definition (not relying on the bijection G).

4.5.2 Generalized Parking Functions

It is convenient to introduce the notion of *generalized parking functions*, which give an alternate notation for describing labelled trapezoidal paths. Some combinatorial properties of generalized parking functions were studied by C. Yan in [31, 32].

Definition 4.34. Fix integers $n \geq 1$, $k \geq 0$, and $m \geq 1$.

(1) Let $TZ_{n,k,m}$ denote the region bounded by the lines $x = 0$, $y = 0$, $x = k + my$, and $y = n$. Number the rows of this region 1 to n, starting at the bottom. Number the columns in each row of this region $1, 2, 3, \ldots$ from left to right. Define

$$B(i) = k + m(i-1) + 1.$$

Note that a labelled lattice path with n labels stays within the region $TZ_{n,k,m}$ if and only if the label in row i appears in one of the columns $1, 2, \ldots, B(i)$ for $1 \leq i \leq n$.

(2) Given any function g with domain $\{1, 2, \ldots, n\}$, set

$$S_j(g) = g^{-1}(j) \text{ and } T_i(g) = g^{-1}(\{1, 2, \ldots, i\}) = \bigcup_{j=1}^{i} S_j(g).$$

(3) A *generalized parking function* or *generalized preference function* of *type* (n, k, m) is a function $g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, B(n)\}$ such that

$$|T_{B(i)}(g)| \geq i \text{ for } 1 \leq i \leq n.$$

Let $\mathcal{P}'_{n,k,m}$ denote the collection of parking functions of type (n, k, m).

Lemma 4.35. *There exists a bijection D_0 between functions*

$$g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots\}$$

and valid labelled lattice paths of height n starting at the origin and ending with a vertical step. This bijection yields a bijection $D : \mathcal{P}'_{n,k,m} \to \mathcal{P}_{n,k,m}$ between generalized parking functions and labelled trapezoidal paths.

Proof. Let g be any function mapping $\{1, 2, \ldots, n\}$ into the positive integers. Starting in the bottom row of the region $\{(x, y) : x \geq 0,\ 0 \leq y \leq n\}$, place the elements of $S_1(g)$ in increasing order in the first column of the diagram, one per row. Starting in the next empty row, place the elements of $S_2(g)$ in increasing order in the second column of the diagram, one per row. Continue similarly: after listing all elements x with $g(x) < i$, start in the next empty row and place the elements of S_i in increasing order in column i. Finally, draw a lattice path starting at $(0, 0)$ by drawing vertical steps immediately left of each label, and then drawing the necessary horizontal steps to get a connected path. $D_0(g)$ is defined to be the resulting labelled path.

The inverse of D_0 is defined as follows. Let P be any valid labelled lattice path of height n starting at the origin and ending with a vertical step. For $1 \leq j \leq n$, define $g(j)$ to be the number of the column in which label j appears. This construction obviously gives an inverse to D_0, hence D_0 is a bijection.

Now, consider a function g and its associated path $P = D_0(g)$. Note that each row in the diagram of P contains exactly one label. We claim that $|T_x(g)| \geq i$ if and only if the label ℓ in row i of P appears in one of the columns $1, 2, \ldots, x$. We prove the contrapositive of each direction. First, assume that label ℓ appears in some column $z > x$. By definition of D_0, there can be at most $i-1$ numbers c such that $g(c) < z$. In particular, the size of $T_x(g)$ is at most $i-1$. Conversely, assume that $|T_x(g)| < i$. In the construction of P, we will have exhausted all numbers c with $g(c) \leq x$ before reaching row i. Thus, the label ℓ in row i must satisfy $g(\ell) > x$, so it appears in a column $z > x$. This proves the claim.

Letting $x = B(i)$ in the claim for $1 \leq i \leq n$, we see that g belongs to $\mathcal{P}'_{n,k,m}$ if and only if the associated path $D_0(g)$ belongs to $\mathcal{P}_{n,k,m}$. (Technically, we may need to add some horizontal steps to the path $D_0(g)$ at the top level $y = n$ to get a path in $\mathcal{P}_{n,k,m}$ ending at $(k + mn, n)$. These extra steps are obviously harmless.) Hence, restricting D_0 to the set of generalized parking functions gives the desired bijection $D : \mathcal{P}'_{n,k,m} \to \mathcal{P}_{n,k,m}$. $\square$

From now on, we will identify the set of generalized parking functions $\mathcal{P}'_{n,k,m}$ with the set of labelled trapezoidal paths $\mathcal{P}_{n,k,m}$.

Example 4.36. For the labelled trapezoidal path $P \in \mathcal{P}_{6,2,3}$ shown in Figure 4.1, the associated parking function g is

$$g(1) = 12,\ g(2) = 17,\ g(3) = 2,$$

$$g(4) = 5,\ g(5) = 2,\ g(6) = 12.$$

Remark 4.37. It is easy to get a recurrence for labelled trapezoidal paths by removing the steps in the first column and their associated labels. If there are $\ell \geq 0$ vertical steps in this column, the associated increasing sequence of labels can be chosen in $\binom{n}{\ell}$ ways. What remains in the upper-right part of the diagram is a labelled trapezoidal path of height $n - \ell$ with the same value of m and a new base length of $k + m\ell - 1$. Setting $P(n, k, m) = |\mathcal{P}_{n,k,m}|$, we obtain the recurrence

$$P(n, k, m) = \sum_{\ell=0}^{n} \binom{n}{\ell} P(n - \ell, k + m\ell - 1, m)$$

with initial conditions

$$P(n, k, m) = 0 \text{ if } n < 0 \text{ or } k < 0;$$

$$P(0,k,m) = 1 \text{ for all } k \geq 0, m \geq 1.$$

From this recurrence, it is easily verified by induction that

$$P(n,k,m) = (k+1)\cdot(mn+k+1)^{n-1}.$$

These calculations (and other more general ones) appear in [32].

Lemma 4.38. *Let $P \in \mathcal{P}_{n,k,m}$ correspond to the generalized parking function g. Then*

$$area(P) = n(k+1) + mn(n-1)/2 - \sum_{i=1}^{n} g(i). \tag{4.18}$$

Proof. It is easy to see that the region $TZ_{n,k,m}$ contains $nk + mn(n-1)/2$ complete lattice cells. Since label i occurs somewhere in column $g(i)$, there are $g(i)-1$ lattice cells inside the region $TZ_{n,k,m}$ and left of label i. These lattice cells lie outside the labelled path associated to g. Subtracting, we find that

$$area(P) = kn + mn(n-1)/2 - \sum_{i=1}^{n}[g(i)-1] = n(k+1) + mn(n-1)/2 - \sum_{i=1}^{n} g(i). \tag{4.19}$$

$\square$

For instance, in the example above we have

$$area(P) = 63 - (12+17+2+5+2+12) = 13.$$

4.5.3 Formal Bounce Paths

Before defining the map G, we need to prove a few technical facts about bounce paths. The basic idea is that a bounce path can be constructed from any sequence of vertical moves v_j by using the usual rule to determine the horizontal moves h_j.

Chapter 3 discussed the *bouncing algorithm* that assigns to each trapezoidal path $P \in \mathcal{T}_{n,k,m}$ its associated bounce path $B(P)$. Recall that $B(P)$ consists of a sequence of alternating vertical and horizontal moves, which we will denote here as $v_j(P)$ and $h_j(P)$. Each vertical move $v_j(P)$ was determined from the path P (and the partial bounce path already constructed), while the horizontal move $h_j(P)$ was calculated from the formula

$$h_j(P) = \sum_{i=0}^{m-1} v_{j-i}(P) + \chi(j < k). \tag{4.20}$$

In the last paragraph, a given path $P \in \mathcal{T}_{n,k,m}$ was used to construct the lists of numbers $v_j(P)$ and $h_j(P)$. Suppose, instead, that we are given only a list of numbers v_j that does not necessarily come from executing the bouncing algorithm on a path P. Then we can still create a "formal bounce path" from the list v_j by using a formula like (4.20) to define numbers h_i in terms of n, k, m, and the v_j's. The precise construction is as follows.

Definition 4.39. Fix integers $n \geq 1$, $k \geq 0$, and $m \geq 1$. Suppose $\{v_j : j \in \mathbb{Z}\}$ is an indexed family of nonnegative integers satisfying the following conditions:

(a) For all $j < 0$, $v_j = 0$.

(b) There exists $j^* \geq 0$ such that $v_{j^*} > 0$ and $v_\ell = 0$ for all $\ell > j^*$.

(c) $\sum_{j=0}^{j^*} v_j = n$.

We introduce the following notation.

(1) Let $J = \max(j^* + (m-1), k-1)$.

(2) For $0 \leq j \leq J$, let
$$h_j = \sum_{i=0}^{m-1} v_{j-i} + \chi(j < k). \tag{4.21}$$

(3) For $0 \leq j \leq J$, let
$$H_j = \sum_{i=0}^{j} h_i \text{ and } V_j = \sum_{i=0}^{j} v_i.$$
It will be convenient to set $H_{-1} = h_{-1} = V_{-1} = 0$.

(4) Let $Q = Q(\{v_j\})$ be a path constructed as follows. Q starts at the origin and makes alternating vertical moves and horizontal moves. For $0 \leq j \leq J$, Q moves up v_j units from its current position and then right h_j units. We refer to this move as the "j^{th} bounce." After the j^{th} bounce, Q has reached coordinates (H_j, V_j). Q is called the *formal bounce path associated to the sequence* $\{v_j\}$.

Example 4.40. Let $(n, k, m) = (7, 2, 2)$. Suppose we are given $v_0 = 3$, $v_1 = 0$, $v_2 = 3$, $v_3 = 1$, and $v_j = 0$ for all other j. Here, $j^* = 3$ and $J = \max(3 + 1, 1) = 4$. Table 4.2 shows the vertical moves and horizontal moves for the formal bounce path $Q(\{v_j\})$.

Table 4.2: The vertical and horizontal moves of a formal bounce path.

j	0	1	2	3	4
v_j	3	0	3	1	0
h_j	4	4	3	4	1

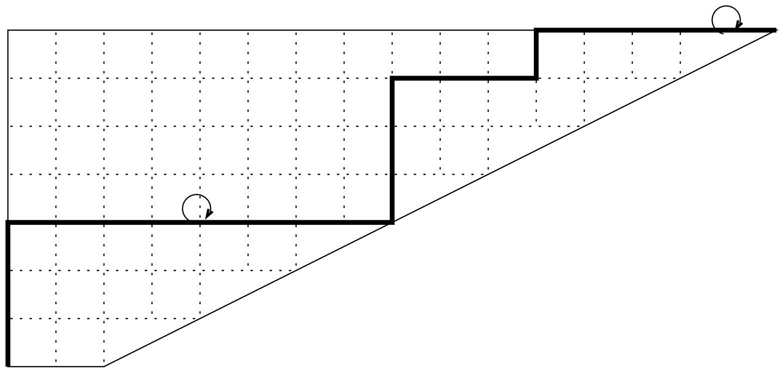


Figure 4.4: A formal bounce path.

The path $Q = Q(\{v_j\})$ is shown in Figure 4.4. Observe that Q happens to lie in the trapezoid $TZ_{n,k,m}$, and Q ends exactly at the upper-right corner $(k+mn, n)$ of this trapezoid. We have $B(Q) = Q$, i.e., the bounce path associated to Q is Q itself. Furthermore, the vertical moves $v_j(Q)$ of this bounce path are precisely the numbers v_j that we were originally given.

Example 4.41. Let $(n, k, m) = (4, 0, 2)$. Suppose we are given $v_0 = 1$, $v_1 = 0$, $v_2 = 0$, $v_3 = 3$, and $v_j = 0$ for all other j. Here, $j^* = 3$ and $J = \max(3+1, 0) = 4$. Table 4.3 shows the vertical moves and horizontal moves for the formal bounce path $Q(\{v_j\})$.

The path $Q = Q(\{v_j\})$ is shown in Figure 4.5. As before, Q lies in the trapezoid $TZ_{n,k,m}$, and ends exactly at the upper-right corner $(k+mn, n)$ of this trapezoid. The vertical moves $v_j(Q)$ of the bounce path $B(Q)$ are

$$v_0(Q) = 1, \quad v_1(Q) = 0, \quad v_2(Q) = 3, \quad v_3(Q) = 0.$$

This is almost the same as the original sequence v_j, except that the element $v_2 = 0$ has disappeared. This occurred because the corresponding horizontal move h_2 was zero.

Table 4.3: The vertical and horizontal moves of another formal bounce path.

j	0	1	2	3	4
v_j	1	0	0	3	0
h_j	1	1	0	3	3

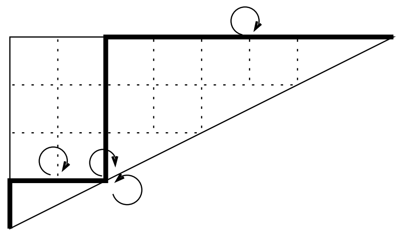


Figure 4.5: Another formal bounce path.

The phenomenon in the two examples above is typical. We will show that the path $Q = Q(\{v_j\})$ is always a valid trapezoidal path. Furthermore, under the additional condition that $h_j > 0$ for $0 \leq j \leq J$, the vertical moves $v_j(Q)$ for the bounce path $B(Q)$ are precisely the original numbers v_j. Hence, in this situation, the list v_j can be recovered from the path Q by performing the bouncing algorithm.

Lemma 4.42. *Let n, k, m, and $\{v_j\}$ be given as in Definition 4.39. Let h_j, V_j, H_j, and $Q = Q(\{v_j\})$ be given as in that definition. Then:*

(1) For $0 \leq j \leq J$, we have

$$H_j = \min(k, j+1) + \sum_{i=0}^{j} \min(m, j+1-i)v_i.$$

(2) For $0 \leq j \leq J$, we have

$$H_j \leq k + mV_j,$$

with equality if and only if $j \geq k-1$ and $v_{j-i} = 0$ for $0 \leq i < m-1$.

(3) Q is a path from $(0,0)$ to $(k+mn, n)$ that always stays inside the trapezoid $TZ_{n,k,m}$.

(4) The path Q reaches the right edge of the trapezoid $TZ_{n,k,m}$ after the j^{th} bounce if and only if $j \geq k-1$ and $v_{j-i} = 0$ for $0 \leq i < m-1$.

(5) Assume that $h_j > 0$ for $0 \leq j \leq J$. Then $B(Q) = Q$ and $v_j(Q) = v_j$ for $0 \leq j \leq J$, so that the original list $\{v_j\}$ can be recovered by performing the bouncing algorithm on Q.

Proof. To prove (1), use (4.21) to compute

$$\begin{aligned} H_j &= \sum_{u=0}^{j} h_u = \sum_{u=0}^{j} \left[\chi(u<k) + \sum_{i=u-(m-1)}^{u} v_i \right] \\ &= \min(k, j+1) + \sum_{i=0}^{j} v_i \sum_{u=0}^{j} \chi(u-(m-1) \leq i \leq u) \\ &= \min(k, j+1) + \sum_{i=0}^{j} \min(m, j+1-i) v_i. \end{aligned}$$

To justify the last equality, fix i with $0 \leq i \leq j$. If $0 \leq i \leq j-(m-1)$, then there are exactly m choices of the index u for which $\chi(u-(m-1) \leq i \leq u) = 1$, namely $u = i$, $u = i+1, \cdots, u = i+(m-1)$. In this case, $j+1-i \geq m$, so that $\min(m, j+1-i) = m$. On the other hand, if $j-(m-1) < i \leq j$, then there are exactly $j+1-i$ choices of the index u for which $\chi(u-(m-1) \leq i \leq u) = 1$, namely $u = i$, $u = i+1, \cdots, u = j$. In this case, $j+1-i < m$, so that $\min(m, j+1-i) = j+1-i$.

Now, we use (1) to compute

$$H_j = \min(k, j+1) + \sum_{i=0}^{j} \min(m, j+1-i) v_i \leq k + m \sum_{i=0}^{j} v_i = k + mV_j.$$

Equality is attained here if and only if $\min(k, j+1) = k$ and $v_i = 0$ for all i such that $\min(m, j+1-i) = j+1-i < m$. In other words, equality is attained if and only if $j \geq k-1$ and $v_{j-i} = 0$ for $0 \leq i < m-1$, giving (2).

Recall that the right boundary of the trapezoid $TZ_{n,k,m}$ is the line $x = k+my$. Q lies inside this trapezoid if and only if all the points (H_j, V_j) lie weakly left of this line, for $0 \leq j \leq J$. This is exactly what the inequality in (2) asserts. Also, equality holds in (2) for some j if and only if (H_j, V_j) lies exactly on the line $x = k+my$. The definition of J guarantees that equality holds in (2) for $j = J$ and that $V_J = n$. Hence, Q ends at the upper-right corner $(k + mn, n)$. We have now proved (3) and (4).

To prove (5), let $0 \le j \le J$. We can assume by induction on j that $v_i(Q) = v_i$ and $h_i(Q) = h_i$ for all i with $0 \le i < j$. In particular, just before the j^{th} bounce, both Q and $B(Q)$ are at coordinates (H_{j-1}, V_{j-1}). Where does Q go from here? By definition of Q, Q goes up $v_j \ge 0$ units and then over h_j units. Now, by the definition of the bouncing algorithm in Chapter 3, $B(Q)$ also goes up v_j units, since it is blocked there by a horizontal step of Q. This step must exist because of the assumption that $h_j > 0$. Therefore, $v_j(Q) = v_j$. Comparing formulas (4.20) and (4.21) and using the induction hypothesis, it is immediate that $h_j(Q) = h_j$ also. This completes the induction. $\square$

4.5.4 The Map $G : \mathcal{I}_{n,k,m} \to \mathcal{P}_{n,k,m}$

We are now ready to define the map G from intermediate objects to generalized parking functions. Let $I = (f; u_1, \ldots, u_n)$ be an intermediate object in $\mathcal{I}_{n,k,m}$. For all integers j, let $S_j = f^{-1}(j)$ and $v_j = |f^{-1}(j)| = |S_j|$. Note that, for $0 \le j \le k+m(n-1)$, S_j consists of the v_j labels appearing in the j^{th} descending block of the word of f. Let j^* be the largest value of j for which $v_j > 0$. It is clear that the sequence $\{v_j : j \in \mathbb{Z}\}$ satisfies conditions (a), (b), and (c) in Definition 4.39.

We will define $G(I)$ in three steps.

- First, draw the formal bounce path $Q = Q(\{v_j\})$ associated to the sequence $\{v_j\}$.
- Second, attach labels to the path Q. Place the v_j labels in S_j in the cells to the right of the v_j vertical segments in the j^{th} vertical move of the path Q, in increasing order from bottom to top. Let g_0 be the function associated to this labelled lattice path via D_0^{-1}.
- Third, define a function g by

$$g(i) = g_0(i) - u_i \text{ for } 1 \le i \le n, \tag{4.22}$$

  and set $G(I) = g$.

In terms of labelled paths, the diagram for g is obtained from the diagram of g_0 as follows. For each label i in the diagram of g_0, move the label $|u_i|$ cells to the right if

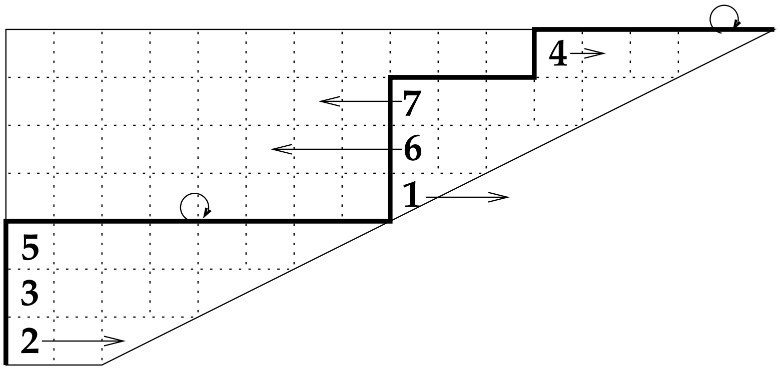


Figure 4.6: Labelled path diagram for g_0.

$u_i < 0$, or u_i cells to the left if $u_i \geq 0$. Then reorder the rows of the diagram to produce a valid configuration of labels (in which labels in a given column increase from bottom to top, and for $r < s$, the labels in column r appear in lower rows than the labels in column s). This construction explains why we called $R_j(f)$ and $L_j(f)$ the right and left limits of j relative to f.

Example 4.43. Let $(n, k, m) = (7, 2, 2)$, and let $I = (f; -2, -2, 0, -1, 0, 3, 2) \in \mathcal{I}_{n,k,m}$, where

$$f(1) = 2,\ f(2) = 0,\ f(3) = 0,\ f(4) = 3,\ f(5) = 0,\ f(6) = 2,\ f(7) = 2.$$

Let us compute $G(I)$. By looking at f, we find that

$$v_0 = 3,\ v_1 = 0,\ v_2 = 3,\ v_3 = 1,\ v_j = 0 \text{ for other } j.$$

The unlabelled path $Q = Q(\{v_j\})$ is shown in Figure 4.4. The corresponding labelled path is shown in Figure 4.6. The arrows in this figure show the motion of the labels caused by the numbers u_j.

Applying D_0^{-1}, we compute

$$g_0(1) = 9, g_0(2) = 1, g_0(3) = 1, g_0(4) = 12, g_0(5) = 1, g_0(6) = 9, g_0(7) = 9.$$

By (4.22), we get

$$g(1) = 11, g(2) = 3, g(3) = 1, g(4) = 13, g(5) = 1, g(6) = 6, g(7) = 7.$$

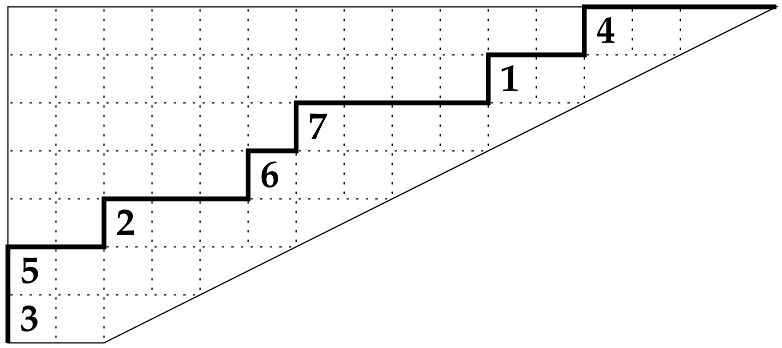


Figure 4.7: Labelled path diagram for g.

The labelled path diagram for g is shown in Figure 4.7. This figure can be obtained from the previous one by moving labels as indicated by the arrows, and then rearranging the rows as explained above. Note that g and g_0 are indeed generalized parking functions of type (n, k, m).

Lemma 4.44. *Let $I = (f; u_1, \ldots, u_n) \in \mathcal{I}_{n,k,m}$, and let Q and g_0 be computed from I as indicated above.*

(1) Q is a path from $(0, 0)$ to $(k + mn, n)$ that always stays inside $TZ_{n,k,m}$. Hence, g_0 is a parking function of type (n, k, m).

(2) For $0 \leq j \leq J$, the horizontal moves h_j of Q satisfy $h_j > 0$.

(3) The function f can be uniquely recovered from g_0.

(4) For $1 \leq x \leq n$, we have

$$g_0(x) = 1 + H_{f(x)-1}.$$

Proof. Statement (1) is immediate from Lemma 4.42(3) and Lemma 4.35.

To prove (2), we suppose that $h_j = 0$ for some j with $0 \leq j \leq J$ and derive a contradiction. First note that the existence of the object $I = (f; u_1, \ldots, u_n) \in \mathcal{I}_{n,k,m}$ implies that $count(f, i) > 0$ for $1 \leq i \leq n$, by definition of u_i. By (4.21), the assumption $h_j = 0$ forces $j \geq k$ and

$$v_j = v_{j-1} = \cdots = v_{j-(m-1)} = 0.$$

Assume that $j^* \leq j \leq J$. Since $v_{j^*} > 0$, the last condition forces $j^* + m \leq j \leq J$. But we also have $j \geq k$, so that $J \geq \max(j^* + m, k)$. This contradicts the definition of J. Therefore, $0 \leq j < j^*$. Since $v_{j^*} > 0$, there exists some $\ell > j$ with $v_\ell > 0$. Choose the minimal ℓ with this property; note that $\ell > k$ since $j \geq k$, and we have

$$v_{\ell-1} = \cdots = v_{\ell-m} = 0.$$

Next, choose i to be the maximum element of the nonempty set $f^{-1}(\ell)$. Recall that

$$\begin{aligned} \mathit{count}(f,i) &= \chi(f(i) \leq k) + |f^{-1}_{<i}(f(i)-m)| + |f^{-1}_{>i}(f(i))| \\ &\quad +|f^{-1}(\{f(i)-1,\ldots,f(i)-(m-1)\})|. \end{aligned}$$

By our choice of i and ℓ, we get $\mathit{count}(f,i) = 0$, which is a contradiction.

Now we can prove that f is uniquely recoverable from g_0. Given g_0, draw the path Q corresponding to g_0 and perform the bouncing algorithm to compute the vertical moves $v_j(Q)$. By Lemma 4.42(5) and part (2), we have $v_j(Q) = v_j$ for $0 \leq j \leq J$. In particular, $v_j(Q) = v_j$ for $0 \leq j \leq j^*$. So, we can recover the numbers $v_j = |f^{-1}(\{j\})|$ from g_0. The labels attached to the j^{th} vertical move of Q are the elements of $f^{-1}(j)$, so we can now recover f itself. Of course, it is possible that $v_j = 0$ for some j; in this case, $f^{-1}(j) = \emptyset$.

To prove (4), consider the labelled path diagram for g_0. In that diagram, all the labels in the set $S_j = f^{-1}(j)$ occur in the column numbered $1 + H_{j-1}$, since the j^{th} vertical move of Q is drawn just to the left of this column, beginning at (H_{j-1}, V_{j-1}). By definition of D_0^{-1}, we must have $g_0(x) = 1 + H_{j-1}$ whenever $f(x) = j$, i.e.,

$$g_0(x) = 1 + H_{f(x)-1} \text{ for } 1 \leq x \leq n.$$

□

The next lemma shows that G does map into the set $\mathcal{P}_{n,k,m}$.

Lemma 4.45. *For each $I \in \mathcal{I}_{n,k,m}$, $g = G(I)$ is a parking function of type (n,k,m).*

Proof. We must check that $g(i) > 0$ for all i and that

$$|T_{B(i)}(g)| \geq i \text{ for } 1 \leq i \leq n.$$

Recall the following definitions:

$$\begin{aligned} R_j(f) &= |f^{-1}_{>j}(\{f(j),\ldots,f(j)-(m-1)\})| = -x_j(f); \\ L_j(f) &= x_j(f) + count(f,j) - 1; \\ count(f,j) &= \chi(f(j)\le k) + |f^{-1}_{<j}(f(j)-m)| + |f^{-1}_{>j}(f(j))| \\ &\quad + |f^{-1}(\{f(j)-1,\ldots,f(j)-(m-1)\})|. \end{aligned}$$

Comparing these formulas, we see that

$$L_j(f) = |f^{-1}_{<j}(\{f(j)-1,\ldots,f(j)-m\})| - \chi(f(j)>k). \tag{4.23}$$

Recall that g_0 was constructed from the formal bounce path $Q = Q(\{v_i\})$, where $v_i = |f^{-1}(i)|$ and

$$h_i = v_i + v_{i-1} + \cdots + v_{i-(m-1)} + \chi(i<k) \text{ for } 0 \le i \le J. \tag{4.24}$$

Let $1 \le j \le n$. If $f(j) \ge 1$, we may take $i = f(j)-1$ in (4.24). Comparing to formula (4.23), we find that

$$L_j(f) \le v_{f(j)-1} + \cdots + v_{f(j)-m} \le h_{f(j)-1}. \tag{4.25}$$

If $f(j) = 0$, then (4.25) holds trivially since $h_{-1} = 0$. Thus, (4.25) holds for all j with $1 \le j \le n$.

Now, since $R_j(f) \ge -u_j \ge -L_j(f)$ by definition of u_j, we have

$$g(j) = g_0(j) - u_j \ge g_0(j) - L_j(f) = 1 + H_{f(j)-1} - L_j(f) \ge 1.$$

The last inequality follows since $L_j(f) \le h_{f(j)-1} \le H_{f(j)-1}$.

Recall that $B(i) = k + m(i-1) + 1$, so $B(1) \le B(2) \le \cdots$. Let $w_1, w_2, \ldots, w_n$ be the permutation of $1, 2, \ldots, n$ obtained from the word of f by erasing all bar symbols. To check that $|T_{B(i)}(g)| \ge i$ for all i, it suffices to show that $g(w_i) \le B(i)$ for all i, for this will imply that

$$\{w_1, \ldots, w_i\} \subset T_{B(i)}(g).$$

Fix $i_0 \in \{1, 2, \ldots, n\}$, and set $j_0 = f(w_{i_0})$. By definition of the word of f, we can write $i_0 = r + t$, where

$$r = |\{x : f(x) < j_0\}| = V_{j_0-1} \text{ and } t = |\{x : f(x) = j_0 \text{ and } x \ge w_{i_0}\}|.$$

Now consider two cases.

Case 1: $H_{j_0-1} \neq k + mV_{j_0-1}$. Let us construct a new formal bounce path Q' from a sequence $\{v'_j : j \in \mathbb{Z}\}$, as follows. Set $v'_j = v_j$ for $0 \leq j < j_0$, and set $v'_j = 0$ for all other j. Let $Q' = Q(\{v'_j\})$, and let h'_j be the horizontal moves of Q'. (Here, $n' = \sum_{j<j_0} v_j$.) Using Lemma 4.42(2) and the assumption $H_{j_0-1} \neq k + mV_{j_0-1}$, it is easy to see that $J' \geq j_0$. From (4.21), we have $h'_j = h_j$ for $-1 \leq j < j_0$, so $H'_j = H_j$ for $-1 \leq j < j_0$. On the other hand,

$$h'_{j_0} = 0 + v_{j_0-1} + \cdots + v_{j_0-(m-1)} + \chi(j_0 < k).$$

Lemma 4.42(2), applied to Q' with $j = j_0 \leq J'$, states that $H'_{j_0} \leq k + mV'_{j_0}$. In other words, $H_{j_0-1} + h'_{j_0} \leq k + mr$, which yields

$$1 + H_{j_0-1} + h'_{j_0} \leq 1 + k + m(r + 1 - 1) = B(r + 1).$$

Case 2: $H_{j_0-1} = k + mV_{j_0-1}$. In this case, lemma (4.42) says that $v_{j_0-1} = \cdots = v_{j_0-(m-1)} = 0$ and $j_0 - 1 \geq k - 1$, hence $j_0 \geq k$. So, if we define h'_{j_0} by the same equation

$$h'_{j_0} = 0 + v_{j_0-1} + \cdots + v_{j_0-(m-1)} + \chi(j_0 < k)$$

used above, then $h'_{j_0} = 0$. So we trivially have $H_{j_0-1} + h'_{j_0} \leq k + mr$, and

$$1 + H_{j_0-1} + h'_{j_0} \leq B(r + 1)$$

in this case as well.

Recalling that $f(w_{i_0}) = j_0$ and using Lemma 4.44(4), we now compute

$$\begin{aligned} g(w_{i_0}) &= g_0(w_{i_0}) - u_{w_{i_0}} \leq g_0(w_{i_0}) + R_{w_{i_0}}(f) \\ &= 1 + H_{j_0-1} + f^{-1}_{>w_{i_0}}(\{j_0, j_0 - 1, \ldots, j_0 - (m-1)\}). \end{aligned}$$

Now, using the definition of t,

$$\begin{aligned} f^{-1}_{>w_{i_0}}(\{j_0, j_0 - 1, \ldots, j_0 - (m-1)\}) &= t - 1 + f^{-1}_{>w_{i_0}}(\{j_0 - 1, \ldots, j_0 - (m-1)\}) \\ &\leq t - 1 + v_{j_0-1} + \cdots + v_{j_0-(m-1)} + \chi(j_0 < k) \\ &= t - 1 + h'_{j_0}, \end{aligned}$$

and so

$$g(w_{i_0}) \le t-1+(1+H_{j_0-1}+h'_{j_0}) \le B(r+1)+(t-1) \le B(r+t) = B(i_0),$$

as desired. ☐

Lemma 4.46. *For each $I = (f; u_1, \ldots, u_n) \in \mathcal{I}_{n,k,m}$, we have $area(G(I)) = tstat(I)$.*

Proof. Let g_0 and $g = G(I)$ be defined as above. In §3.2.3, we saw that the area of the bounce path corresponding to g_0 is precisely $x_0(f)$. Let C denote the constant $n(k+1)+mn(n-1)/2$. Using Lemma 4.38 for g and for g_0, we get

$$\begin{aligned} area(g) &= C - \sum_{i=1}^{n} g(i) = C - \sum_{i=1}^{n} g_0(i) + \sum_{i=1}^{n} u_i \\ &= area(g_0) + \sum_{i=1}^{n} u_i = x_0(f) + \sum_{i=1}^{n} u_i \\ &= tstat(I). \end{aligned}$$

☐

4.5.5 The map $G^{-1} : \mathcal{P}_{n,k,m} \to \mathcal{I}_{n,k,m}$

We now define a map $H : \mathcal{P}_{n,k,m} \to \mathcal{I}_{n,k,m}$ that will turn out to be the inverse of G. The basic idea is to use a labelled path $P \in \mathcal{P}_{n,k,m}$ to recover the symbols in the word of f (including bars) from left to right. As $w(f)$ is being reconstructed, we obtain partial information about f and g_0 that is used to continue the reconstruction process. When the full word has been found, we know f and g_0. We then define

$$u_i = g_0(i) - g(i), \tag{4.26}$$

where g is the parking function corresponding to P, and set $H(P) = (f; u_1, \ldots, u_n)$. Comparing (4.26) to (4.22) makes it clear that $G \circ H = Id_{\mathcal{P}_{n,k,m}}$. It is less clear that $H \circ G$ is also an identity map, and that H does map into $\mathcal{I}_{n,k,m}$. The former will follow from the latter by a pigeonhole-type argument, as in the $m = 1$ case worked out earlier.

A key observation is the following "prefix property" of $w(f)$: if j is any label, then the quantities

$$x_j(f),\ count(f, j),\ L_j(f),\ R_j(f)$$

can all be computed using only the symbols preceding j (including bars) in the word of f. This observation is immediate from the definitions of these quantities and the word of f. In particular, here we use the fact that the elements of each set $f^{-1}(i)$ appear in *decreasing* order in $w(f)$.

We now give the detailed definition of H. Let $P \in \mathcal{P}_{n,k,m}$ be a given labelled path, and let $g = g(P)$ be the corresponding generalized parking function. We compute $H(P)$ using the algorithm given below. The algorithm uses variables $pw(f)$, pf, and pg_0 to represent partially reconstructed versions of $w(f)$, f, and g_0, respectively. The observation in the previous paragraph says that if j occurs in $pw(f)$ at some step, then we can compute $x_j(f) = x_j(pf)$, etc., and the answer obtained is independent of how $pw(f)$ is extended in later steps to $w(f)$. Note that every time we add a label j to $pw(f)$, the definitions of $w(f)$ and g_0 allow us to deduce the values of $f(j)$ and $g_0(j)$. Thus, pf and pg_0 can now be defined for input j. The following example illustrates this process.

Example 4.47. Let $(n,k,m) = (7,2,2)$. Suppose that we are given the following partial reconstruction of the word of f:

$$pw(f) = 5\ 3\ 2 \mid \mid 7.$$

From this prefix of $w(f)$, we can deduce that $v_0 = 3$, $v_1 = 0$, and $v_2 \geq 1$. We can further deduce that $h_0 = H_0 = 4$, $h_1 = 4$, and $H_1 = 8$. Therefore, regardless of the value of v_2 or h_2, Lemma 4.44(4) gives

$$pg_0(5) = 1, pg_0(3) = 1, pg_0(2) = 1, pg_0(7) = 9.$$

Moreover,

$$pf(5) = 0, pf(3) = 0, pf(2) = 0, pf(7) = 2.$$

The domain of definition for pf and pg_0 is currently $\{2,3,5,7\}$.

Now suppose that we are told the next symbol in $w(f)$ is 6. Then we can conclude that $v_2 \geq 2$, that $pg_0(6) = 9$, and that $pf(6) = 2$.

Figure 4.8 gives the algorithm defining H. It is obvious from this algorithm and the prefix property of $w(f)$ that the required inequalities

$$-R_j(f) \leq u_j \leq L_j(f)$$

Algorithm for H: Input: a labelled path $P \in \mathcal{P}_{n,k,m}$.

1. Initialize $pw(f)$ be the empty word. Let pf and pg_0 be undefined for every input. Initialize a variable v to be 0. Let $g = D^{-1}(P)$ be the parking function associated to the labelled path P.

2. While $pw(f)$ does not contain all the labels from 1 to n, perform the following steps.

   (a) Loop through all labels $x \in \{1, 2, \ldots, n\}$ that do not yet appear in $pw(f)$, from largest to smallest. For each such label x, perform the following step.

   Temporarily assume that the next symbol in $pw(f)$ is x. Use this assumption to compute $pg_0(x)$ using the formal bouncing rules. Set $u_x = pg_0(x) - g(x)$.

   - If $u_x > L_x(pf)$, declare an error condition and abort the algorithm.
   - If $u_x < -R_x(pf)$, assert that $f(x) > v$ and discard the assumption that the next symbol in $pw(f)$ is x. At this point, reset $pg_0(x)$ and u_x to be undefined again.
   - Otherwise, we must have $-R_x(pf) \leq u_x \leq L_x(pf)$. In this case, assert that $f(x) = v$. Retain the assumption that the next symbol in $pw(f)$ is x, and retain the values of $pg_0(x)$ and u_x already computed. Define $pf(x) = v$.

   (b) We have now (greedily) determined all values x for which $f(x) = v$. Append a bar symbol at the end of $pw(f)$, and increment v by 1.

3. Add the appropriate number of trailing bar symbols to the end of $pw(f)$, so that the total number of bars is $k + m(n-1) + 1$. We now know $w(f)$ and f and g_0. The output of the algorithm is the object

$$H(P) = (f; u_1, \ldots, u_n),$$

where $u_i = g_0(i) - g(i)$ for $1 \leq i \leq n$.

Figure 4.8: Definition of H.

all hold. To see that $H(P)$ does belong to $\mathcal{I}_{n,k,m}$, we must still prove that the algorithm produces a function $f \in \mathcal{F}_{n,k,m}$, i.e., that $0 \leq f(x) \leq k + m(n-1)$ for $1 \leq x \leq n$. We must also show that the algorithm always terminates and that it never encounters the error condition.

Lemma 4.48. *If $P \in \mathcal{P}_{n,k,m}$, then the algorithm defining $H(P)$ never declares an error.*

Proof. The proof is by induction on the value of the variable v. First, assume $v = 0$. Suppose that the processing of label x causes an error. Note that $pg_0(x) = 1$, since the bounce path starts in the first column. On the other hand, $g(x) \geq 1$, so that $u_x = pg_0(x) - g(x) \leq 0$. But the error occurred because $u_x > L_x(pf)$, where $L_x(pf) = 0$ by (4.23). This is a contradiction.

Next, assume by induction that $v = j > 0$ and that the algorithm has executed the loop iterations for $v = 0, 1, \ldots, j-1$ with no error. Suppose that the algorithm declares an error in the loop iteration for $v = j$ while processing label x. Then, in particular, label x was not added to $pw(f)$ in the previous iteration when $v = j-1$. So, in iteration $j-1$ we must have had $u_x < -R_x(pf)$. Using Lemma 4.44(4) and expanding the definitions, this says that

$$H_{j-2} + 1 - g(x) < -|pf_{>x}^{-1}(\{j-1, \ldots, j-m\})|. \tag{4.27}$$

Multiplying by -1 and noting that all quantities are integers, we can rewrite this as

$$-H_{j-2} - 1 + g(x) \geq |pf_{>x}^{-1}(\{j-1, \ldots, j-m\})| + 1. \tag{4.28}$$

Next, the assumption that x caused an error in iteration j means that $u_x > L_x(pf)$ in iteration j. Translating the definitions gives

$$H_{j-1} + 1 - g(x) > |pf_{<x}^{-1}(\{j-1, \ldots, j-m\})| - 1 + \chi(j \leq k). \tag{4.29}$$

Adding the last two inequalities, we conclude that

$$h_{j-1} > |pf^{-1}(\{j-1, \ldots, j-m\})| + \chi(j < k+1). \tag{4.30}$$

(To justify the simplification of the right side, observe that we cannot have $pf(x) \in \{j-1, \ldots, j-m\}$; otherwise the algorithm would not be considering label x during iteration j.) But, on the other hand, the definition of the bounce path gives

$$h_{j-1} = |pf^{-1}(\{j-1, \ldots, j-m\})| + \chi(j-1 < k), \tag{4.31}$$

which contradicts the preceding inequality and completes the induction proof. Note that the prefix property of the word of f is needed to ensure that certain quantities appearing in the equations above do not change from one iteration to the next. $\square$

Lemma 4.49. *Let $P \in \mathcal{P}_{n,k,m}$. Suppose that, at some stage of the algorithm defining $H(P)$, $pw(f)$ contains exactly i numbers, where $0 \leq i \leq n-1$. Then $pw(f)$ contains at most $k+mi$ bar symbols.*

Proof. We use induction on the length ℓ of $pw(f)$. The result obviously holds when $pw(f)$ is the empty word. Suppose that the result holds when $pw(f)$ has length $\ell \geq 0$. Let $pw(f)$ have i numbers and b bar symbols. If $b < k+mi$ and the algorithm appends a number next, then the result still holds since $b < k+m(i+1)$. If $b < k+mi$ and the algorithm appends a bar symbol next, then the result still holds since $b+1 \leq k+mi$. We are reduced to the case where $b = k+mi$. It suffices to show that, in this case, the next symbol appended by step 2 of the algorithm will be a number, not a bar.

To prove this, we establish a number of claims.

Claim 1: If $i > 0$ and y is the rightmost label in $pw(f)$, then $pw(f)$ has at least m bar symbols following y. Proof: If, instead, there were $s < m$ bar symbols after y, consider the prefix p' with y and these s bar symbols erased. This shorter prefix has $i-1 \geq 0$ numbers and $k+mi-s > k+m(i-1)$ bar symbols in it, which contradicts the induction hypothesis.

For the next few claims, assume x is a label not already appearing in $pw(f)$. Let us tentatively append x to $pw(f)$ to obtain a new partial word $pw'(f)$, as in step 2 of the algorithm defining H. Then $pf(x) = b$, since there are b bars preceding x in the word of f.

Claim 2: $V_{b-1} = i$. Recall that $V_{b-1} = |\{x : pf(x) \leq b-1\}|$. The claim is clear when $i = 0$, since x is the first number in $pw'(f)$ and $pf(x) = b > b-1$. If $i > 0$, claim 1 shows that $pw(f)$ ends in a bar symbol. So, the $i-1$ numbers y preceding x in $pw'(f)$ must satisfy $pf(y) \leq b-1$. The numbers following x (and x itself) have function values at least b, so claim 2 follows.

Claim 3: $b-1 \geq k-1$ and $v_{b-1-u} = 0$ for $0 \leq u < m-1$. We have $b-1 = k+mi-1 \geq k-1$. If $i = 0$, so that x is the first number in $pw'(f)$, then we certainly have $v_{b-1-u} = 0$ for all $u \geq 0$. If $i > 0$, claim 1 shows that $pw(f)$ ends in m (or more) bar symbols. It again follows that $v_{b-1-u} = 0$ for $0 \leq u < m-1$.

Claim 4: $pg_0(x) = b+1$. Recalling that $pf(x) = b$, Lemma 4.44(4) gives $pg_0(x) = 1 + H_{b-1}$. Next, Lemma 4.42(2) and claim 3 show that $H_{b-1} = k + mV_{b-1}$. Combining this with claim 2, we get $pg_0(x) = 1 + k + mi = b + 1$.

Now we can prove the earlier assertion that the next symbol appended to $pw(f)$ by the algorithm will be a number, not a bar. By claim 1, the last symbol (if any) generated by the algorithm was a bar symbol. So, without loss of generality, we can assume the algorithm is at the beginning of step 2(a).

To get a contradiction, suppose that all labels considered in this iteration of step 2(a) are rejected. This happens if and only if $u_x < -R_x(pf)$ for all unusued labels x. Now, by definition of $\mathcal{P}_{n,k,m}$, $|T_{B(i+1)}(g)| \geq i+1$. So there exist at least $i+1$ labels $x \in \{1, \ldots, n\}$ such that

$$g(x) \leq B(i+1) = 1 + k + mi = b + 1.$$

Choose such an x that does not already appear in $pw(f)$. Consider what happens when step 2(a) tentatively appends this x to $pw(f)$ to give $pw'(f)$. We have $pg_0(x) = b+1$ by claim 4, and so

$$u_x = pg_0(x) - g(x) \geq b + 1 - (b+1) = 0.$$

But the assumption that x was rejected means that

$$u_x < -R_x(pf) \leq 0.$$

We obtain the contradiction $u_x < 0$ and $u_x \geq 0$. $\square$

Corollary 4.50. *Let $P \in \mathcal{P}_{n,k,m}$.*

(1) When executing the algorithm defining $H(P)$, all n labels in $\{1, 2, \ldots, n\}$ are eventually added to $pw(f)$. Consequently, the algorithm always terminates.

(2) If f is the function produced by the algorithm defining $H(P)$, then

$$f(x) \in \{0, 1, \ldots, k + m(n-1)\} \text{ for } 1 \leq x \leq n.$$

Consequently, H is a well-defined map from $\mathcal{P}_{n,k,m}$ to $\mathcal{I}_{n,k,m}$.

Proof. To prove (1), suppose that the algorithm only adds $i < n$ labels to $pw(f)$. After the i^{th} label is appended, each subsequent iteration of step 2 of the algorithm will add one more bar symbol to $pw(f)$. Eventually, there will be more than $k + mi$ bar symbols, contradicting the previous lemma. Thus, all n labels are eventually added to $pw(f)$, at which point the algorithm exits the loop in step 2 and terminates after step 3.

To prove (2), consider the value of $pw(f)$ just before the n^{th} label x is appended to it. This prefix of $w(f)$ contains $i = n-1$ labels. By the lemma, the number of bars in $pw(f)$ is at most $k + m(n-1)$. Since $f(x)$ is always the number of bars preceding x in $w(f)$, we have $f(x) \leq k+m(n-1)$. For the same reason, we have $f(y) \leq k+m(n-1)$ for all labels y preceding x in $w(f)$. So, the image of f is contained in $\{0, 1, \ldots, k+m(n-1)\}$. This also shows, incidentally, that step 3 of the algorithm defining H makes sense. We observed earlier that the numbers u_j produced by the algorithm satisfy the required inequalities. Hence, we finally conclude that H is a well-defined map from $\mathcal{P}_{n,k,m}$ to $\mathcal{I}_{n,k,m}$. □

As remarked earlier, it is clear that $G \circ H = Id_{\mathcal{P}_{n,k,m}}$. Our final theorem says that H is the two-sided inverse for G.

Theorem 4.51. *The maps $G : \mathcal{I}_{n,k,m} \to \mathcal{P}_{n,k,m}$ and $H : \mathcal{P}_{n,k,m} \to \mathcal{I}_{n,k,m}$ are bijections with $H = G^{-1}$. For $P \in \mathcal{P}_{n,k,m}$, define $pmaj(P) = qstat(H(P))$. Then:*

$$pmaj(P) = qstat(H(P)) \text{ and } area(P) = tstat(H(P)); \tag{4.32}$$

$$qstat(I) = pmaj(G(I)) \text{ and } tstat(I) = area(G(I)). \tag{4.33}$$

Consequently,

$$\begin{aligned} \sum_{P\in\mathcal{P}_{n,k,m}} q^{pmaj(P)} t^{area(P)} &= \sum_{I\in\mathcal{I}_{n,k,m}} q^{qstat(I)} t^{tstat(I)} \\ &= \sum_{P\in\mathcal{P}_{n,k,m}} q^{area(P)} t^{h(P)} \\ &= CH_{n,k,m}(q,t), \end{aligned} \tag{4.34}$$

and so all these statistics have the same univariate distribution.

Proof. We have already shown that G maps into $\mathcal{P}_{n,k,m}$, H maps into $\mathcal{I}_{n,k,m}$, and $G \circ H = Id_{\mathcal{P}_{n,k,m}}$. The last equation implies that H is an injection and G is a surjection. But Corollary 4.32 showed that

$$|\mathcal{I}_{n,k,m}| = |\mathcal{P}_{n,k,m}| < \infty.$$

Since the sets are finite, H is automatically a surjection, G is automatically an injection, and $H = G^{-1}$. The properties in (4.33) follow from Lemma 4.46 and the very definition of $pmaj$, and (4.32) follows by replacing I by $H(P)$ and simplifying. The equalities in (4.34) follow from the existence of the weight-preserving bijections G and F. Letting $q = 1$ or $t = 1$ in (4.34) gives the final assertion of the theorem. $\square$

Acknowledgement: This chapter is essentially a reprint, with minor modifications, of the paper "Conjectured Combinatorial Models for the Hilbert Series of Generalized Diagonal Harmonics Modules" by N. Loehr and J. Remmel, which is now in preparation for publication. The dissertation author was the primary investigator and author of this paper.

5

More Results in Lattice Path Enumeration

This chapter presents some further results and open problems in the theory of lattice path enumeration. First, we use a generalization of André's famous reflection principle to count trapezoidal lattice paths. Second, we prove various identities for counting lattice paths including a new determinantal formula for the Carlitz-Riordan generating function $C_n^{area}(q)$, which enumerates Dyck paths by area. Third, we discuss bijections that connect Haglund's combinatorial q,t-Catalan sequence (see Chapter 1) to classical permutation statistics. As a byproduct of these bijections, we obtain two new collections of permutations that are enumerated by the Catalan numbers.

5.1 André's Reflection Principle and Trapezoidal Paths

This section presents a generalization of André's reflection principle, which gives a new combinatorial proof of a formula for the number of lattice paths lying within certain trapezoids. This section is completely self-contained.

Consider paths in the xy-plane that go from (W,H) to $(0,0)$ by taking W west steps and H south steps of length one. The number of such paths is

$$\binom{H+W}{H,W} = \frac{(H+W)!}{H!W!},$$

since each such path has $H+W$ steps total, and we can choose any set of W steps to be west steps. We prefer to write the binomial coefficient as a multinomial coefficient, so that both the height H and width W explicitly appear.

Next, consider lattice paths going from (n,n) to $(0,0)$ by taking n west steps and n south steps that never go strictly below the diagonal line $y=x$. As is well-known, the number of such paths is the Catalan number

$$\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n,n} - \binom{2n}{n-1,n+1}.$$

The famous reflection principle of André [1] gives a combinatorial proof of this last result, in which paths descending below the diagonal are matched off bijectively with paths fitting in an $(n-1)\times(n+1)$ rectangle.

We will prove a more general result for lattice paths contained in certain trapezoids. Let $k \geq 0$, $H > 0$, and $m > 0$ be integers. Set $W = k + mH$, and let D denote the diagonal line whose equation is $x = k + my$. How many lattice paths go from (W,H) to $(0,0)$ by taking W west steps and H south steps that never go strictly below the diagonal D? The well-known answer is

$$\binom{H+W}{H,W} - m\binom{H+W}{H-1,W+1}.$$

It is hard to use reflections to prove this result, since the symmetry group of a (non-square) rectangle does not include reflection through a diagonal. However, this symmetry group does include a half-turn, which sends each corner of the rectangle to the diagonally opposite corner. Hence, our new combinatorial proof of this identity will be based on a "rotation principle."

Label each lattice point $P=(x,y)$ with the integer $x-(k+my)$, which is the signed horizontal distance from D to P. Note that lattice points on D have label zero, lattice points left of D have negative labels, and lattice points right of D have positive labels. Let S denote the set of *all* paths from (W,H) to $(0,0)$, and let T denote the set of all paths from $(W+1,H-1)$ to $(0,0)$. Let S_0 denote the set of paths in S that never go strictly below D; so S_0 consists of paths that only visit points having nonpositive labels. For $1 \leq i \leq m$, let S_i denote the set of paths in S that do go below D, and whose first positive label (reading southwest from (W,H)) is i. Since taking a single south step causes the label to increase by m, the first positive label (if any) for each path in S must be an element of $\{1, 2, \ldots, m\}$. Therefore, S is the disjoint union of $S_0, S_1, \ldots, S_m$.

We will define m bijections $f_i : S_i \to T$. This will prove the desired result, since $|S| = \binom{H+W}{H,W}$ and $|T| = \binom{H+W}{H-1,W+1}$ and $|S| = \sum_{j=0}^{m} |S_j|$. To define f_i, let π be a path in S_i. Augment the beginning of π with a horizontal step from (W, H) to the point $Q = (W+1, H)$, whose label is 1. By assumption, there exists a point R where π descends below D for the first time, and the label of R is i. Rotate the portion of the augmented path between R and Q by 180° about the midpoint of the line segment QR. Then erase the resulting vertical step from $(W+1, H-1)$ to $(W+1, H)$ to obtain a path $f_i(\pi)$ in T. If we encode the augmented path as a sequence of h's and v's representing horizontal and vertical steps, the rotation corresponds to reversing the part of this sequence between Q and R and then erasing the initial v. This v must be present, since π can only go below D for the first time by taking a vertical step. See Figure 5.1 for an example where $m = 3$, $k = 5$, $H = 9$, and $W = 32$.

To see that f_i is a bijection, we display a two-sided inverse map $g_i : T \to S_i$. Given a path $\tau \in T$ from $(W + 1, H - 1)$ to $(0, 0)$, augment the beginning of this path with a vertical move to $Q = (W + 1, H)$. Call this augmented path τ'. Let S be the first point on τ' after Q whose label is i. Such a point must exist, since the label of $(W+1, H-1)$ is $m+1$, the label of $(0, 0)$ is nonpositive, and each west step decrements the current label by one. Form a path π' by rotating the portion of τ' between S and Q by 180°. Finally, define the path $\pi = g_i(\tau)$ to be the path π' with the initial horizontal step from (W, H) to $(W + 1, H)$ deleted. This step must exist, since τ arrives at S by taking a west step.

We must check that $\pi = g_i(\tau)$ is an element of the set S_i. Let $d_0, \ldots, d_s$ be the labels of the lattice points visited by τ', starting from Q. Similarly, let $e_0, \ldots, e_s$ be the labels of the lattice points visited by π', starting from Q. Let $j \geq 1$ be the smallest index such that $d_j = i$. Thus, the path τ' reaches the point S after taking j steps from Q. Note that $d_0 = 1$, $d_1 = m + 1$, $d_{j-1} = i + 1$, and $d_j = i$. Also, $e_0 = 1$, $e_1 = 0$, $e_{j-1} = i - m$, and $e_j = i$. More generally, we claim that $e_k = (i+1) - d_{j-k}$ for $0 \leq k \leq j$. This is certainly true for $k = 0$. Suppose it is true for some $k < j$. Since the rotation map reverses the sequence of j steps in τ' leading from Q to S, it follows that

$$e_{k+1} - e_k = d_{j-k} - d_{j-(k+1)}.$$

More specifically, both sides are m if the $(j-k)^{\text{th}}$ step of τ' is vertical, and both sides are -1 if the $(j-k)^{\text{th}}$ step of τ' is horizontal. Either way,

$$e_{k+1} = (e_k + d_{j-k}) - d_{j-(k+1)} = (i+1) - d_{j-(k+1)},$$

which proves that the claim holds with $k+1$ in place of k. Now, since $d_{j-k} > i$ for $0 < j-k < j$, the claim shows that $e_k \leq 0$ for $0 < k < j$. On the other hand, $e_j = i > 0$. This says that the path $\pi = g_i(\tau)$ goes below the diagonal D for the first time at step j, where it hits the point S whose label is i. This shows that π does belong to S_i. Furthermore, it is now clear that $f_i(g_i(\tau)) = \tau$, since f_i will just rotate the portion of π' between Q and S and produce τ' again. Similarly, it is easy to see from the labelling rules that $g_i(f_i(\pi)) = \pi$ for any $\pi \in S_i$. The crucial observation is that every point in $f_i(\pi)$ between Q and S has a label larger than i. This follows from the claim, since every point in π between Q and S has a nonpositive label. We conclude that each f_i is a bijection with inverse g_i, and the proof is complete.

The rotation technique given here fails if we consider a diagonal line D whose equation is $x = k + (r/s)y$, where $r > 1$ and $s > 1$. We leave it as an open problem to find a combinatorial proof of the appropriate formula in this more general case.

5.2 Enumerating Lattice Paths by Area and Major Index

This section presents a number of recursions and formulas for counting special collections of lattice paths. Most of these identities involve the *area generating function* for these paths, which is the sum of terms $q^{area(P)}$ over all paths P in the collection. One new result proved here is a determinantal formula for the Carlitz-Riordan q-analogue of the Catalan number, which was defined in Chapter 1 as

$$C_n^{area}(q) = \sum_{P \in \mathcal{D}_n} q^{area(P)}.$$

This section is essentially self-contained.

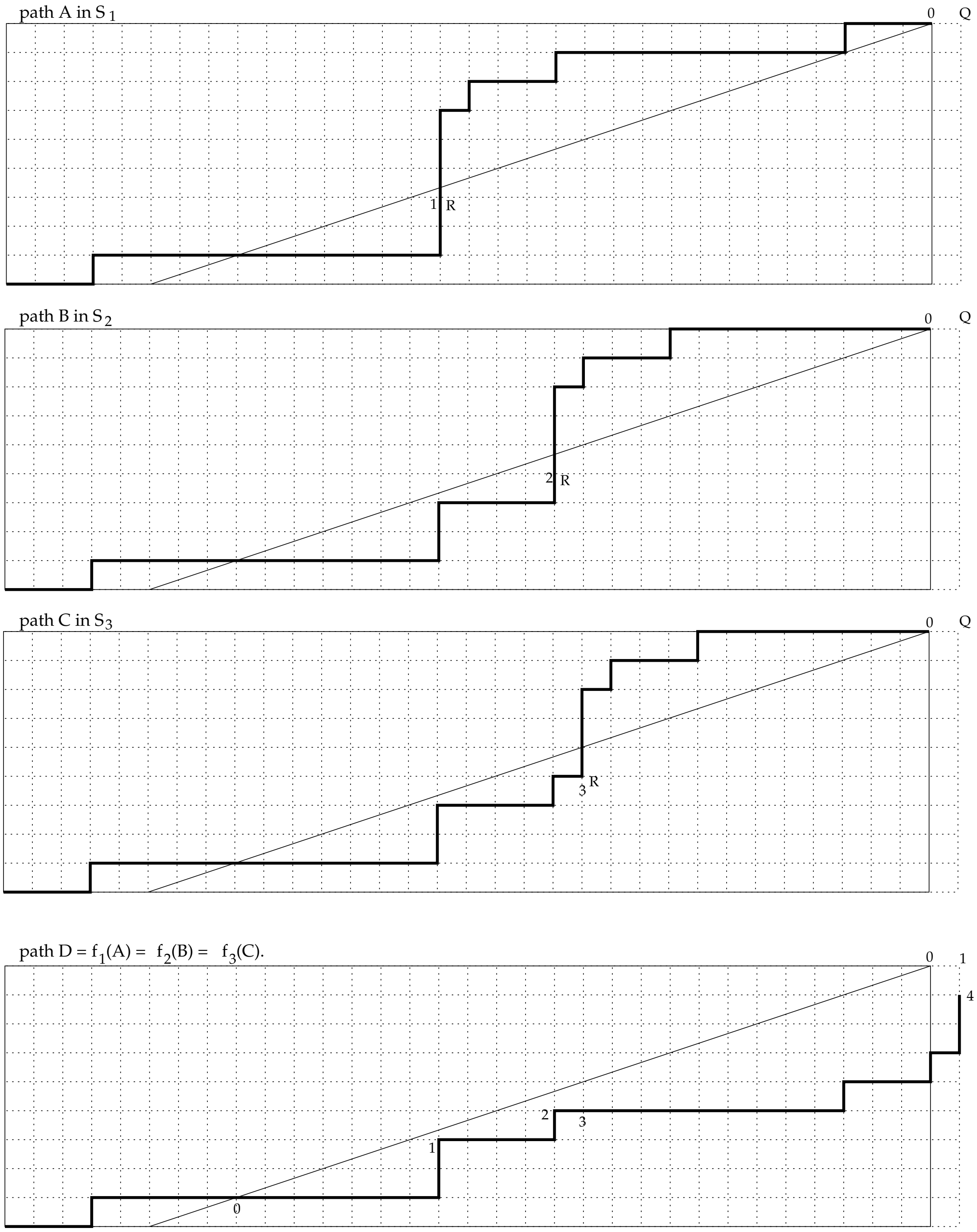


Figure 5.1: Example of the rotation maps.

5.2.1 Counting Paths inside Arbitrary Shapes

Definition 5.1. Let $n \geq 0$, and suppose $B : \{0, 1, \ldots, n\} \to \mathbb{Z}$ is a function such that $0 \leq B(0) \leq B(1) \leq \cdots \leq B(n)$. The function B determines a region in the first quadrant

$$R(B) = \{(x, y) : 0 \leq y \leq n \text{ and } 0 \leq x \leq B(y)\}.$$

A lattice path that goes from $(0, 0)$ to (x, y) by taking x horizontal steps and y vertical steps is said to be *valid relative to* B if and only if it is contained in the region $R(B)$. This means that $0 \leq y \leq n$ and $0 \leq c \leq B(d)$ for all points (c, d) on the path. Let $\mathcal{P}(x, y, B)$ denote the collection of such paths.

There is a simple recursion for computing the numbers $|\mathcal{P}(x, y, B)|$.

Proposition 5.2. *For $x > 0$ and $y > 0$, we have*

$$|\mathcal{P}(x, y, B)| = |\mathcal{P}(x - 1, y, B)| + |\mathcal{P}(x, y - 1, B)|.$$

The initial conditions are

$$|\mathcal{P}(0, y, B)| = 1 = |\mathcal{P}(x, 0, B)| \textit{ for all } x, y \geq 0,$$

$$|\mathcal{P}(x, y, B)| = 0 \textit{ whenever } (x, y) \notin R(B).$$

Proof. We can uniquely construct each path $P \in \mathcal{P}(x, y, B)$ either by appending a horizontal step to a valid path from the origin to $(x-1, y)$, or by appending a vertical step to a valid path from the origin to $(x, y - 1)$. The recursion above follows immediately. The initial conditions are clear, since there is only one path from $(0, 0)$ to $(0, y)$, consisting of y vertical steps. Similarly, there is only one path from $(0, 0)$ to $(x, 0)$. Finally, if (x, y) is outside the region of validity, then there are no valid paths ending at (x, y). □

The previous result can also be rewritten

$$|\mathcal{P}(x, y, B)| = \chi((x-1, y) \in R(B))|\mathcal{P}(x-1, y, B)| + \chi((x, y-1) \in R(B))|\mathcal{P}(x, y-1, B)|$$

for $(0, 0) \neq (x, y) \in R(B)$, with initial condition $|\mathcal{P}(0, 0, B)| = 1$.

Example 5.3. Fix an integer $m \geq 1$, and let $B(i) = \lfloor i/m \rfloor$ for $i \geq 0$. The set $\mathcal{P}(H, W, B)$ consists of lattice paths from $(0, 0)$ to (H, W) contained in the trapezoid bounded by the lines $x = 0$, $y = W$, $x = H$ and $y = mx$. (This is essentially the same collection of paths considered in §5.1, but the orientation and position of the trapezoid is different.) Using the recursion above, we can now give an algebraic proof of the formula

$$|\mathcal{P}(H, W, B)| = \binom{H+W}{H, W} - m\binom{H+W}{H-1, W+1} \text{ for } W \geq mH \tag{5.1}$$

This formula evaluates to 1 when $H = 0$, which is the correct value for the initial condition. Given $H > 0$ and $W \geq mH$, assume by induction that (5.1) holds for paths ending at $(H-1, W)$ and $(H, W-1)$. If $W = mH$, the point $(H, W-1)$ is outside $R(B)$, and the recursion gives

$$\begin{aligned} |\mathcal{P}(H, W, B)| = |\mathcal{P}(H-1, W, B)| &= \binom{H-1+W}{H-1, W} - m\binom{H-1+W}{H-2, W+1} \\ &= \binom{(m+1)H-1}{H-1, mH} - m\binom{(m+1)H-1}{H-2, mH+1}. \end{aligned}$$

Routine manipulation of factorials shows that this expression does equal

$$\binom{(m+1)H}{H, mH} - m\binom{(m+1)H}{H-1, mH+1},$$

as desired. If $W > mH$, the point $(H, W-1)$ is inside $R(B)$, and the recursion gives

$$|\mathcal{P}(H, W, B)| = \binom{H+W-1}{H-1, W} - m\binom{H+W-1}{H-2, W+1} + \binom{H+W-1}{H, W-1} - m\binom{H+W-1}{H-1, W}.$$

Two applications of the identity $\binom{a+b}{a,b} = \binom{a+b-1}{a-1,b} + \binom{a+b-1}{a,b-1}$ show that this expression does equal

$$\binom{H+W}{H, W} - m\binom{H+W}{H-1, W+1},$$

which completes the induction.

Example 5.4. The recursion can be used to rapidly compute values of $|\mathcal{P}(x, y, B)|$ even when no explicit formula is available. Figure 5.2 illustrates such a computation when $B(i) = \lfloor 3 + (3/2)i \rfloor$. Each lattice point in $R(B)$ is labelled by the number $|\mathcal{P}(x, y, B)|$; the label is drawn just below and left of the lattice point. Points on the far left and bottom of the figure have label 1. For a point inside the region, its label is the sum of the label of the point to its left and the point below it, provided that the latter point is still within the region $R(B)$. From Figure 5.2, we see that $|\mathcal{P}(9, 4, B)| = 241$.

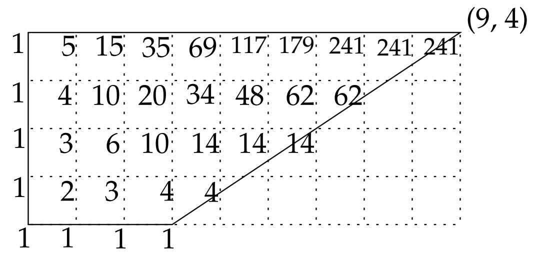


Figure 5.2: Counting lattice paths with the recursion.

Definition 5.5. For a path $P \in \mathcal{P}(x, y, B)$, define the *area* of P to be the number of complete lattice cells lying to the right of P and inside the region $R(B)$. These cells are called *area cells for P*. Define the *area generating function*

$$F(x, y, B; q) = \sum_{P \in \mathcal{P}(x,y,B)} q^{area(P)}.$$

The recursion above generalizes easily to a recursion for these area generating functions.

Proposition 5.6. *Assume* $B(i-1) \leq B(i)$ *for* $1 \leq i \leq n$*. For* $x > 0$ *and* $y > 0$*, we have*

$$F(x, y, B; q) = F(x-1, y, B; q) + q^{B(y-1)-x} F(x, y-1, B; q).$$

The initial conditions are

$$F(x, 0, B; q) = 1 \text{ for all } x \geq 0,$$

$$F(0, y, B; q) = q^{B(0)+B(1)+\cdots+B(y-1)} \text{ for all } y \geq 0.$$

Proof. Let $P \in \mathcal{P}(x, y, B)$. If P is obtained by appending a horizontal step to some path $P' \in \mathcal{P}(x-1, y, B)$, then clearly $area(P) = area(P')$. This gives the term $F(x-1, y, B; q)$ in the recursion. Suppose instead that P is obtained by appending a vertical step to some path $P' \in \mathcal{P}(x, y-1, B)$. Since B is increasing, P will have $B(y-1) - x$ new area cells in its top row, in addition to all the area cells of P'. This gives the other term in the recursion. The initial conditions follow since the path ending at $(x, 0)$ has no area cells, while the path ending at $(0, y)$ has $B(0) + B(1) + \cdots + B(y-1)$ area cells. □

The reader can readily generalize this result to the case where B is not necessarily increasing.

5.2.2 Recursions for Counting Dyck Paths by Area

We now return to the special case of Dyck paths.

Definition 5.7. For each integer $n > 0$, let $\mathcal{D}_n$ denote the collection of Dyck paths ending at (n, n). For $n > 0$ and all k, let $\mathcal{D}_{n,k}$ denote the collection of paths in D_n terminating in exactly k horizontal steps. Let $C_n = |\mathcal{D}_n|$ and $C_{n,k} = |\mathcal{D}_{n,k}|$.

Clearly, $C_{n,k} = 0$ for $k \leq 0$ and $k > n$, while $C_{n,n} = 1$. It is easy to see that $C_{n+1,1} = C_n = \sum_{k=1}^{n} C_{n,k}$. Recall from Chapter 1 that C_n is the n^{th} Catalan number.

Definition 5.8. For $n \geq 1$ and $1 \leq k \leq n$, set

$$C_{n,k}(q) = \sum_{P \in \mathcal{D}_{n,k}} q^{area(P)},$$

$$C_n(q) = \sum_{P \in \mathcal{D}_n} q^{area(P)}.$$

$C_n(q)$ is the Carlitz-Riordan q-analogue of the Catalan numbers, which was called $C_n^{area}(q)$ in Chapter 1. It is convenient to set $C_0(q) = 1$.

The following recursion characterizes the quantities $C_n(q)$.

Proposition 5.9. *We have $C_0(q) = C_1(q) = 1$ and, for $n > 1$,*

$$C_n(q) = \sum_{k=1}^{n} q^{k-1} C_{k-1}(q) C_{n-k}(q).$$

Proof. We have $C_0(q) = 1$ by definition, while $C_1(q) = 1 = q^0$ because the unique Dyck path of order 1 has zero area cells. To get the recursion for $C_n(q)$, classify the paths $P \in \mathcal{D}_n$ based on the smallest value $k > 0$ such that (k, k) is on the path P. Such a k must exist, since (n, n) is on the path P. Let P_1 be the portion of P going from $(0, 1)$ to $(k - 1, k)$, and let P_2 be the portion of P going from (k, k) to (n, n). See Figure 5.3.

If we shift P_1 down one unit, we get an element of $\mathcal{P}_{k-1}$; the minimality of k guarantees that this shifted path does not go below the line $y = x$. Similarly, if we shift

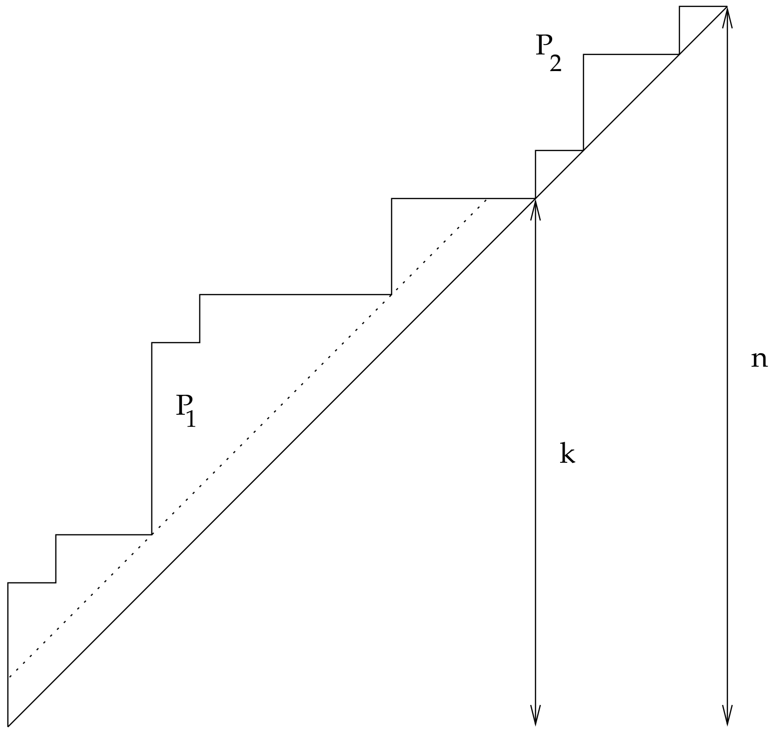


Figure 5.3: Dissecting a Dyck path based on the first return to the main diagonal.

P_2 left k units and down k units, we get an element of $\mathcal{P}_{n-k}$. This process is obviously reversible. We have

$$area(P) = area(P_1) + area(P_2) + (k-1),$$

since the path P has $k-1$ area cells in rows 2 through k that do not count as area cells of P_1 after shifting it down one unit. The recursion follows immediately from the product rule for generating functions [3]. (The reader may check that the convention $C_0(q) = 1$ gives the correct summand in the extreme cases where $k = 1$ or $k = n$.) □

We can get an even simpler recursion for the quantities $C_{n,k}(q)$.

Theorem 5.10.

$$C_{n,k}(q) = q^{k-1}C_{n-1,k-1}(q) + q^{-1}C_{n,k+1}(q) \textit{ for } 0 < k < n \tag{5.2}$$

with initial conditions $C_{n,n} = q^{n(n-1)/2}$, $C_{n,k} = 0$ *for* $k > n$ *and* $k \le 0$.

Proof. For $0 < k < n$, each path in $\mathcal{D}_{n,k}$ can be constructed uniquely in one of the following two ways. First, we can append a new top row containing $k-1$ area cells to an arbitrary path of order $n-1$ that ends in exactly $k-1$ horizontal steps. This produces a path in $\mathcal{D}_{n,k}$ in which there are at least two vertical steps just before the last k horizontal steps. Second, we can remove the leftmost area cell in the top row of an arbitrary path of order n that ends in exactly $k+1$ horizontal steps. This produces a path in $\mathcal{D}_{n,k}$ in which there is exactly one vertical step just before the last k horizontal steps. The recursion follows immediately. (Alternatively, letting $B(i) = i$ for all i, the recursion can be derived from (5.6) by noting that $C_{n,k}(q) = F(n-k, n, B; q)$.) For the initial conditions, note that the unique path of order n ending in n horizontal steps has area $n(n-1)/2$. Also, if $k > n$ or $k \leq 0$, then there are no Dyck paths of order n ending in k horizontal steps. $\square$

This recursion can be used to compute $C_n(q)$ too, since we clearly have

$$C_n(q) = C_{n+1,1}(q).$$

We now derive some consequences of recursion (5.2), which follow by iteratively expanding (5.2) in various ways.

Proposition 5.11.

$$C_{n,k}(q) = q^{k-1} \sum_{m=k-1}^{n-1} C_{n-1,m}(q). \tag{5.3}$$

$$C_{n,k}(q) = q^{(k-1)+(k-2)+\cdots+(k-i)} \sum_{m=0}^{n-k} \begin{bmatrix} m+i-1 \\ i-1 \end{bmatrix}_q C_{n-i,k-i+m}(q) \text{ for } 1 \leq i \leq k. \tag{5.4}$$

$$C_{n,k}(q) = \sum_{j=0}^{k-1} q^{kj-1-((j+1)j/2)} C_{n-j,k-j+1}(q). \tag{5.5}$$

$$C_{n,k}(q) = \sum_{0 \leq j < k/2} (-1)^j q^{k+j^2-j-1} \begin{bmatrix} k-1-j \\ j \end{bmatrix}_q C_{n-j,1}(q) \text{ for } 1 \leq k < 2n. \tag{5.6}$$

Proof. For each identity, we outline both an algebraic proof and a combinatorial proof. To get equation (5.3), we iterate (5.2) to eliminate terms involving $C_{n,k+1}(q)$, then

$C_{n,k+2}(q)$, $C_{n,k+3}(q)$, and so on:

$$\begin{aligned}
C_{n,k}(q) &= q^{k-1}C_{n-1,k-1}(q) + q^{-1}C_{n,k+1}(q) \\
&= q^{k-1}C_{n-1,k-1}(q) + q^{-1}\left(q^{k}C_{n-1,k}(q) + q^{-1}C_{n,k+2}(q)\right) \\
&= q^{k-1}\left(C_{n-1,k-1}(q) + C_{n-1,k}(q)\right) \\
&\quad + q^{-2}\left(q^{k+1}C_{n-1,k+1}(q) + q^{-1}C_{n,k+3}(q)\right) \\
&= q^{k-1}\left(C_{n-1,k-1}(q) + C_{n-1,k}(q) + C_{n-1,k+1}(q)\right) \\
&\quad + q^{-3}\left(q^{k+2}C_{n-1,k+2}(q) + q^{-1}C_{n,k+4}(q)\right) = \cdots .
\end{aligned}$$

Ultimately, we obtain

$$\begin{aligned}
C_{n,k}(q) &= q^{k-1}\left(C_{n-1,k-1}(q) + \cdots + C_{n-1,n-2}(q)\right) \\
&\quad + q^{k-n}\left(q^{n-1}C_{n-1,n-1}(q)\right) \\
&= q^{k-1}\sum_{m=k-1}^{n-1} C_{n-1,m}(q).
\end{aligned}$$

To get a combinatorial proof of the same identity, observe that the top row of a path counted by $C_{n,k}(q)$ contains $k-1$ area cells. Removing this row, we get a path counted by $C_{n-1,m}(q)$, where the number of horizontal steps m at the end of this smaller path must satisfy $k-1 \leq m \leq n-1$. Identity (5.3) now follows immediately from the sum rule.

To prove the equations (5.4), use induction on i. The base case $i = 1$ is just (5.3). Assuming that (5.4) holds for some $i < k$, we can get the analogous identity for $i+1$ by substituting

$$C_{n-i,k-i+m}(q) = q^{k-(i+1)+m} \sum_{j=k-(i+1)+m}^{n-i-1} C_{n-(i+1),j}(q),$$

interchanging the order of summation, and simplifying. Alternatively, Figure 5.4 shows how to prove (5.4) combinatorially. Fix i with $1 \leq i \leq k$. To construct an arbitrary path $P \in \mathcal{D}_{n,k}$, first choose an integer m between 0 and $n-k$ inclusive. Second, choose a path $P' \in \mathcal{D}_{n-i,k-i+m}$. The generating function for this choice is $C_{n-i,k-i+m}(q)$. Third, draw a vertical line from $(n-k, n-i)$ to $(n-k, n)$, and shade in the cells to the right of this line. There are $(k-i) + \cdots + (k-2) + (k-1)$ such cells. Fourth, draw a lattice path Q

in the rectangle with southwest corner $(n-i-(k-i+m), n-i)$ and northeast corner $(n-k, n-1)$. This rectangle has height $i-1$ and width m, so the generating function for this fourth choice is $\left[{m+i-1 \atop m,i-1}\right]_q$. Multiplying the generating functions for these choices and adding over all m, we obtain (5.4).

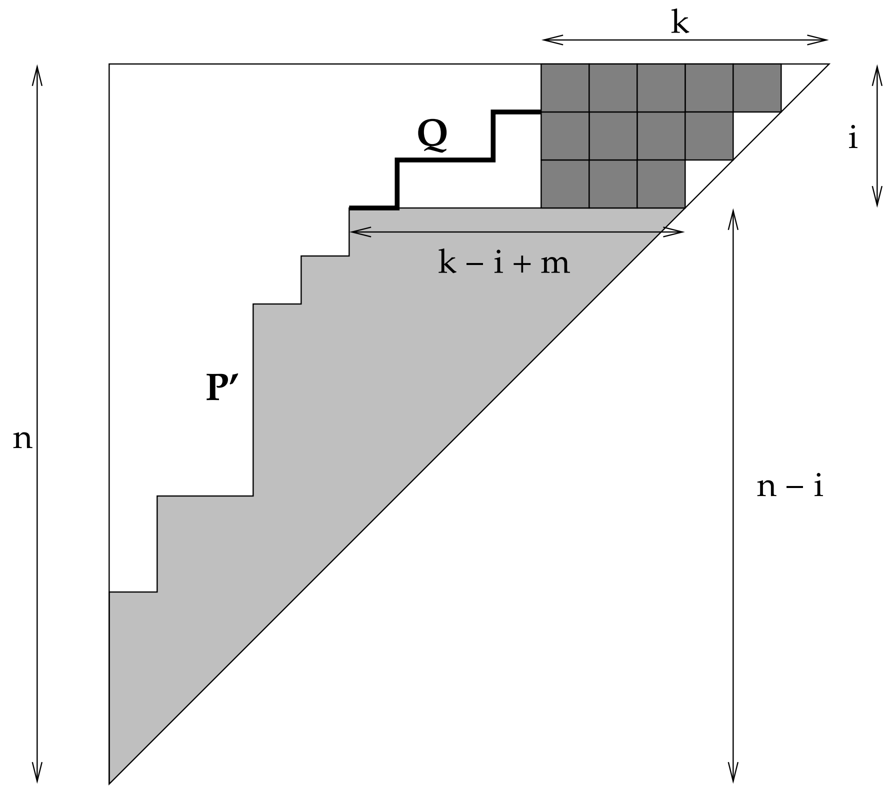


Figure 5.4: Dissecting a Dyck path by removing the top i rows.

We note that the case $i = k$ of (5.4) yields the relation

$$C_{n,k}(q) = \sum_{m=1}^{n-k} \begin{bmatrix} m+k-1 \\ m \end{bmatrix}_q q^{k(k-1)/2} C_{n-k,m}(q),$$

which is Haglund's recursion for the q,t-Catalan sequence with $t = 1$ (see Chapter 1).

To prove (5.5), iterate (5.2) to eliminate the terms involving $C_{n-1,k-1}(q)$, then

$C_{n-2,k-2}(q)$, $C_{n-3,k-3}(q)$, and so on:

$$\begin{aligned}
C_{n,k}(q) &= q^{k-1}C_{n-1,k-1}(q) + q^{-1}C_{n,k+1}(q) \\
&= q^{k-1}\left(q^{k-2}C_{n-2,k-2}(q) + q^{-1}C_{n-1,k}(q)\right) + q^{-1}C_{n,k+1}(q) \\
&= q^{2k-3}\left(q^{k-3}C_{n-3,k-3}(q) + q^{-1}C_{n-2,k-1}(q)\right) \\
&\quad + q^{k-2}C_{n-1,k}(q) + q^{-1}C_{n,k+1}(q) = \cdots .
\end{aligned}$$

Note that this iteration produces terms of the form $q^{pow_j}C_{n-j,k-j+1}(q)$, for $j \geq 0$. It is easily checked by induction that

$$pow_j = -1 + (k-1) + (k-2) + \cdots + (k-j) = kj - 1 - j(j+1)/2.$$

During the iteration, there is also an extra term $q^{pow}C_{n-m,k-m}(q)$, which is expanded using (5.2) to obtain the next formula in the iteration. This process ends when $m = k$, since $C_{n-k,0}(q) = 0$. What remains is the sum of terms $q^{pow_j}C_{n-j,k-j+1}(q)$ for $0 \leq j \leq k-1$. This completes the algebraic proof of (5.5).

To prove (5.5) combinatorially, classify paths in $\mathcal{D}_{n,k}$ by the number of vertical steps immediately preceding the last run of k horizontal steps in the path. If there are $j+1$ such vertical steps, then we must have $0 \leq j \leq k-1$ by definition of a Dyck path. We can uniquely construct each path in $\mathcal{D}_{n,k}$ as follows.

- Choose j with $0 \leq j \leq k-1$.
- Choose a path P' in $\mathcal{D}_{n-j,k-j+1}$. The generating function for this choice is
$$C_{n-j,k-j+1}(q).$$
- Remove the leftmost area cell in the top row of P'. The generating function for this step is q^{-1}, since the area decreases by one.
- Add a trapezoidal region consisting of j rows of area cells, whose lengths from top to bottom are $k-1, k-2, \ldots, k-j$. The generating function for this step is
$$q^{(k-1)+\cdots+(k-j)} = q^{kj-(j+1)j/2}.$$

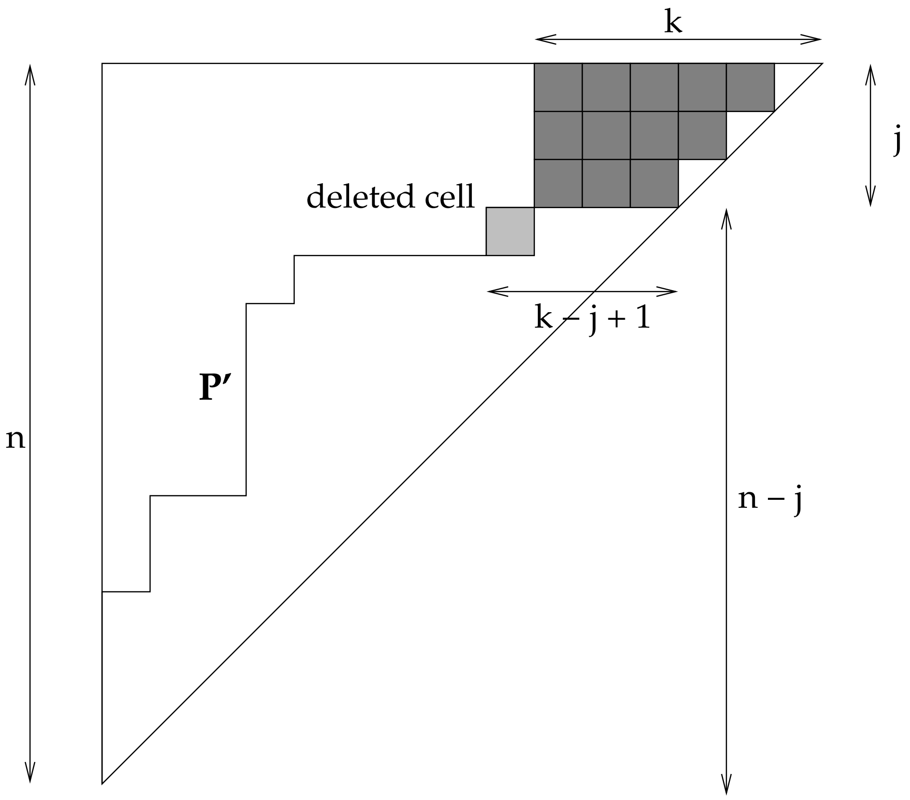


Figure 5.5: Classifying Dyck paths based on the size of the last vertical segment.

See Figure 5.5. Observe that this process of combining P' and j to produce P is reversible. Multiplying the generating functions from each step and adding over all j, we obtain (5.5).

Equation (5.6) is proved by induction on n and k. Note that (5.6) is equivalent to

$$C_{n,k}(q) = \sum_{j=0}^{\infty} (-1)^j q^{k+j^2-j-1} \begin{bmatrix} k-1-j \\ j \end{bmatrix}_q C_{n-j,1}(q),$$

since $\begin{bmatrix} a \\ b \end{bmatrix}_q = 0$ if $b > a$. If $k = 1$ and n is arbitrary, this equation says $C_{n,1}(q) = C_{n,1}(q)$, which is true. In particular, the equation holds when $n = 1$ and $1 \leq k < 2$. For the induction step, first rewrite (5.2) as

$$C_{n,k+1}(q) \quad = \quad qC_{n,k}(q) - q^k C_{n-1,k-1}(q).$$

Use the induction hypothesis to rewrite the right-hand side:

$$\begin{aligned} C_{n,k+1}(q) &= q\sum_{j=0}^{\infty}(-1)^j q^{k+j^2-j-1}\begin{bmatrix}k-1-j\\ j\end{bmatrix}_q C_{n-j,1}(q) \\ &\quad -q^k\sum_{j=0}^{\infty}(-1)^j q^{k-1+j^2-j-1}\begin{bmatrix}k-2-j\\ j\end{bmatrix}_q C_{n-1-j,1}(q). \end{aligned}$$

In the first summation here, separate the summand corresponding to $j = 0$. In the second summation, make the substitution $i = j + 1$. The result is

$$\begin{aligned} C_{n,k+1}(q) &= q^k C_{n,1}(q) + \sum_{i=1}^{\infty}(-1)^i q^{k+i^2-i}\begin{bmatrix}k-1-i\\ i\end{bmatrix}_q C_{n-i,1}(q) \\ &\quad +\sum_{i=1}^{\infty}(-1)^i q^{2k-1+(i-1)^2-(i-1)-1}\begin{bmatrix}k-1-i\\ i-1\end{bmatrix}_q C_{n-i,1}(q). \end{aligned}$$

The last two summations combine to give

$$\begin{aligned} C_{n,k+1}(q) &= q^k C_{n,1}(q) \\ &\quad +\sum_{i=1}^{\infty}(-1)^i q^{(k+1)+i^2-i-1}\left(\begin{bmatrix}k-1-i\\ i\end{bmatrix}_q + q^{k-2i}\begin{bmatrix}k-1-i\\ i-1\end{bmatrix}_q\right) C_{n-i,1}(q). \end{aligned}$$

Applying Theorem 1.51(3) with $C = i$ and $D = k - 1 - 2i$, we see that

$$\begin{bmatrix}k-1-i\\ i\end{bmatrix}_q + q^{k-2i}\begin{bmatrix}k-1-i\\ i-1\end{bmatrix}_q = \begin{bmatrix}k-i\\ i\end{bmatrix}_q.$$

Using this above, we get

$$C_{n,k+1}(q) = +\sum_{i=0}^{\infty}(-1)^i q^{(k+1)+i^2-i-1}\begin{bmatrix}(k+1)-1-i\\ i\end{bmatrix}_q C_{n-i,1}(q).$$

This completes the induction step. Note that the induction breaks down when we reach $k = 2n$, because one of the summands for that k is $C_{0,1}(q)$, which is undefined. Thus, the stated formula (5.6) for $C_{n,k}(q)$ is valid for $1 \le k < 2n$.

A combinatorial proof of (5.6) is less obvious, since both positive and negative terms appear in this equation. Such a proof can be given by defining a weight-preserving, sign-reversing involution on a suitable collection of objects. This will be done in a more general setting in the next subsection. □

Example 5.12. Setting $q = 1$ in (5.6), we find that

$$C_{n,k} = \sum_{0 \le j < k/2} (-1)^j \binom{k-1-j}{j} C_{n-j,1} \text{ for } 1 \le k < 2n.$$

Since $C_{n,n} = 1$ and $C_{n,k} = 0$ for $k > n$, we obtain recursions that only involve the Catalan numbers (recall $C_{n-j-1} = C_{n-j,1}$). In particular, replacing n by $n+1$, k by $n+2$, and solving for $C_n = C_{n+1,1}$ we obtain a recursion

$$C_n = \sum_{1 \le j \le \frac{n+1}{2}} (-1)^{j-1} \binom{n+1-j}{j} C_{n-j}.$$

Similarly, replacing n by $n+1$ and k by $2n+1 < 2(n+1)$ gives a recurrence

$$0 = \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} C_{n-j},$$

from which we get an expression for C_n in terms of *all* of the preceding Catalan numbers. Of course, (5.6) also gives q-analogues of these recursions for Carlitz's q-Catalan numbers.

Example 5.13. Let $n = 3$ and $k = 5 = 2n - 1$. Then (5.6) gives the true identity

$$0 = q^4 \begin{bmatrix} 4 \\ 0 \end{bmatrix}_q (1+q) - q^4 \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \cdot 1 + q^6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \cdot 1.$$

If, instead, $n = 3$ and $k = 6 = 2n$, (5.6) becomes

$$0 = q^5 \begin{bmatrix} 5 \\ 0 \end{bmatrix}_q (1+q) - q^5 \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \cdot 1 + q^7 \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \cdot 1,$$

which is false since the right side is q^9. This shows that the requirement $1 \le k < 2n$ imposed in the proposition cannot be relaxed.

5.2.3 Using Involutions to Count Lattice Paths

This subsection describes some sign-reversing, weight-preserving involutions that can be used to prove additional identities for counting lattice paths. One application is an explicit determinantal formula for the Carlitz-Riordan q-analogue of the Catalan numbers.

We begin with an involution involving the numbers $C_{n,k}$, which will later be extended to the area generating functions $C_{n,k}(q)$. First, recall that $C_{n,k} = |\mathcal{D}_{n,k}|$, where

$\mathcal{D}_{n,k}$ is the set of Dyck paths of height n ending in exactly k horizontal steps. Let $\mathcal{D}'_{n,k}$ denote the set of Dyck paths of height n beginning with exactly k vertical steps. It is easy to see that $C_{n,k} = |\mathcal{D}'_{n,k}|$, since reflection about the line $y = n - x$ gives a bijection between $\mathcal{D}_{n,k}$ and $\mathcal{D}'_{n,k}$. Since this reflection preserves area, we also have

$$C_{n,k}(q) = \sum_{P \in \mathcal{D}_{n,k}} q^{area(P)} = \sum_{P \in \mathcal{D}'_{n,k}} q^{area(P)}.$$

Theorem 5.14. *For all $i, j \geq 1$, we have*

$$\chi(i = j)(-1)^i = \sum_{k \geq 1} (-1)^k \binom{k}{i-k,\, 2k-i} C_{k,j}. \tag{5.7}$$

Proof. Fix i and j. If $j > i$, it is easy to see that both sides of the given identity are zero, so we can assume $i \geq j$. We will interpret the right side of this identity as counting a collection S of signed objects. By definition, an object of S is a triple (k, P, w), where $k \geq 1$ is an integer, P is a path in $\mathcal{D}'_{k,j}$, and $w \in R(0^{2k-i}1^{i-k})$ is a rearrangement of $2k - i$ zeroes and $i - k$ ones. We may as well take $k \leq i$ here (lest w not exist), so S is finite with cardinality $\sum_{k=1}^{i} \binom{k}{i-k,\, 2k-i} C_{k,j}$. Next, define the *sign* of the object (k, P, w) to be $(-1)^k$. Then the sum of the signs of all objects in S is precisely the right side of (5.7).

We now define an involution $\phi : S \to S$ such that for all $x \in S$, either $\phi(x) = x$ or the sign of $\phi(x)$ is the opposite of the sign of x. Intuitively, ϕ pairs off objects with opposite signs, and ϕ may also have some unpaired *fixed points* of unspecified sign. Since the sign of x cancels the sign of $\phi(x)$ when $\phi(x) \neq x$, we have

$$\sum_{x \in S} sign(x) = \sum_{x \in S : x = \phi(x)} sign(x).$$

This will imply the desired result, since the left side of (5.7) will turn out to be the sum of the signs of the fixed points of ϕ.

Let us describe at the outset the fixed points of ϕ. If $i > j$, then ϕ will have no fixed points. If $i = j$, then ϕ will have one fixed point, which is the object $x_0 = (k_0, P_0, w_0)$ such that $k_0 = i = j$, P_0 is a path consisting of i north steps and i east steps, and $w = 0^i$. Note that x_0 does belong to S, and its sign is $(-1)^i$. So, whatever

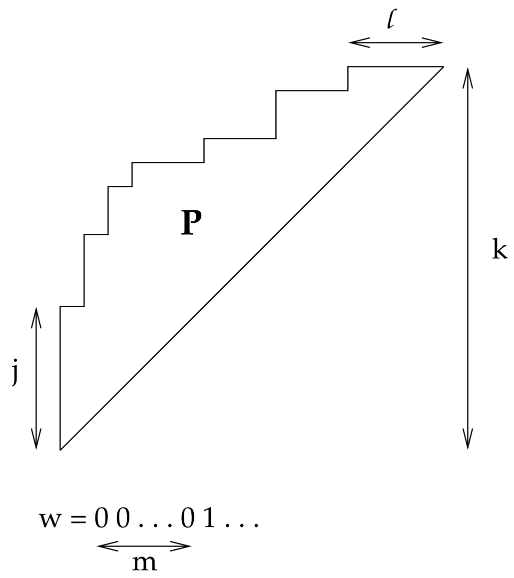


Figure 5.6: Quantities used to define the involution.

the values of i and j are,

$$\sum_{x \in S: x=\phi(x)} sign(x) = \chi(i=j)(-1)^i$$

as required.

To define ϕ in general, let $x = (k, P, w) \in S$ with $x \neq x_0$. Let $\ell \geq 1$ be the number of east steps at the end of the path P, so that P has $\ell - 1$ area cells in its top row. Let $m \geq 0$ be the number of zeroes at the beginning of the word w. See Figure 5.6. Consider two cases.

Case 1: $m \leq \ell$**.** Define $\phi(x) = (k', P', w') \in S$ as follows. Let $k' = k+1$. Let P' be P with a new top row that contains exactly m area cells. Let w' be w with the leftmost one replaced by two zeroes.

Case 2: $m > \ell$**.** Define $\phi(x) = (k', P', w') \in S$ as follows. Let $k' = k-1$. Let P' be P with its top row erased. (Thus, P' has $\ell - 1$ fewer area cells than P.) Let w' be w with the ℓ^{th} and $(\ell+1)^{\text{th}}$ zeroes replaced by a single one.

Since $k' = k \pm 1$, it is clear that ϕ is sign-reversing. However, we need to check that the definition of ϕ makes sense and that ϕ is an involution.

Consider the actions performed on $x = (k, P, w)$ in case 1.

- First, $k' = k + 1 \geq 1$ since $k \geq 1$.

- Second, we claim that P' does belong to $\mathcal{D}'_{k',j}$. Proof: Since $m \leq \ell$, there is enough room above the top row of P to put m area cells in the top row of P' with no overhanging cells. So P' is a Dyck path of height $k' = k + 1$. Does P' begin with exactly j vertical steps, as required? The only way this could fail is if the m new area cells in the top row of P' extend to the leftmost column of the diagram. This happens if and only if P consists of j vertical steps followed by j horizontal steps, and moreover $j = \ell = m = k$. Now, $m \leq 2k - i$, since m is the number of zeroes at the beginning of w. So, $i \leq 2k - m = 2j - j = j$. But we are already assuming $i \geq j$, so that $i = j$. Note that $w \in R(0^{2k-i}1^{i-k}) = R(0^k)$, so w must be 0^k. Finally, we see that $x = x_0$, the fixed point of ϕ, contrary to our assumption. Therefore, P' does begin with exactly j vertical steps.

- Third, we claim that w does have at least one 1 in it. For otherwise, we must have $i = k$, forcing $w = 0^k = 0^i$ and $m = k$. Since $P \in \mathcal{D}'_{k,j}$ and $\ell \geq m = k$, we must in fact have $\ell = k$ and P ends in k horizontal steps. The total height of P is k, so P must be the path consisting of k vertical steps followed by k horizontal steps. But then $j = k = i$, as P starts with j vertical steps, and we again reach the contradiction $x = x_0$.

- Fourth, we claim that $w' \in R(0^{2k'-i}1^{i-k'})$. By definition, w has $i - k$ ones and $2k - i$ zeroes. According to the definition of ϕ, w' has $i - k - 1 = i - k'$ ones and $2k - i + 2 = 2k' - i$ zeroes, as required.

- Fifth, let us compute $\phi(x') = \phi((k', P', w'))$. Let m' and ℓ' be computed from x' in the same way that m and ℓ were computed from x. We have $\ell' = m + 1$, since the top row of P' has m area cells. We have $m' > m + 1$, since w' must begin with at least $m + 2$ zeroes by definition of m. Thus $m' > \ell'$, so we follow the instructions in case 2 to compute $\phi(x') = (k'', P'', w'')$. We have $k'' = k' - 1 = k$. We obtain P'' by erasing the top row of P', which gives the original path P again. Finally, we obtain w'' by replacing the $(\ell')^{\text{th}}$ and $(\ell' + 1)^{\text{th}}$ zeroes in w' by a single one. But

$\ell' = m + 1$, so the two zeroes being replaced are exactly the ones that were added to w in case 1. Thus, $w'' = w$. Finally, we see that $\phi(x') = x$, as desired.

Next, we must check the analogous assertions for the actions performed in case 2. So assume that $x = (k, P, w)$ was such that $m > \ell$.

- First, we claim that $k' = k - 1$ is at least 1. Since $k \geq 1$, we need only rule out the possibility $k = 1$. Now, $k = 1$ forces $j = 1 = \ell$, and we are assuming $\ell < m$. Thus, w begins with $m > 1$ zeroes. But w has only $2k - i \leq 1$ zeroes total, which is a contradiction.

- Second, we claim that P' does belong to $\mathcal{D}'_{k',j}$. Proof: P' is clearly a Dyck path of height $k' = k - 1$. Does P' begin with exactly j vertical steps, as required? The only way this could fail is if the top row of P extends to the leftmost column of the diagram. This happens if and only if P consists of j vertical steps followed by j horizontal steps, and moreover $j = \ell = k$. Now, $m \leq 2k - i$, since m is the number of zeroes at the beginning of w. Also recall that $i \geq k$ since the number of ones in w cannot be negative. So, $m \leq 2k - i \leq 2k - k = k = \ell$, contradicting the fact that we are in case 2. Therefore, P' does begin with exactly j vertical steps.

- Third, note that the initial string of zeroes in w has length at least $\ell + 1$, by definition of m. So it makes sense to replace the ℓ^{th} and $(\ell+1)^{\text{th}}$ zeroes by a single one to obtain w'. Note that this new one is the leftmost one in w'.

- Fourth, we claim that $w' \in R(0^{2k'-i}1^{i-k'})$. By definition, w has $i - k$ ones and $2k - i$ zeroes. According to the definition of ϕ, w' has $i - k + 1 = i - k'$ ones and $2k - i - 2 = 2k' - i$ zeroes, as required.

- Fifth, let us compute $\phi(x') = \phi((k', P', w'))$. Let m' and ℓ' be computed from x' in the same way that m and ℓ were computed from x. We have $m' = \ell - 1$, by definition of w'. Also, $\ell' \geq \ell - 1$, lest the top row of P have overhanging area cells that are forbidden by the definition of Dyck path. So, $\ell' \geq m'$, and we use the directions in case 1 to compute $\phi(x') = (k'', P'', w'')$. We have $k'' = k' + 1 = k$. We obtain P'' by adding a new top row with $m' = \ell - 1$ area cells to P', which gives the original path P again. Finally, we obtain w'' by replacing the leftmost one in

w' by two zeroes. This exactly reverses the transformation used to create w' from w, so $w'' = w$. Finally, we see that $\phi(x') = x$, as desired.

□

Identity (5.7) can be interpreted as a matrix inversion result.

Corollary 5.15. *Let $M \geq 1$. Consider the lower triangular matrices*

$$C = ||C_{n,k}||_{1\leq n,k\leq M} \text{ and } B = ||b_{k,j}||_{1\leq k,j\leq M}, \text{ where } b_{k,j} = (-1)^{k-j}\binom{j}{k-j}.$$

B and C are inverse matrices, i.e.,

$$CB = BC = I = ||\chi(i=j)||_{1\leq i,j\leq M}.$$

Proof. Writing out what the identity $BC = I$ means entry by entry, we get

$$\chi(i=j) = \sum_{k=1}^{i}(-1)^{i-k}\binom{k}{i-k}C_{k,j} \text{ for } 1 \leq i,j \leq M.$$

This is just a rearrangement of (5.7). It follows that $CB = I$ as well, which says that

$$\sum_{k=1}^{j} C_{i,k}(-1)^{k-j}\binom{j}{k-j} = \chi(i=j) \text{ for } 1 \leq i,j \leq M.$$

The reader may enjoy giving a direct algebraic derivation of $CB = I$ by induction on i, using (5.2) (with $q = 1$) and the identity

$$\sum_{u=0}^{m-j+1}(-1)^u\binom{j}{u} = (-1)^{m-(j-1)}\binom{j-1}{m-(j-1)} \tag{5.8}$$

as intermediate steps. □

We now give q-analogues of the last two results.

Theorem 5.16. *For each $i, j \geq 1$, we have*

$$\chi(i=j)(-1)^i q^{-i(i-1)/2} = \sum_{k\geq 1}(-1)^k \begin{bmatrix} k \\ i-k,\, 2k-i \end{bmatrix}_q C_{k,j}(q)q^{k(k+1)-2ki}. \tag{5.9}$$

Proof. We use the set of signed objects S and the sign-reversing involution ϕ from the proof of Theorem 5.14. We will assign *weights* to these objects in such a way that ϕ is weight-preserving. This means that $weight(\phi(x)) = weight(x)$ for all $x \in S$. It follows that

$$\sum_{x\in S} sign(x)q^{weight(x)} = \sum_{x\in S: x=\phi(x)} sign(x)q^{weight(x)}. \tag{5.10}$$

For $x = (k, P, w) \in S$, define the weight of x by

$$weight((k, P, w)) = area(P) + coinv(w) + k(k + 1) - 2ki.$$

Let us check that ϕ is weight-preserving. First, using the notation in the proof of Theorem 5.14, assume $x \neq x_0$ and that $m \leq \ell$. Following the instructions in case 1, we see that $area(P') = area(P) + m$, since the new path P' has m additional area cells in its top row. Note that the leftmost one in w is preceded by m zeroes, so deleting this one will decrease the coinversion count by m. However, the two zeroes that replace this one will increase the coinversion count by $2(i-k-1)$, since there are $i-k-1$ ones following the position where these zeroes are inserted. Thus, $coinv(w') = coinv(w) - m + 2(i-k-1)$. Finally, $k' = k + 1$ in this case. Hence,

$$\begin{aligned} weight(\phi(x)) &= [area(P) + m] + [coinv(w) - m + 2i - 2k - 2] \\ &\quad +[(k + 1)(k + 2) - 2(k + 1)i] \\ &= area(P) + coinv(w) + k(k + 1) - 2ki \\ &= weight(x). \end{aligned}$$

Next, assume that $x \neq x_0$ and that $m > \ell$. Following the instructions in case 2, we see that $area(P') = area(P) - (\ell - 1)$, since the new path P' does not include the $\ell - 1$ area cells in the top row of P. Arguing as above, removing the two zeroes from w at positions ℓ and $\ell + 1$ (which occur before the leftmost one in w) will decrease the coinversion count by $2(i - k)$. Putting a one in place of these zeroes increases the coinversion count by $\ell - 1$. Thus, $coinv(w') = coinv(w) - 2(i - k) + (\ell - 1)$. Finally,

$k' = k - 1$ in this case. Hence,

$$\begin{aligned} weight(\phi(x)) &= [area(P) - (\ell - 1)] + [coinv(w) - 2i + 2k + (\ell - 1)] \\ &\quad +[(k-1)k - 2(k-1)i] \\ &= area(P) + coinv(w) + k(k+1) - 2ki \\ &= weight(x). \end{aligned}$$

So, ϕ is weight-preserving.

Finally, let us compute each side of (5.10). Fix $k \geq 1$. To construct an object $(k, P, w) \in S$, we choose any path $P \in \mathcal{D}'_{k,j}$ and any word $w \in R(0^{2k-i}1^{i-k})$. The generating functions for these choices, relative to *area* and *coinv* respectively, are $C_{k,j}(q)$ and $\left[\begin{smallmatrix} k \\ 2k-i, i-k \end{smallmatrix}\right]_q$ (see Chapter 1). Adding over all k, we see that

$$\sum_{x \in S} sign(x) q^{weight(x)} = \sum_{k \geq 1} (-1)^k \begin{bmatrix} k \\ i-k, 2k-i \end{bmatrix}_q C_{k,j}(q) q^{k(k+1)-2ki}. \tag{5.11}$$

On the other hand, ϕ has no fixed points if $i \neq j$. If $i = j$, one easily checks that

$$weight(x_0) = i(i-1)/2 + 0 + i(i+1) - 2i^2 = -i(i-1)/2,$$

where x_0 is the unique fixed point of ϕ. Therefore,

$$\sum_{x \in S: \phi(x) = x} sign(x) q^{weight(x)} = \chi(i = j)(-1)^i q^{-i(i-1)/2}. \tag{5.12}$$

Combining these calculations with (5.10), the theorem follows. □

This last result can also be phrased in terms of matrices.

Corollary 5.17. *Consider the lower triangular matrices* $C_q = ||C_{i,j}(q)||_{1 \leq i,j \leq M}$ *and* $B_q = ||b_{i,j}(q)||_{1 \leq i,j \leq M}$, *where*

$$b_{i,j}(q) = (-1)^{j-i} \begin{bmatrix} j \\ i-j, 2j-i \end{bmatrix}_q q^{j(j+1)-2ji+i(i-1)/2}.$$

Then B_q *and* C_q *are inverse matrices, i.e.,*

$$C_q B_q = B_q C_q = I = ||\chi(i = j)||_{1 \leq i,j \leq M}.$$

Proof. Writing out what $B_qC_q = I$ means, entry by entry, we get

$$\sum_{k=1}^{i}(-1)^{k-i}\begin{bmatrix} k \\ i-k, 2k-i \end{bmatrix}_q q^{k(k+1)-2ki+i(i-1)/2}C_{k,j}(q) = \chi(i=j).$$

But this is just a rearrangement of (5.9). □

We now obtain the promised determinantal formula for the Carlitz-Riordan q-analogues of the Catalan numbers.

Theorem 5.18. *For all $n \geq 1$, we have*

$$C_n(q) = det\left|\left|(-1)^{m-k}q^{(m-k)^2+(m-k)}\begin{bmatrix} k \\ m+1-k \end{bmatrix}_q\right|\right|_{1\leq m,k\leq n}.$$

Proof. Let $M = n+1$ in the previous corollary, so that B_q and C_q are $(n+1)\times(n+1)$ matrices. On the one hand, the $n+1, 1$-entry of the inverse of B_q is the $n+1, 1$-entry of C_q, which is $C_{n+1,1}(q) = C_n(q)$. On the other hand, the adjoint formula for the $n+1, 1$-entry of the inverse of B_q yields

$$C_n(q) = \frac{(-1)^n\det||b_{m+1,k}(q)||_{1\leq m,k\leq n}}{\det||b_{m,k}(q)||_{1\leq m,k\leq n+1}},$$

where

$$b_{m+1,k}(q) = (-1)^{m+1-k}q^{(k+1)k+(m+1)m/2-2k(m+1)}\begin{bmatrix} k \\ m+1-k \end{bmatrix}_q.$$

The lower triangular matrix $||b_{m,k}(q)||_{1\leq m,k\leq n+1}$ has determinant

$$\det||b_{m,k}(q)||_{1\leq m,k\leq n+1} = \prod_{m=1}^{n+1} q^{m^2+m+(m^2-m)/2-2m^2} = \prod_{m=1}^{n+1} q^{-m(m-1)/2}.$$

Using these relations and doing routine simplifications of the determinants, we arrive at the stated formula for $C_n(q)$. □

Of course, there are similar formulas for $C_{n,k}(q)$ with $k > 1$.

Next, we discuss lattice paths contained in more general shapes. Fix i and j with $i \geq j$, and fix integers $n_1, \ldots, n_i \geq 0$. Set $n_{i+1} = 0$, and set $N_s = n_1 + \cdots + n_s$ for $0 \leq s \leq i+1$. Define a shape Λ whose successive rows, counting from the bottom, have N_s cells. See Figure 5.7 for an example.

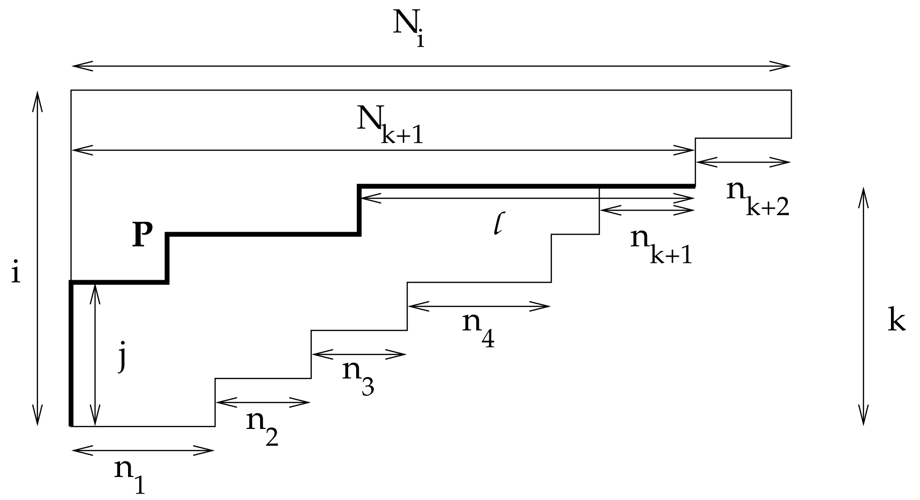


Figure 5.7: A lattice path inside a general shape.

Consider lattice paths starting at $(0,0)$ consisting of a series of north and east steps that never go strictly outside the shape Λ. For $j \leq k \leq i$ define $\mathcal{E}_{k,j}$ to be the set of all such paths ending at (N_{k+1}, k) that begin with exactly j north steps. Also define

$$E_{k,j}(q) = \sum_{P \in \mathcal{E}_{k,j}} q^{area(P)},$$

where $area(P)$ is the number of cells right of the path P contained in the shape Λ.

Theorem 5.19.

$$\chi(i=j)(-1)^i q^{|\Lambda|+\sum_{t=1}^{i}(t-i)(n_{t+1}+1)} = \sum_{k=j}^{i} (-1)^k \begin{bmatrix} N_{k+1} \\ i-k \end{bmatrix}_q q^{\sum_{t=1}^{k}(t-i)(n_{t+1}+1)} E_{k,j}(q). \tag{5.13}$$

Proof. We define a sign-reversing, weight-preserving involution on a suitable collection S of objects. Define an object of S to be a triple (k,P,w), where $j \leq k \leq i$, $P \in \mathcal{E}_{k,j}$, and $w \in R(0^{N_{k+1}+k-i}1^{i-k})$. Define the *sign* of the object (k,P,w) to be $(-1)^k$. Define the *weight* of the object (k,P,w) by

$$weight((k,P,w)) = area(P) + coinv(w) + \sum_{t=1}^{k}(t-i)(n_{t+1}+1).$$

The generating function for all the objects in S, taking into account the signs and weights, is precisely the right side of (5.13).

Next, we define the involution ϕ. Let us describe the fixed points first. If $i \neq j$, then ϕ has no fixed points. If $i = j$, then ϕ will have one fixed point $x_0 = (i, p_0, w_0)$, where p_0 is the path that goes up i steps and then right $N_i = N_{i+1}$ steps, and w_0 consists entirely of zeroes. Note that $sign(x_0) = (-1)^i$ and $weight(x_0) = |\Lambda| + \sum_{t=1}^{i}(t-i)(n_{t+1}+1)$. Hence, the generating function for the fixed points of ϕ is precisely the left side of (5.13).

Now, we describe the action of ϕ on elements $x \neq x_0$ in S. Suppose $x = (k, P, w) \neq x_0$. Let ℓ be the number of horizontal steps at the end of P, so that $n_{k+1} \leq \ell \leq N_{k+1}$. See Figure 5.7. Let m be the number of zeroes at the beginning of w, so that $0 \leq m \leq N_{k+1} + k - i$. We construct $\phi(x) = (k', P', w')$ according to the following rules.

Case 1: $m \leq \ell$**.** Let $k' = k+1$. Form P' from P by adding a new top row with exactly m (right-justified) area cells. Form w' from w by replacing the leftmost one in w with $n_{k+2}+1$ zeroes.

Case 2: $m > \ell$**.** Let $k' = k-1$. Form P' from P by erasing the top row, which has $\ell - n_{k+1}$ area cells. Form w' from w by replacing the $n_{k+1}+1$ zeroes at positions $\ell+1$, $\ell+1-1$, $\cdots$, $\ell+1-n_{k+1}$ with a single one. Note that these deleted zeroes all occur in the initial string of zeroes, since $m > \ell$. Also, there are enough zeroes to delete, since $\ell \geq n_{k+1}$.

It is easy to check that the rules given in the two cases above make sense and always produce a well-defined element of S. For instance, we would encounter problems in case 1 if the given object $x = (k, P, w)$ had $k = i$ or (equivalently) if w contained no one. But, in this situation, the definitions of m and ℓ imply that $m = N_{k+1} \leq \ell \leq N_{k+1}$, so that $m = \ell = N_{k+1}$. This forces $j = k = i$ and $P = P_0$ and $w = w_0$, so that $x = x_0$, contrary to our assumption. Another possible problem in case 1 occurs if the new path P' does not start with exactly j vertical steps. This happens if and only if $m = \ell = N_{k+1}$, from which we can deduce that $k = j$, $k = i$ (lest w have fewer than m zeroes), $P = P_0$, $w = w_0$, and finally $x = x_0$. Problems occur in case 2 if $k = j$ or (equivalently) if the new path P' does not start with exactly j vertical steps. But in this situation,

$$\ell = N_{k+1} < m \leq N_{k+1} + k - i \leq N_{k+1} = \ell,$$

which is a contradiction.

It is clear that ϕ is sign-reversing. To check that ϕ is an involution, consider what happens in each case. Let ℓ' be the number of horizontal steps at the end of P', and let m' be the number of zeroes at the beginning of w'. If we were originally in case 1 (where $m \leq \ell$), note that

$$m' \geq m + n_{k+2} + 1 = \ell' + 1 > \ell',$$

so we use the rules in case 2 to compute $\phi((k', P', w'))$. It is easy to see that the actions here just reverse the actions performed in case 1. Similarly, suppose we were originally in case 2 (where $m > \ell$). Then $m' = \ell - n_{k+1}$, and inspection of Figure 5.7 shows that $\ell' \geq \ell - n_{k+1} = m'$, so we use the rules in case 1 to compute $\phi((k', P', w'))$. Again, these rules just reverse the actions performed in case 2.

Finally, we check that ϕ is weight-preserving. In case 1, the weight of $\phi(x)$ is easily seen to be

$$area(P) + m + coinv(w) - m + (n_{k+2} + 1)(i - (k+1)) + \sum_{t=1}^{k+1} (t - i)(n_{t+1} + 1)$$

by comparing $coinv(w)$ to $coinv(w')$ as in the proof of Theorem 5.16. This expression simplifies to

$$area(P) + coinv(w) + \sum_{t=1}^{k} (t - i)(n_{t+1} + 1) = weight(x).$$

Similarly, in case 2, the weight of $\phi(x)$ is

$$area(P) - (\ell - n_{k+1}) + coinv(w) + (\ell - n_{k+1}) - (n_{k+1} + 1)(i - k) + \sum_{t=1}^{k-1} (t - i)(n_{t+1} + 1),$$

which again simplifies to

$$area(P) + coinv(w) + \sum_{t=1}^{k} (t - i)(n_{t+1} + 1) = weight(x).$$

The theorem now follows from the existence of the sign-reversing, weight-preserving involution ϕ. $\square$

Of course, this result can also be rephrased in terms of matrices, and one can solve for the quantities $E_{k,j}(q)$ using Cramer's Rule. We leave these routine tasks to the interested reader.

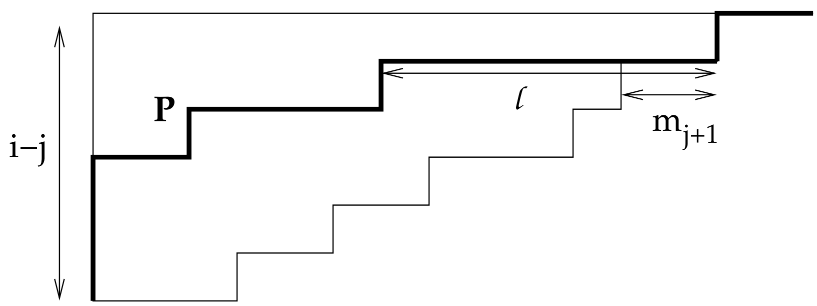


Figure 5.8: Setup for Theorem 5.20.

We end this subsection with a variant of the preceding involution. Let a shape Λ be constructed from given numbers $n_1, \ldots, n_{i+1}$ as in Figure 5.7. For convenience, set $m_s = n_{i+1-s}$ and $M_s = m_1 + \cdots + m_s$ for $0 \le s \le i$. Let $\mathcal{F}^{\Lambda}_{n,k}$ denote the set of lattice paths contained in Λ that go from $(0,0)$ to (N_{n+1}, n) whose top row contains exactly k area cells, for $k \ge 0$. Note that $\mathcal{F}^{\Lambda}_{n,k}$ is empty if $k > N_n$. Let

$$F_{n,k}(q) = \sum_{P \in \mathcal{F}_{n,k}} q^{area(P)}.$$

Theorem 5.20. *For $0 \le k \le N_i + i - 1$, we have*

$$F^{\Lambda}_{i,k}(q)q^{-k} = \sum_{j=0}^{i-1} (-1)^j q^{\sum_{s=1}^{j}(m_s+1)(s-1)} \begin{bmatrix} k - M_j \\ j \end{bmatrix}_q F^{\Lambda}_{i-j,0}(q).$$

Proof. Fix i, k, and Λ. Define a set S of objects consisting of all triples (j, P, w), where $0 \le j \le i-1$, $P \in \mathcal{F}^{\Lambda}_{i-j,0}$, and $w \in R(1^j 0^{k-M_j-j})$. See Figure 5.8. Define the sign of (j, P, w) to be $(-1)^j$, and define

$$weight((j, P, w)) = area(P) + coinv(w) + \sum_{s=1}^{j} (m_s + 1)(s-1).$$

The right side of the equation in the theorem is the generating function for S.

Define an involution ϕ on S as follows. First, the fixed points of ϕ consist of all objects $x = (0, P, 0^k)$ such that $P \in \mathcal{F}^{\Lambda}_{i,0}$ and the last vertical step of P is preceded by exactly $\ell \ge k$ horizontal steps. By adding k area cells to the top row of P, we obtain a bijection between the set of all such objects x and the set $\mathcal{F}^{\Lambda}_{i,k}$, such that the weight of x is k less than the area of the corresponding path in $\mathcal{F}_{i,k}^{\Lambda}$. Therefore, the left side of the equation in the theorem is the generating function for the set of fixed points of ϕ. In the special case when $k > N_i$, note that ϕ has no fixed points, and the left side of the equation in the theorem is zero in this case.

Now, let us define ϕ for a non-fixed point $x = (j, P, w)$. Let ℓ be the number of east steps in P preceding the last vertical step of P, so that P has $\ell - m_{j+1}$ area cells in the second row from the top. Let m be the number of initial zeroes in w.

Case 1: $m > \ell$**.** Let $j' = j + 1$. Form P' from P by removing the empty top row of P and erasing all $\ell - m_{j+1}$ area cells in the second row of P. Form w' from w by replacing the $m_{j+1} + 1$ zeroes at positions $\ell + 1, \ldots, \ell + 1 - m_{j+1}$ in w by a single one.

Case 2: $m \leq \ell$**.** Let $j' = j - 1$. Form P' from P by putting m new area cells in the empty top row of P, and then adding a new empty top row above it. Replace the leftmost one in w with $m_j + 1$ zeroes.

The usual computations, which we leave to the reader, show that ϕ is a well-defined sign-reversing, weight-preserving involution. To see how the choice of k dictates the fixed points of ϕ, note that the rules in case 2 make no sense if w does not contain a 1. This happens when $j = 0$ and $w = 0^k$. So, $x = (0, P, 0^k)$ for some path $P \in \mathcal{F}_{i,0}^{\Lambda}$. Since we are in case 2, $k = m \leq \ell$, which shows that the number of horizontal steps in P preceding the last vertical step must be at least k. We have arrived at the description of the fixed points of ϕ given above. Next, let us see why we require $k \leq N_i + i - 1$. Consider an object $x = (j, P, w)$ with $j = i - 1$. For such an object, P must be a path of height 1 with no area cells, so that $\ell = m_i$. We need to rule out the possibility $m > \ell$, which would cause us to set $j' = j + 1 = i$, a forbidden value. Now, $m \leq k - M_{i-1} - (i - 1)$, and equality can occur if all zeroes in w occur at the beginning. Thus, we require that

$$k - M_{i-1} - (i - 1) \leq m_i,$$

which is equivalent to $k \leq M_i + i - 1 = N_i + i - 1$. $\square$

Corollary 5.21. *(1) Let $m_0 = m_i = 0$ and $m_j = m$ for $0 < j < i$, so that Λ is an m-staircase. Then*

$$F_{i,k}^{\Lambda}(q)q^{-k} = \sum_{j=0}^{i-1} (-1)^j q^{(m+1)\binom{j}{2}} \begin{bmatrix} k - mj \\ j \end{bmatrix}_q F_{i-j,0}^{\Lambda}(q) \textit{ for } 0 \leq k \leq (m+1)(i-1).$$

(2) For $i \geq 1$ and $\ell \geq 1$,

$$q^{i\ell-\ell} = \sum_{j=0}^{i-1} (-1)^j q^{\binom{j}{2}} \begin{bmatrix} \ell \\ j \end{bmatrix}_q \begin{bmatrix} i-j-1+\ell \\ i-j-1, \ell \end{bmatrix}_q.$$

Proof. Equation (1) follows immediately by substituting $m_j = m$ into the previous theorem.

To get equation (2), let $m_i = \ell$ and let all other m_j's be zero. Then Λ is a rectangle of width ℓ and height i. Choosing $k = \ell$, we have $F^{\Lambda}_{i,\ell}(q) = q^{i\ell}$. Also, $F^{\Lambda}_{i-j,0}(q)$ is the generating function for lattice paths in a rectangle of width ℓ and height $i-j-1$ (since the top row must be empty), which is $\left[\begin{smallmatrix} i-j-1+\ell \\ i-j-1, \ell \end{smallmatrix}\right]_q$. □

Remark 5.22. Suppose we let $m = 1$ in equation (1). Setting $i = n$, replacing k by $k-1$, and noting that $C_{a,b}(q) = F^{\Lambda}_{a,b-1}(q)$, we obtain

$$C_{n,k}(q)q^{-(k-1)} = \sum_{j=0}^{n-1} (-1)^j q^{j(j-1)} \begin{bmatrix} k-1-j \\ j \end{bmatrix}_q C_{n-j,1}(q) \text{ for } 0 \leq k-1 \leq 2n-2.$$

This is just a rearrangement of (5.6) from Proposition 5.11. Thus, we have now given a combinatorial proof of that identity.

Example 5.23. Letting $\ell = 4$ and $i = 5$ in (2), we get the identity

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix}_q \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q \begin{bmatrix} 7 \\ 4 \end{bmatrix}_q + q \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \begin{bmatrix} 6 \\ 4 \end{bmatrix}_q - q^3 \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \begin{bmatrix} 5 \\ 4 \end{bmatrix}_q + q^6 \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q = q^{16}.$$

5.2.4 Counting Dyck Paths by Major Index

We conclude this section by presenting a recursion similar to (5.2) that keeps track of the major index of paths in $\mathcal{D}_{n,k}$. Recall that any $P \in \mathcal{D}_n$ can be encoded as a string of n zeroes and n ones by replacing vertical steps by zeroes and horizontal steps by ones. Let $w(P)$ denote the word encoding P. A path $P \in \mathcal{D}_n$ belongs to the subcollection $P \in \mathcal{D}_{n,k}$ if and only if $w(P)$ ends in 01^k. We define the *major index of P* by setting $maj(P) = maj(w(P))$, which is the sum of the positions i in $w(P)$ where a 1 is followed by a 0. We define

$$M_{n,k}(q) = \sum_{P \in \mathcal{D}_{n,k}} q^{maj(P)}.$$

Theorem 5.24. *For $1 \leq k < n$, we have*

$$M_{n,k}(q) = M_{n-1,k-1}(q) + qM_{n,k+1}(q) + M_{n-1,k}(q) \cdot (q^{2n-k-1} - q). \tag{5.14}$$

The initial condition is

$$M_{n,n}(q) = 1.$$

Proof. The initial condition is clear, since the only path in $\mathcal{D}_{n,n}$ corresponds to the word $0^n 1^n$, which has no descents. To derive the recursion, we introduce some temporary notation. Let

$$\mathcal{A}_{n,k} = \{P \in \mathcal{D}_{n,k} : w(P) = v001^k \text{ for some } v\},$$

$$\mathcal{B}_{n,k} = \{P \in \mathcal{D}_{n,k} : w(P) = v1101^k \text{ for some } v\},$$

$$\mathcal{H}_{n,k} = \{P \in \mathcal{D}_{n,k} : w(P) = v0101^k \text{ for some } v\},$$

$$A_{n,k}(q) = \sum_{P \in \mathcal{A}_{n,k}} q^{maj(P)},$$

$$B_{n,k}(q) = \sum_{P \in \mathcal{B}_{n,k}} q^{maj(P)},$$

$$H_{n,k}(q) = \sum_{P \in \mathcal{H}_{n,k}} q^{maj(P)}.$$

Evidently, each $\mathcal{D}_{n,k}$ can be written as a disjoint union

$$\mathcal{D}_{n,k} = \mathcal{A}_{n,k} \cup \mathcal{B}_{n,k} \cup \mathcal{H}_{n,k},$$

where some of these sets may be empty. We therefore have

$$M_{n,k}(q) = A_{n,k}(q) + B_{n,k}(q) + H_{n,k}(q). \tag{5.15}$$

We claim that there is a weight-preserving bijection $\alpha : \mathcal{A}_{n,k} \to \mathcal{D}_{n-1,k-1}$. If $n \leq 1$ or $k \leq 1$, both sets mentioned are empty and the result is trivial. Now assume $n > 1$ and $k > 1$. If $P \in \mathcal{A}_{n,k}$ has word $w(P) = v001^k$, we define $\alpha(P)$ to be the path P' whose word is $v01^{k-1}$. It is easy to see that this path does belong to $\mathcal{D}_{n-1,k-1}$, and $maj(P) = maj(\alpha(P))$. Furthermore, α is a bijection: the inverse map α^{-1} simply replaces the word $v01^{k-1}$ of a path $P' \in \mathcal{D}_{n-1,k-1}$ by $v001^k$. It follows that

$$A_{n,k}(q) = M_{n-1,k-1}(q). \tag{5.16}$$

Next, we define a bijection $\beta : \mathcal{B}_{n,k} \to \mathcal{D}_{n,k+1} - \mathcal{A}_{n,k+1}$ as follows. If $P \in \mathcal{B}_{n,k}$ has word $w(P) = v1101^k$, let $\beta(P)$ be the path whose word is $v1011^k = v101^{k+1}$. It is immediate that $\beta(P)$ does lie in $\mathcal{D}_{n,k+1} - \mathcal{A}_{n,k+1}$ and that β maps $\mathcal{B}_{n,k}$ one-to-one onto this set. Furthermore,

$$maj(\beta(P)) = maj(P) - 1,$$

since applying β causes the last descent to move one position to the left. Expressing this fact in terms of generating functions and using (5.16), we see that

$$B_{n,k}(q) = q(M_{n,k+1}(q) - A_{n,k+1}(q)) = qM_{n,k+1}(q) - qM_{n-1,k}(q). \tag{5.17}$$

Finally, we define a bijection $\gamma : \mathcal{H}_{n,k} \to \mathcal{D}_{n-1,k}$ as follows. If $P \in \mathcal{H}_{n,k}$ has word $w(P) = v0101^k$, let $\gamma(P)$ be the path whose word is $w' = v01^k$. Again, it is easy to check that γ maps $\mathcal{H}_{n,k}$ one-to-one onto $\mathcal{D}_{n-1,k}$. Note that the only difference between the descent set of $w(P)$ and the descent set of w' is that $w(P)$ has an extra descent just before its last zero. The position of this descent is $2n - k - 1$, and so

$$maj(P) = maj(\gamma(P)) + 2n - k - 1.$$

Since γ is a bijection, we conclude that

$$H_{n,k}(q) = q^{2n-k-1}M_{n-1,k}(q). \tag{5.18}$$

Putting (5.16), (5.17), and (5.18) into (5.15) and rearranging, we obtain the desired recursion. $\square$

Using this recursion, one can prove by induction that

$$M_{n,k}(q) = q^{n-k}\left(\begin{bmatrix} 2n-k-1 \\ n-k, n-1 \end{bmatrix}_q - \begin{bmatrix} 2n-k-1 \\ n-k-1, n \end{bmatrix}_q\right) \text{ for } k < n. \tag{5.19}$$

The proof is a dreary manipulation of q-binomial coefficients making heavy use of Theorem 1.51(3) and (4). We leave this manipulation to the interested reader. An elegant combinatorial proof of formula (5.19) appears in [13].

Finally, we observe that $C_n^{maj}(q) = \sum_{D \in \mathcal{D}_n} q^{maj(D)}$ can be recovered from the quantities $M_{n,k}(q)$ using the identity

$$M_{n+1,1}(q) = q^{2n}C_n^{maj}(q),$$

which follows easily from the definitions. Using (5.19) in this equation and simplifying, one can derive MacMahon's identity [25]

$$C_n^{maj}(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_q,$$

which was mentioned in Chapter 1.

5.3 Permutation Statistics and Catalan Numbers

The Catalan numbers occur ubiquitously in combinatorics. R. Stanley's book *Enumerative Combinatorics* and its addendum [29, 30] list over 95 collections of objects counted by the Catalan numbers. This section augments this list with two additional collections of permutations that are enumerated by the Catalan numbers. Furthermore, we show that the generating function for either collection, relative to the classical coinversion and major index statistics, is precisely the q,t-Catalan sequence $OC_n(q,t)$ discussed in Chapter 1. This is proved by exhibiting weight-preserving bijections between the given collections and the set of Dyck paths.

Recall from §1.4.1 the following three statistics on the collection $\mathcal{D}_n$ of Dyck paths of order n.

1. The *area* statistic, denoted $a(D)$ in this section, is the number of lattice cells between the path D and the line $y = x$.
2. The *bounce* statistic, denoted $b(D)$ in this section, is the sum of the x-coordinates (excluding n) where the bounce path of D hits the line $y = x$. As in Chapter 1, the bounce paths in this section will go from (n,n) to $(0,0)$.
3. The *number of bounces*, denoted $c(D)$ in this section, is the number of times the bounce path of D touches the line $y = x$ strictly between (n,n) and $(0,0)$.

Consider the trivariate generating function

$$C_n(q,t,z) = \sum_{P \in \mathcal{D}_n} q^{a(P)} t^{b(P)} z^{c(P)}.$$

We will show that this generating function coincides with the generating function for the triple of permutation statistics $(coinv, maj, des)$ on two special subcollections of S_n.

5.3.1 Special Permutations

This subsection discusses the two special collections of permutations that are counted by the Catalan numbers. The definition of these collections involves the factorization of a permutation into ascending runs.

Definition 5.25. Let $\sigma = \sigma_1 \cdots \sigma_n$ be a permutation of $\{1, 2, \ldots, n\}$. Let $i_1, i_2, \ldots, i_s$ be the set of all indices $i < n$ such that $\sigma_i > \sigma_{i+1}$, where $i_1 < i_2 < \ldots < i_s$.

(1) The $s + 1$ lists of contiguous elements

$$R_{s+1} = \sigma_1\sigma_2 \cdots \sigma_{i_1}; \quad R_s = \sigma_{i_1+1} \cdots \sigma_{i_2}; \quad \cdots; \quad R_1 = \sigma_{i_s+1} \cdots \sigma_n$$

are called the *ascending runs* of σ. We have labelled these runs $R_1, \ldots, R_{s+1}$ *from right to left.* Let n_i be the number of symbols in R_i. Let $N_0 = 0$ and $N_i = n_1 + \cdots + n_i$ for $i > 0$.

(2) Let m_i be the smallest (leftmost) entry in ascending run R_i, and let M_i be the largest (rightmost) entry in R_i. We call m_i and M_i the *minimum* and *maximum* of run R_i, respectively.

Example 5.26. Suppose $\sigma = 4, 7, 1, 5, 8, 3, 2, 6$. Then

$$\begin{array}{lllll} R_1 = 2, 6; & m_1 = 2; & M_1 = 6; & n_1 = 2; & N_1 = 2. \\ R_2 = 3; & m_2 = 3; & M_2 = 3; & n_2 = 1; & N_2 = 3. \\ R_3 = 1, 5, 8; & m_3 = 1; & M_3 = 8; & n_3 = 3; & N_3 = 6. \\ R_4 = 4, 7; & m_4 = 4; & M_4 = 7; & n_4 = 2; & N_4 = 8. \end{array}$$

Definition 5.27. Fix a permutation σ of $\{1, 2, \ldots, n\}$. Assume σ has s ascending runs. For $i > s$, set $n_i = 0$ and $N_i = N_s = n$.

(1) We say σ has the *decreasing-minimum property* if and only if

$$m_s > m_{s-1} > \cdots > m_2 > m_1.$$

(In particular, m_1, the smallest entry in the rightmost ascending run of σ, must equal 1 in this case.)

(2) We say σ has the *bounded-maximum property* if and only if

$$M_i \leq N_{i+1} \text{ for } 1 \leq i < s.$$

In words, the maximum element in the i^{th} run from the right is no larger than the total length of the rightmost $i+1$ runs.

(3) We say σ has the *bounded-minimum property* if and only if

$$m_{i+1} > N_{i-1} + 1 \text{ for } 1 < i < s.$$

In words, the minimum element in the $(i+1)^{\text{th}}$ run from the right is strictly larger than the total length of the rightmost $i-1$ runs plus one.

(4) Define the *first special collection of permutations* to be the set $\mathcal{U}_n$ of permutations of $\{1, 2, \ldots, n\}$ that have the decreasing-minimum property and the bounded-maximum property.

(5) Define the *second special collection of permutations* to be the set $\mathcal{V}_n$ of permutations of $\{1, 2, \ldots, n\}$ that have the decreasing-minimum property and the bounded-minimum property.

We will show that $|\mathcal{U}_n| = C_n = |\mathcal{V}_n|$ for all n. The proof gives a bijection between each collection and the collection of Dyck paths of order n.

Example 5.28. Consider the following permutations of $\{1, 2, \ldots, 9\}$.

(1) The permutation $\alpha = 4, 6, 9, 2, 8, 1, 3, 5, 7$ has the decreasing-minimum property, but not the bounded-maximum or bounded-minimum property.

(2) The permutation $\beta = 7, 9, 5, 8, 1, 2, 3, 4, 6$ has the decreasing-minimum property, the bounded-maximum property, and the bounded-minimum property.

(3) The permutation $\gamma = 3, 4, 2, 8, 9, 1, 5, 6, 7$ has the decreasing-minimum property and the bounded-maximum property, but not the bounded-minimum property.

(4) The permutation $\delta = 6, 7, 2, 3, 4, 1, 5, 8, 9$ has the decreasing-minimum property and the bounded-minimum property, but not the bounded-maximum property.

5.3.2 Statistics on Permutations

This subsection reviews the definitions of some classical permutation statistics. If A is any logical statement, we set $\chi(A) = 1$ if A is true, and $\chi(A) = 0$ if A is false. It is convenient to define these statistics for lists of distinct integers that are not necessarily permutations of $\{1, 2, \ldots, n\}$.

Definition 5.29. Let $w = w_1 w_2 \ldots w_n$ be a sequence of n distinct integers w_i.

(1) The *coinversion count* of w is defined by

$$coinv(w) = \sum_{1 \le i < j \le n} \chi(w_i < w_j).$$

(2) The *descent set* of w is defined by

$$Des(w) = \{i : 1 \le i < n \text{ and } w_i > w_{i+1}\}.$$

(3) The *descent count* of w is defined by

$$des(w) = |Des(w)| = \sum_{i=1}^{n-1} \chi(w_i > w_{i+1}).$$

(4) The *major index* of w is defined by

$$maj(w) = \sum_{i \in Des(w)} i = \sum_{i=1}^{n-1} i\chi(w_i > w_{i+1}).$$

Example 5.30. Let $w = 4, 7, 1, 2, 9, 8, 11$. Then

$$\begin{aligned} coinv(w) &= 4 + 3 + 4 + 3 + 1 + 1 + 0 = 16 \\ Des(w) &= \{2, 5\} \\ des(w) &= 2 \\ maj(w) &= 7. \end{aligned}$$

Definition 5.31. Define two trivariate generating functions

$$U_n(q, t, z) = \sum_{\sigma \in \mathcal{U}_n} q^{coinv(\sigma)} t^{maj(\sigma)} z^{des(\sigma)},$$

$$V_n(q, t, z) = \sum_{\sigma \in \mathcal{V}_n} q^{coinv(\sigma)} t^{maj(\sigma)} z^{des(\sigma)}.$$

5.3.3 Pictures of Permutation Statistics

We will show that each of the special collections of permutations $\mathcal{U}_n$ and $\mathcal{V}_n$ has size C_n, the Catalan number. This follows from the trivariate identity

$$U_n(q,t,z) = C_n(q,t,z) = V_n(q,t,z),$$

which we prove by exhibiting weight-preserving bijections between the collections $\mathcal{U}_n$, $\mathcal{D}_n$, and $\mathcal{V}_n$. Setting $z = 1$, this result implies that the q,t-Catalan sequence of Garsia and Haiman [15] can be defined in terms of classical permutation statistics.

The idea behind our weight-preserving bijections comes from the following observation. We can draw a picture that illustrates permutation statistics in a suggestive way. For example, consider the permutation

$$w = 10, 5, 6, 12, 14, 2, 7, 9, 11, 13, 1, 3, 4, 8 \in \mathcal{U}_{14}.$$

In Figure 5.9, we have entered the entries of w in a diagonal line of lattice cells, going northeast from $(0,0)$ to $(14,14)$. A capital D marks each descent of w. In this case, the descents occur at coordinates $(1,1)$, $(5,5)$, and $(10,10)$. Note that $maj(w) = 16$, which is the sum of the x-coordinates where the D's are located. Also, $des(w) = 3$, which is the total number of D's.

Next, for all $i < j$, shade in the unique lattice cell located above w_i and left of w_j if and only if $w_i < w_j$. Clearly, the number of cells shaded is exactly $coinv(w)$. Also note that each ascending run of w will cause a certain triangular group of cells to be shaded. These cells are shaded darker in Figure 5.9. Now, compare this figure to the Dyck path and bounce path shown in Figure 1.4. The three statistics agree, and there is an obvious correspondence between the bounce path for D and the darkly shaded cells in the diagram for w. On the other hand, the cells above the bounce path and below D do not appear in the same place as the lightly shaded cells in the diagram for w, although the *number* of these cells is the same.

These remarks suggest the following strategy for defining a weight-preserving bijection. First, we show how to convert a bounce path into a permutation in such a way that the positions of the bounces on the main diagonal correspond to descents of the permutation. Second, we describe how to modify two consecutive ascending runs

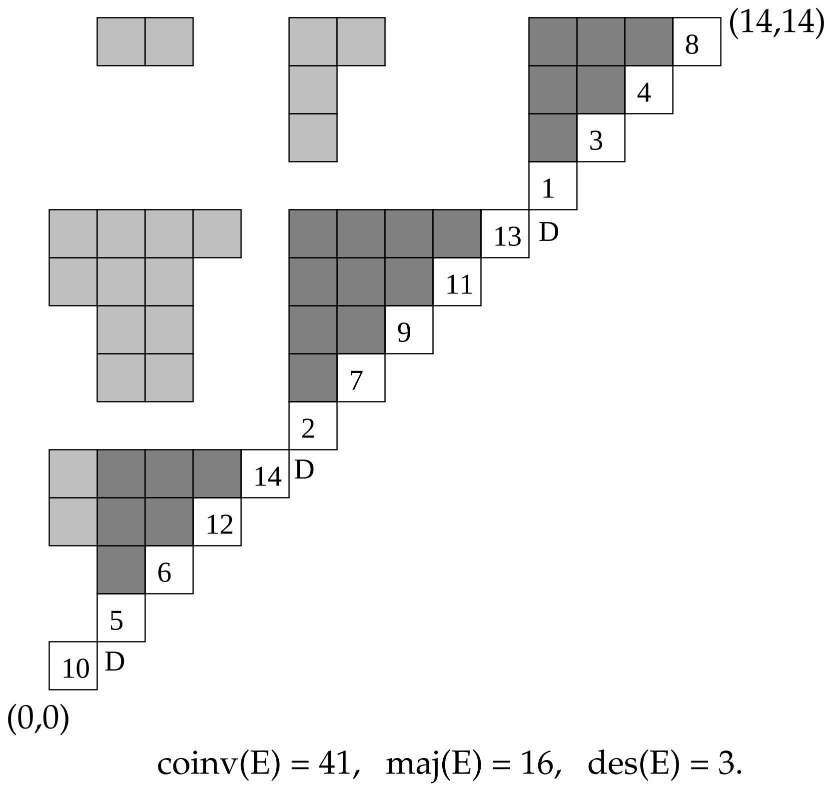


Figure 5.9: Picturing permutation statistics.

in a permutation to account for area cells above a given horizontal move in the bounce path. Third, we use this "local" modification of the permutation repeatedly to account for all the area cells above the entire bounce path. The final permutation we get depends on the order in which these modifications are performed. The two most natural orders, which roughly correspond to going forwards or backwards along the bounce path, lead us to the special collections $\mathcal{U}_n$ and $\mathcal{V}_n$ defined above.

The rest of this section fills in the details of this strategy. In §5.3.4, we set up notation to describe how the bounce path dissects the diagram of a Dyck path into smaller components. In §5.3.5, we discuss bounce paths and their corresponding permutations. In §5.3.6, we describe a local modification to a word that increases its coinversions by the area of a given partition. §5.3.7 uses the preceding ideas to define the required weight-preserving bijections. Finally, §5.3.8 contains some further remarks and open problems involving these bijections.

5.3.4 Dissecting Dyck Paths

We begin by rephrasing the bounce path construction in a more convenient form. Recall that a *composition* of n is an ordered list $(v_1, v_2, \ldots, v_s)$ of positive integers such that $v_1 + v_2 + \cdots + v_s = n$.

Definition 5.32. Fix a positive integer n. A *dissected Dyck path* of order n consists of a composition $(v_1, \ldots, v_s)$ of n, together with a list of $s-1$ partitions $(\lambda^1, \ldots, \lambda^{s-1})$ such that the Ferrers diagram of λ^i is contained in a rectangle of height $v_i - 1$ and width v_{i+1}. (The diagram consists of right-justified rows of cells, with the lowest row corresponding to the largest part of λ^i.) Let $\mathcal{D}'_n$ denote the set of all dissected Dyck paths of order n. A typical element of $\mathcal{D}'_n$ looks like

$$P = ((v_1, \ldots, v_s), (\lambda^1, \ldots, \lambda^{s-1})).$$

Lemma 5.33. *There is a bijection between $\mathcal{D}_n$ and $\mathcal{D}'_n$. If the Dyck path $D \in \mathcal{D}_n$ corresponds to the dissected path P as in the definition above, then*

$$a(D) = \sum_{i=1}^{s} v_i(v_i - 1)/2 + \sum_{i=1}^{s-1} |\lambda^i|$$

$$b(D) = \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} v_j$$

$$c(D) = s - 1.$$

Proof. Given a Dyck path D, draw its derived bounce path starting from (n, n). Define P by letting s be the number of horizontal moves in the bounce path, letting v_i be the length of the i^{th} horizontal move, and letting λ^i be the partition whose diagram consists of the cells below the path D that are above the $(i+1)^{\text{th}}$ horizontal move of the bounce path and left of the i^{th} vertical move of the bounce path. It is immediate from the definitions that P does belong to $\mathcal{D}'_n$. Note that $v_i(v_i - 1)/2$ is the number of area cells of D in the triangle bounded by the i^{th} horizontal move of the bounce path, the i^{th} vertical move of the bounce path, and the diagonal $y = x$; whereas $|\lambda^i|$ is the number of area cells of D above the $i+1^{\text{th}}$ horizontal move of the bounce path. Adding up all the area cells, we obtain the stated formula for $a(D)$. When the bounce path returns to the main diagonal for the i^{th} time, its vertical coordinate is just the sum of the remaining vertical moves in the bounce path $v_{i+1}, \ldots, v_s$. This observation yields the stated formula for $b(D)$. The formula for $c(D)$ is clear, since $c(D)$ was defined to be one less than the number of horizontal moves in the bounce path.

Furthermore, the process of creating π from D is reversible. Given any $P \in \mathcal{D}'_n$, use the numbers v_i to draw a bounce path inside an empty triangle. Then stack the diagrams of the partitions λ^i above the horizontal moves of the bounce path in the obvious way to recover the Dyck path D. $\square$

Henceforth, we will identify a Dyck path $D \in \mathcal{D}_n$ with its associated dissected path $P \in \mathcal{D}'_n$, regarding D and P as the same object.

Example 5.34. The Dyck path D shown in Figure 1.4 in Chapter 1 corresponds to the dissected path

$$P = ((4,5,4,1),((1,1,4),(2,2,3,4),(0,1,1))).$$

The bounce path corresponding to this path is

$$B = ((4,5,4,1),((0,0,0),(0,0,0,0),(0,0,0))).$$

More generally, to go from an arbitrary Dyck path P to its associated bounce path B, we simply replace all partitions λ^i in the dissected path by partitions consisting of all zero parts.

5.3.5 Bounce Paths and Skeletal Permutations

Definition 5.35. Let $\pi = (v_1, \ldots, v_s)$ be a composition of n. Set $V_0 = 0$ and $V_i = v_1 + v_2 + \cdots + v_i$ for $i > 0$.

(1) The *bounce path determined by* π is

$$P(\pi) = ((v_1, \ldots, v_s), (0, 0, \ldots, 0)),$$

where each 0 denotes a partition with the appropriate number of zero parts. Clearly, this is a bijection between the set of all compositions of n and the set of all bounce paths of order n.

(2) The *skeletal permutation determined by* π is the permutation $\sigma^0(\pi) = R_s \cdots R_2 R_1$, where

$$R_i(\sigma^0) = V_{i-1}+1, V_{i-1}+2, \cdots, V_i \text{ for } 1 \le i \le s.$$

(Since $V_{i+1} > V_{i-1}+1$, this notation is consistent with that used in Definition 5.25.)

Example 5.36. If $n = 14$ and $\pi = (4,5,4,1)$, then

$$P(\pi) = ((4,5,4,1),((0,0,0),(0,0,0,0),(0,0,0))),$$

$$\sigma^0(\pi) = 14,10,11,12,13,5,6,7,8,9,1,2,3,4.$$

If $n = 11$ and $\pi = (2,6,3)$, then

$$\sigma^0(\pi) = 9,10,11,3,4,5,6,7,8,1,2.$$

Lemma 5.37. *Let w be a skeletal permutation. Then w has the decreasing-minimum property, the bounded-maximum property, and the bounded-minimum property. If $i < j$ and $x \in R_i$ and $y \in R_j$, then $x < y$.*

Proof. Write $w = R_s \cdots R_2 R_1$, where $R_i = V_{i-1}+1, V_{i-1}+2, \cdots, V_i$, as in the definition above. Using the notation of Definition 5.25, we have $n_i = v_i$, $N_i = V_i$, $m_i = V_{i-1}+1$, and $M_i = V_i = N_i$ for all i. In particular, $m_s > \cdots > m_2 > m_1$ by definition of the V_j's, so w has the decreasing-minimum property. Since $M_i = N_i \le N_{i+1}$, w has the bounded-maximum property. Since $m_{i+1} = V_i + 1 = N_i + 1 > N_{i-1} + 1$ for $1 < i < s$, w has the bounded-minimum property. The last assertion of the lemma follows immediately from the definition of R_i and R_j. □

The next lemma shows that the path statistics for $P(\pi)$ agree with the permutation statistics for $\sigma^0(\pi)$.

Lemma 5.38. *Let $\pi = (v_1, \ldots, v_s)$ be a composition of n. Let $P = P(\pi)$ be the associated bounce path, and let $w = \sigma^0(\pi)$ be the associated permutation. Then*

$$a(P) = coinv(w),\ b(P) = maj(w),\ c(P) = des(w).$$

Proof. Write $w = R_s \cdots R_2 R_1$, where $R_i = V_{i-1}+1, V_{i-1}+2, \cdots, V_i$, as usual. To compute the coinversion count of w, consider two indices $r < s$. Let r belong to run R_j and let s belong to run R_i, so that $j \geq i$. If $j > i$, then the previous lemma implies that $w_r > w_s$, so that the pair (r, s) does not contribute to the coinversion count. On the other hand, if $j = i$, then $w_r < w_s$ since $R_j = R_i$ is an ascending run. So the pair (r, s) does contribute to the coinversion count. There are $n_j(n_j - 1)/2 = v_j(v_j - 1)/2$ such pairs (r, s) coming from each run R_j. Therefore,

$$coinv(w) = \sum_{j=1}^{s} v_j(v_j - 1)/2 = a(P),$$

where the last equality follows from Lemma 5.33. Next, the definition of $w = \sigma^0(\pi)$ shows that w has $s - 1$ descents, so

$$des(w) = s - 1 = c(P).$$

Moreover, the i^{th} descent of w (counting descents from right to left) occurs at position $\sum_{j=i+1}^{s} v_j$. Hence,

$$maj(w) = \sum_{i=1}^{s-1} \sum_{j=i+1}^{s} v_j = b(P),$$

where we have again used Lemma 5.33. □

5.3.6 The Local Modification Algorithm

Definition 5.39. Let a and b be fixed positive integers.

(1) Let $P(a, b)$ denote the set of partitions λ such that

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_a = b.$$

In other words, $\lambda \in P(a, b)$ if and only if λ is a partition consisting of a parts of size at most b, where the smallest part (and possibly others too) has size 0. These are exactly the partitions λ whose diagrams fit in a rectangle of width b and height $a - 1$.

(2) Let S be a set of $a+b$ distinct integers. Choose notation so that the elements of S are

$$x_1 < x_2 < \cdots < x_a < x_{a+1} < \cdots < x_{a+b}.$$

The *basic word* associated to S, a, and b is defined to be

$$w^0 = w^0(S,a,b) = x_{a+1}, x_{a+2}, \cdots, x_{a+b}, x_1, x_2, \ldots, x_a.$$

Let $G(S,a,b)$ denote the set of rearrangements w of the elements of S such that $Des(w) = \{b\}$ and $w_{b+1} = x_1$. This means that

$$w = w_1 < w_2 < \cdots < w_b > w_{b+1} < w_{b+2} < \cdots < w_{b+a} \text{ and } w_{b+1} = x_1.$$

In particular, the basic word is an element of $G(S,a,b)$.

Lemma 5.40. *Fix a, b, S as in the definition above. There is a bijection h between $P(a,b)$ and $G(S,a,b)$ such that, if $h(\lambda) = w$, then $coinv(w) = coinv(w^0) + |\lambda|$. Furthermore, $w_1 \leq w_1^0$ and $w_{a+b} \geq w_{a+b}^0$ and $w_{b+1} = w_{b+1}^0 = x_1$, the minimum element of S.*

Proof. The map h is defined as follows. Given $\lambda \in P(a,b)$, set $\mu_i = \lambda_i + i$ for $1 \leq i \leq a$. Then $\mu_1 = 1$ and $\mu_1, \ldots, \mu_a$ are a distinct indices between 1 and $a+b$. We construct $w = h(\lambda)$ as follows. Let the rightmost ascending run of w be

$$x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_a}, \tag{5.20}$$

and let the leftmost ascending run of w consist of the remaining b elements of S in ascending order. Since $x_{\mu_1} = x_1$ is the minimum element of S, it follows that $Des(w) = \{b\}$ and that $w \in G(S,a,b)$.

We claim that $w_1 \leq w_1^0 = x_{a+1}$. For, if this were not true, then $w_1, \ldots, w_b$ would be a list of b distinct integers belonging to the set $\{x_{a+2}, \ldots, x_{a+b}\}$ of $b-1$ elements, which is absurd. The inequality $w_{a+b} \geq w_{a+b}^0 = x_a$ is proved in the same way.

Next, let us compare the coinversion counts of w^0 and w. Direct calculation shows that

$$coinv(w^0) = b(b-1)/2 + a(a-1)/2,$$

since all elements in the left ascending run of w^0 exceed all elements in the right ascending run of w^0. As for w, the two ascending runs of lengths b and a in w will also give us

$b(b-1)/2+a(a-1)/2$ coinversions. However, we must also count coinversions caused by elements in the left ascending run of w that are less than elements in the right ascending run of w. Observe that there are $\mu_1 - 1$ elements in the left run less than x_{μ_1}, namely $x_1, \ldots, x_{\mu_1-1}$. Next, there are $\mu_2 - 2$ elements in the left run less than x_{μ_2}, namely, those in the set

$$\{x_1, \ldots, x_{\mu_2}\} - \{\mu_1, \mu_2\}.$$

In general, the i^{th} element of the right run of w is x_{μ_i}. The set of elements in the left run of w less than this element is

$$\{x_1, \ldots, x_{\mu_i}\} - \{\mu_1, \mu_2, \ldots, \mu_i\},$$

so there are $\mu_i - i = \lambda_i$ such elements. Adding up all these extra coinversions, we see that

$$coinv(w) = b(b-1)/2 + a(a-1)/2 + \lambda_1 + \cdots + \lambda_a = coinv(w^0) + |\lambda|,$$

as desired.

Finally, h is a bijection, because we can recover λ from w as follows. Given any $w \in G(S, a, b)$, look at which elements x_i appear in the rightmost ascending run of w to determine the numbers $\mu_1, \ldots, \mu_a$, as in (5.20). Then set $\lambda_i = \mu_i - i$ to recover λ. It is easy to check that $w \in G(S, a, b)$ implies that $\lambda \in P(a, b)$; in particular, $w_{b+1} = x_1$ ensures that $\lambda_1 = 0$. □

Example 5.41. (1) Let $a = 4$, $b = 5$, $S = \{1, 2, 3, 4, 5, 8, 9, 11, 14\}$, and $\lambda = (0, 1, 1, 4)$. We have

$$w^0(a, b, S) = 5, 8, 9, 11, 14, 1, 2, 3, 4; \quad coinv(w^0) = 16.$$

We compute $\mu = (1, 3, 4, 8)$, and hence

$$h(\lambda) = x_2, x_5, x_6, x_7, x_9 > x_1, x_3, x_4, x_8 = 2, 5, 8, 9, 14, 1, 3, 4, 11.$$

Note that $coinv(h(\lambda)) = 22 = coinv(w^0) + |\lambda|$. Also, as promised in the lemma, $2 \leq 5$, $11 \geq 4$, and the minimum element 1 of S is the first element of the rightmost run of $h(\lambda)$.

(2) Let $a = 3$, $b = 4$, $S = \{1, 2, 3, 4, 5, 6, 7\}$, and $w = 2, 4, 5, 7, 1, 3, 6 \in G(S, a, b)$. We have $\mu = (1, 3, 6)$ and so $\lambda = (0, 1, 3) \in P(a, b)$. Note that $w = h(\lambda)$.

Let $\sigma = R_s \cdots R_2 R_1$ be a permutation with ascending runs R_j. Assume there is some i such that $n_{i+1} = a$, $n_i = b$, and $m_{i+1} > M_i$. The last inequality implies that every element of R_{i+1} is larger than every element of R_i. So, letting S be the set of elements in R_i or R_{i+1}, $w^0 = R_{i+1}R_i$ is the basic word $w^0(S, a, b)$.

Now, let $\lambda \in P(a, b)$, and let $w = h(\lambda)$. Let σ' be the permutation obtained from σ by replacing the subord $w^0 = R_{i+1}R_i$ in σ by its rearrangement w. The following lemma shows how this local modification affects the permutation statistics.

Lemma 5.42. *Let σ' be obtained from σ and λ as described above. Then*

$$\begin{aligned} coinv(\sigma') &= coinv(\sigma) + |\lambda|, \\ Des(\sigma') &= Des(\sigma), \\ des(\sigma') &= des(\sigma), \\ maj(\sigma') &= maj(\sigma). \end{aligned}$$

Moreover, if σ has the decreasing-minimum property, then σ' also has the decreasing-minimum property.

Proof. In general, given any word abc consisting of subwords a, b, and c, and given a rearrangement b' of the letters of b, the definition of coinversion shows that

$$coinv(ab'c) - coinv(abc) = coinv(b') - coinv(b).$$

In particular, letting $abc = \sigma$, $b = w^0$, $b' = w$, so that $ab'c = \sigma'$, we obtain

$$coinv(\sigma') - coinv(\sigma) = coinv(w) - coinv(w^0) = |\lambda|,$$

which is the first assertion of the lemma.

Assume temporarily that $i > 1$ and $i + 1 < s$, so that w^0 occurs somewhere in the middle of σ:

$$\sigma = \cdots M_{i+2} > \underbrace{m_{i+1} < \cdots < M_{i+1} > m_i < \cdots < M_i}_{w^0} > m_{i-1} \cdots .$$

To check that $Des(\sigma') = Des(\sigma)$, it clearly suffices to show that the three descents displayed here are preserved. Now,

$$\sigma' = \cdots M_{i+2}\ ,\ \underbrace{w_1 < \cdots < w_b > w_{b+1} < \cdots < w_{b+a}}_{w}\ ,\ m_{i-1} \cdots ,$$

where the ascents and descents shown within w are correct by definition of w. The previous lemma shows that $w_1 \leq w_1^0 = m_{i+1} < M_{i+2}$ and $w_{a+b} \geq w_{a+b}^0 = M_i > m_{i-1}$, so that there *are* descents at the required positions in σ'. The same argument obviously works in the special cases $i = 1$ and $i + 1 = s$. Therefore, $Des(\sigma') = Des(\sigma)$, which implies that $des(\sigma') = des(\sigma)$ and $maj(\sigma') = maj(\sigma)$.

Finally, assume that σ has the decreasing-minimum property. Then $m_s > m_{s-1} > \cdots > m_1$. Now, the minima of the new permutation σ' are

$$m_s, \ldots, m_{i+2}, w_1, w_{b+1}, m_{i-1}, \ldots, m_1.$$

It suffices to check that $m_{i+2} > w_1 > w_{b+1} > m_{i-1}$. First, $m_{i+2} > m_{i+1} = w_1^0 \geq w_1$. Second, $w_1 > w_{b+1}$ since the latter is the minimum element of S. Third, $w_{b+1} = w_{b+1}^0 = m_i > m_{i-1}$. □

5.3.7 Bijections from Paths to Permutations

We now have all the tools needed to produce the weight-preserving bijections from Dyck paths to special permutations.

Theorem 5.43. *Fix $n \geq 1$.*

(1) There exists a bijection $f : \mathcal{D}_n \to \mathcal{U}_n$ such that, for all $P \in \mathcal{D}_n$,

$$coinv(f(P)) = a(P), \quad maj(f(P)) = b(P), \quad des(f(P)) = c(P).$$

(2) There exists a bijection $g : \mathcal{D}_n \to \mathcal{V}_n$ such that, for all $P \in \mathcal{D}_n$,

$$coinv(g(P)) = a(P), \quad maj(g(P)) = b(P), \quad des(g(P)) = c(P).$$

(3) We have

$$U_n(q,t,z) = C_n(q,t,z) = V_n(q,t,z).$$

Therefore, each of the collections $\mathcal{U}_n$ and $\mathcal{V}_n$ has cardinality equal to the Catalan number C_n.

Proof. *Step 1.* We define a function $f : \mathcal{D}_n \to \mathcal{U}_n$ that preserves the three statistics. Let D be a path in $\mathcal{D}_n$, which we write in dissected form as

$$D = ((v_1, \ldots, v_s), (\lambda^1, \ldots, \lambda^{s-1})).$$

Let $\sigma^0 = R_s^0 \cdots R_2^0 R_1^0$ be the skeletal permutation determined by $\pi = (v_1, \ldots, v_s)$, as discussed in §5.3.5. Let P be the bounce path for D. By Lemma 5.38, $\sigma^0 \in \mathcal{U}_n$ and

$$coinv(\sigma^0) = a(P),\ maj(\sigma^0) = b(P) = b(D),\ des(\sigma^0) = c(P) = c(D).$$

We now perform a sequence of modifications on consecutive ascending runs of σ^0, as described in the last section. For $i = 1, 2, \ldots, s-1$ (in that order), consider the ascending runs $w = R_{i+1}R_i$ in the current permutation σ. We replace w by its rearrangement $h(\lambda^i)$ to obtain the new value of σ (more details below). The σ obtained after doing modification $i = s-1$ is defined to be $f(D)$.

To see that this procedure works, we make the following induction hypothesis. Let σ denote the value of the permutation just before modification i is performed. Assume that:

(a) $\sigma = R_s \cdots R_2 R_1$ factors into s ascending runs.

(b) The entries of R_i belong to the set $R_1^0 \cup \cdots \cup R_i^0$.

(c) For all $j > i$, $R_j = R_j^0$;

(d) σ has the same values of Des, des, and maj as σ^0.

(e) $coinv(\sigma) = coinv(\sigma^0) + |\lambda^1| + \cdots + |\lambda^{i-1}|$.

(f) $\sigma \in \mathcal{U}_n$.

Lemma 5.37 shows that these conditions hold when $i = 1$, for $\sigma = \sigma^0$ in this case. Next, assume that $i \geq 1$ and that these conditions hold. Modification i will replace the subword $w = R_{i+1}R_i = m_{i+1} < \cdots < M_{i+1} > m_i < \cdots < M_i$ by a rearrangement depending on λ^i, as in Lemma 5.42. To use that lemma, we need to know that $m_{i+1} > M_i$. Since the induction hypothesis guarantees that $m_{i+1} \in R_{i+1}^0$ and $M_i \in R_1^0 \cup \cdots \cup R_i^0$, this inequality follows from Lemma 5.37. So, we may replace w by its rearrangement $h(\lambda^i) = R_{i+1}' R_i'$.

Properties (d), (e), and (a) still hold for $i+1$, by Lemma 5.42. Note that all entries in R'_{i+1} come from $R_1^0 \cup \cdots \cup R_i^0 \cup R_{i+1}^0$, so (b) holds for $i+1$. Also, for $j > i+1$, the entries of R_j were not changed, so (c) holds for $i+1$. Finally, consider (f). The new σ still has the decreasing-minimum property, by Lemma 5.42. To check the bounded-maximum property, it suffices to check that the two new maxima M'_i and M'_{i+1} satisfy $M'_i \leq N'_{i+1}$ and $M'_{i+1} \leq N'_{i+2}$. Note that M_{i+1} is the largest value occurring in w, so that M'_i and M'_{i+1} are at most M_{i+1}. Also, $M_{i+1} = M_{i+1}^0 = N_{i+1}$ by (c) and the definition of R_{i+1}^0. Note that $N'_j = N_j = N_j^0$ for all j, by (a) and (d). Since (f) holds for i, we have

$$M'_i \leq M_{i+1} = N_{i+1} = N'_{i+1} \text{ and } M'_{i+1} \leq M_{i+1} = N_{i+1} \leq N_{i+2} = N'_{i+2},$$

and so (f) holds for $i+1$. This completes the induction.

After doing all $s-1$ local modifications, (d) and (e) show that the final permutation $\sigma = f(D)$ has

$$coinv(\sigma) = a(D), \; maj(\sigma) = b(D), \; des(\sigma) = c(D),$$

since $a(D) = a(P) + \sum_{i=1}^{s-1} |\lambda^i|$. By (f), the function f does map into the set $\mathcal{U}_n$.

Step 2. We exhibit $f^{-1} : \mathcal{U}_n \to \mathcal{D}_n$, which shows that f is a bijection. Start with $\beta = R_s \cdots R_2 R_1 \in \mathcal{U}_n$. The run lengths $n_i = |R_i|$ of β allow us to recover the bounce path $(n_1, \ldots, n_s)$ of $f^{-1}(\beta)$. For $s > 1$, we then recover the partitions $\lambda^{s-1}, \lambda^{s-2}, \ldots, \lambda^1$ in this order. First, look at the subword $w = R_s R_{s-1}$ of β. This is an element of $G(S, a, b)$ for $S = R_s \cup R_{s-1}$, $a = n_{s-1}$, $b = n_s$, because $m_s > m_{s-1}$. So, we can compute $\lambda^{s-1} = h^{-1}(w) \in P(a, b)$. Next, replace the subword w in β by the corresponding basic word $w^0(S, a, b) = R'_s R'_{s-1}$. Iterate this process, considering the subword $w = R_{s-1} R_{s-2}$ of the current β to obtain λ^{s-2}, etc. The last step uses $w = R_2 R_1$ to recover λ^1. If the process succeeds, it clearly reverses the action of f, hence gives an inverse map for f.

We show that this process succeeds for $\beta \in \mathcal{U}_n$ by induction on s. If $s = 1$, so that $\beta = 1, 2, \ldots, n \in \mathcal{U}_n$, then $f^{-1}(\beta) = ((n), \emptyset) \in \mathcal{D}_n$. Now assume $s > 1$. As noted above, the decreasing-minimum property of β shows that $m_s > m_{s-1}$, so that the first subword $w = R_s R_{s-1}$ does belong to $G(S, a, b)$. We claim that the largest b elements $n, n-1, \ldots, n-(b-1)$ all appear in $S = R_{s-1} \cup R_s$. For, suppose $x > n - b$ belonged to R_j where $j < s-1$. Then

$$M_j \geq x > n - b = N_{s-1} \geq N_{j+1},$$

and this contradicts the bounded-maximum property of β. Now, in the basic word $w^0(S,a,b) = R'_s R'_{s-1}$, the largest b elements of S all occur in R'_s. Hence, the new β after this step looks like

$$\beta = n-(b-1),\cdots,n-1,n > R'_{s-1}, R_{s-2},\ldots,R_1.$$

Consider $\beta^* = R'_{s-1}, R_{s-2},\ldots,R_1 \in S_{n-b}$. Since $m'_{s-1} = m_{s-1}$ and $\beta \in \mathcal{U}_n$, β^* still has the decreasing-minimum property. Furthermore, β^* still has the bounded-maximum property since that property does not constrain the value of M'_{s-1}. To summarize, β^* belongs to $\mathcal{U}_{n-b}$.

Note that the rest of the algorithm for computing f^{-1} depends only on β^*, which has $s-1$ runs. By induction, the rest of the algorithm proceeds successfully. This completes the proof that $f^{-1} : \mathcal{U}_n \to \mathcal{D}_n$ exists.

Step 3. We define a function $g : \mathcal{D}_n \to \mathcal{V}_n$ that preserves the three statistics. Let D be a path in $\mathcal{D}_n$, which we write in dissected form as

$$D = ((v_1,\ldots,v_s),(\lambda^1,\ldots,\lambda^{s-1})).$$

Let $\sigma^0 = R_s^0 \cdots R_2^0 R_1^0$ be the skeletal permutation determined by $\pi = (v_1,\ldots,v_s)$, as discussed in §5.3.5. Let P be the bounce path for D. By Lemma 5.38, $\sigma^0 \in \mathcal{V}_n$ and

$$coinv(\sigma^0) = a(P),\ maj(\sigma^0) = b(P) = b(D),\ des(\sigma^0) = c(P) = c(D).$$

We now perform a sequence of modifications on consecutive ascending runs of σ^0, as described in the last section. For $i = s-1, s-2, \ldots, 1$ (in that order), consider the ascending runs $w = R_{i+1}R_i$ in the current permutation σ. We replace w by its rearrangement $h(\lambda^i)$ to obtain the new value of σ (more details below). The σ obtained after doing modification $i = 1$ is defined to be $g(D)$.

To see that this procedure works, we make the following induction hypothesis. Let σ denote the value of the permutation just before modification i is performed. Assume that:

(a) $\sigma = R_s \cdots R_2 R_1$ factors into s ascending runs.

(b) The entries of R_{i+1} belong to the set $R_{i+1}^0 \cup \cdots \cup R_s^0$.

(c) For all $j \leq i$, $R_j = R_j^0$;

(d) σ has the same values of Des, des, and maj as σ^0.

(e) $coinv(\sigma) = coinv(\sigma^0) + |\lambda^{s-1}| + \cdots + |\lambda^{i+1}|$.

(f) $\sigma \in \mathcal{V}_n$.

Lemma 5.37 shows that these conditions hold when $i = s-1$, for $\sigma = \sigma^0$ in this case. Next, assume that $i \leq s-1$ and that these conditions hold. Modification i will replace the subword $w = R_{i+1}R_i = m_{i+1} < \cdots < M_{i+1} > m_i < \cdots < M_i$ by a rearrangement depending on λ^i, as in Lemma 5.42. To use that lemma, we need to know that $m_{i+1} > M_i$. Since the induction hypothesis guarantees that $m_{i+1} \in R_{i+1}^0 \cup \cdots \cup R_s^0$ and $M_i \in R_i^0$, this inequality follows from Lemma 5.37. So, we may replace w by its rearrangement $h(\lambda^i) = R'_{i+1}R'_i$. Properties (d), (e), and (a) still hold for $i-1$, by Lemma 5.42. Note that all entries in R'_i come from $R_i^0 \cup \cdots \cup R_s^0$, so (b) holds for $i-1$. Also, for $j \leq i-1$, the entries of R_j were not changed, so (c) holds for $i-1$. Finally, consider (f). The new σ still has the decreasing-minimum property, by Lemma 5.42. To check the bounded-minimum property, it suffices to check that the two new minima m'_i and m'_{i+1} satisfy $m'_i > N'_{i-2}+1$ (when $i > 2$) and $m'_{i+1} > N'_{i-1}+1$ (when $i < s$). Note that $N'_j = N_j = N_j^0$ for all j, by (a) and (d). Also, m_i is the smallest value occurring in w. By Lemma 5.42, $m'_i = m_i$, and $m_i = m_i^0 = N_{i-1}+1$ by (c) and the definition of R_i^0. Since $m'_i = m_i$ is the smallest value in w, we have

$$m'_{i+1} \geq m_i + 1 = N_{i-1} + 2 > N_{i-1} + 1.$$

Since (f) holds for i, we have (when $i > 2$)

$$m'_i = m_i = N_{i-1} + 1 > N'_{i-2} + 1$$

and so (f) holds for $i-1$. This completes the induction.

After doing all $s-1$ local modifications, (d) and (e) show that the final permutation $\sigma = g(D)$ has

$$coinv(\sigma) = a(D),\ maj(\sigma) = b(D),\ des(\sigma) = c(D),$$

since $a(D) = a(P) + \sum_{i=1}^{s-1} |\lambda^i|$. By (f), the function g does map into the set $\mathcal{V}_n$.

Step 4. We exhibit $g^{-1} : \mathcal{V}_n \to \mathcal{D}_n$, which shows that g is a bijection. Start with $\beta = R_s \cdots R_2 R_1 \in \mathcal{V}_n$. The run lengths $n_i = |R_i|$ of β allow us to recover the bounce path $(n_1, \ldots, n_s)$ of $f^{-1}(\beta)$. For $s > 1$, we then recover the partitions $\lambda^1, \lambda^2, \ldots, \lambda^{s-1}$ in this order. First, look at the subword $w = R_2 R_1$ of β. This is an element of $G(S, a, b)$ for $S = R_2 \cup R_1$, $a = n_1$, $b = n_2$, because $m_2 > m_1$. So, we can compute $\lambda^1 = h^{-1}(w) \in P(a, b)$. Next, replace the subword w in β by the corresponding basic word $w^0(S, a, b) = R_2' R_1'$. Iterate this process, considering the subword $w = R_3 R_2$ of the current β to obtain λ^2, etc. The last step uses $w = R_s R_{s-1}$ to recover λ^{s-1}. If the process succeeds, it clearly reverses the action of g, hence gives an inverse map for g.

We show that this process succeeds for $\beta \in \mathcal{V}_n$ by induction on s. If $s = 1$, so that $\beta = 1, 2, \ldots, n \in \mathcal{V}_n$, then $g^{-1}(\beta) = ((n), \emptyset) \in \mathcal{D}_n$. Now assume $s > 1$. As noted above, the decreasing-minimum property of β shows that $m_2 > m_1$, so that the first subword $w = R_2 R_1$ does belong to $G(S, a, b)$. We claim that the smallest $a + 1$ elements $1, 2, \ldots, a + 1$ all appear in $S = R_2 \cup R_1$. For, suppose $x \leq a + 1$ belonged to R_j where $j > 2$. Then

$$m_j \leq x \leq a + 1 = N_1 + 1 \leq N_{j-2} + 1,$$

and this contradicts the bounded-minimum property of β.

Now, in the basic word $w^0(S, a, b) = R_2' R_1'$, the smallest a elements of S all occur in R_1'. Hence, the new β after this step looks like

$$\beta = R_s, R_{s-1}, \ldots, R_3, R_2', 1, 2, \ldots, a.$$

Since $a + 1 \in R_1 \cup R_2 = R_1' \cup R_2'$ and $R_1' = 1, 2, \ldots, a$, we must have $m_2' = a + 1$. Since all elements smaller than m_2' occur in R_1', we must have $m_3 > m_2'$. Now, consider $\beta^* = R_s, \ldots, R_3, R_2'$. Subtract a from every element of β^*. This is a harmless notation change, since the decisions made later in the algorithm depend only on the relative ordering of the symbols in the permutation. It is easy to see that β^* still has the decreasing-minimum property. Furthermore, subtracting a from each element of β and deleting the last run of size a does not destroy the bounded-minimum property, since m_j and N_{j-2} will both decrease by a. To summarize, β^* belongs to $\mathcal{V}_{n-a}$.

Note that the rest of the algorithm for computing g^{-1} depends only on β^*, which has $s - 1$ runs. By induction, the rest of the algorithm proceeds successfully. This completes the proof that $g^{-1}: \mathcal{V}_n \to \mathcal{D}_n$ exists.

Step 5. We prove (3). The existence of the weight-preserving bijection f shows that $U_n(q,t,z) = C_n(q,t,z)$. Similarly, the existence of g shows that $C_n(q,t,z) = V_n(q,t,z)$. The final assertion follows by setting $q = t = z = 1$ in these generating functions, and noting that $|\mathcal{D}_n| = C_n$. □

Corollary 5.44. *Let $OC_n(q,t)$ denote the q,t-Catalan number of Garsia and Haiman. Then*

$$OC_n(q,t) = \sum_{\sigma\in\mathcal{U}_n} q^{coinv(\sigma)} t^{maj(\sigma)} = \sum_{\sigma\in\mathcal{V}_n} q^{coinv(\sigma)} t^{maj(\sigma)}.$$

Proof. As noted in Chapter 1, the identity $C_n(q,t,1) = OC_n(q,t)$ follows from a theorem of Garsia and Haglund [14]. Thus, the corollary follows by setting $z = 1$ in part (3) of the last theorem. □

Example 5.45. (1) Consider the path

$$D = ((4,5,4,1),((1,1,4),(2,2,3,4),(0,1,1)))$$

shown in Figure 1.4. We compute $f(D)$ in steps, as follows:

$$\sigma^0 = 14 > 10,11,12,13 > 5,6,7,8,9 > 1,2,3,4$$

$i=1:$	$w = 5,6,7,8,9 > 1,2,3,4;\ \lambda^1 = (0,1,1,4),\ \mu^1 = (1,3,4,8)$
	$\sigma = 14 > 10,11,12,13 > 2,5,6,7,9 > 1,3,4,8$
$i=2:$	$w = 10,11,12,13 > 2,5,6,7,9;\ \lambda^2 = (0,2,2,3,4),\ \mu^2 = (1,4,5,7,9)$
	$\sigma = 14 > 5,6,10,12 > 2,7,9,11,13 > 1,3,4,8$
$i=3:$	$w = 14 > 5,6,10,12;\ \lambda^3 = (0,0,1,1),\ \mu^2 = (1,2,4,5)$
	$\sigma = 10 > 5,6,12,14 > 2,7,9,11,13 > 1,3,4,8 = f(D).$

This is the permutation shown in Figure 5.9.

(2) Let us compute $g(D)$, for D as in (1).

$$\sigma^0 = 14 > 10, 11, 12, 13 > 5, 6, 7, 8, 9 > 1, 2, 3, 4$$

$i = 3:$	$w = 14 > 10, 11, 12, 13;\ \lambda^3 = (0, 0, 1, 1),\ \mu^2 = (1, 2, 4, 5)$
	$\sigma = 12 > 10, 11, 13, 14 > 5, 6, 7, 8, 9 > 1, 2, 3, 4$
$i = 2:$	$w = 10, 11, 13, 14 > 5, 6, 7, 8, 9;\ \lambda^2 = (0, 2, 2, 3, 4),\ \mu^2 = (1, 4, 5, 7, 9)$
	$\sigma = 12 > 6, 7, 10, 13 > 5, 8, 9, 11, 14 > 1, 2, 3, 4$
$i = 1:$	$w = 5, 8, 9, 11, 14 > 1, 2, 3, 4;\ \lambda^1 = (0, 1, 1, 4),\ \mu^1 = (1, 3, 4, 8)$
	$\sigma = 12 > 6, 7, 10, 13 > 2, 5, 8, 9, 14 > 1, 3, 4, 11 = g(D)$.

(3) Consider $\beta = 7, 9 > 5, 8 > 1, 2, 3, 4, 6 \in \mathcal{U}_9 \cap \mathcal{V}_9$. We can compute $f^{-1}(\beta)$ as follows:

$$\beta = 7, 9 > 5, 8 > 1, 2, 3, 4, 6$$

$i = 2:$	$\mu^2 = (1, 3), \lambda^2 = (0, 1), w^0 = 8, 9 > 5, 7$
	$\beta = 8, 9 > 5, 7 > 1, 2, 3, 4, 6$
$i = 1:$	$\mu^1 = (1, 2, 3, 4, 6), \lambda^1 = (0, 0, 0, 0, 1), w^0 = 6, 7 > 1, 2, 3, 4, 5$
	$\beta^0 = 8, 9 > 6, 7 > 1, 2, 3, 4, 5$
	$f^{-1}(\beta) = ((5, 2, 2), ((0, 0, 0, 1), (1)))$.

Similarly, we can compute $g^{-1}(\beta)$:

$$\beta = 7, 9 > 5, 8 > 1, 2, 3, 4, 6$$

$i = 1:$	$\mu^1 = (1, 2, 3, 4, 6), \lambda^1 = (0, 0, 0, 0, 1), w^0 = 6, 8 > 1, 2, 3, 4, 5$
	$\beta = 7, 9 > 6, 8 > 1, 2, 3, 4, 5$
$i = 2:$	$\mu^2 = (1, 3), \lambda^2 = (0, 1), w^0 = 8, 9 > 6, 7$
	$\beta = 8, 9 > 6, 7 > 1, 2, 3, 4, 5$
	$g^{-1}(\beta) = ((5, 2, 2), ((0, 0, 0, 1), (1)))$.

By chance, we have $f^{-1}(\beta) = g^{-1}(\beta)$ in this example.

5.3.8 Further Remarks on Paths and Permutations

We can obtain an infinite family of weight-preserving bijections from Dyck paths to certain collections of permutations, as follows. For each $s \geq 1$, let τ_s be a fixed permutation of $\{1, 2, \ldots, s\}$. Define a bijection based on the collection τ_s as follows. Beginning with a Dyck path D, find its derived bounce path and construct the associated skeletal permutation σ^0, as usual. Next, visit each descent of σ^0 in turn, and modify the two ascending runs before and after this descent using the local modification algorithm. If σ^0 has s descents, use τ_s to determine the order in which the descents of σ^0 are visited. It is easy to see that this process does yield a weight-preserving bijection from $\mathcal{D}_n$ to some subcollection of the permutations of n letters. The two bijections above correspond to the cases where $\tau_s = 1, 2, \ldots, s$ for all s or where $\tau_s = s, \ldots, 2, 1$ for all s. These bijections are particularly nice because their images have relatively simple descriptions. It is unclear how to describe the image of the bijection constructed from an arbitrary collection τ_s, although it is easy to see that every permutation in this image must have the decreasing-minimum property.

One of the motivations for considering these bijections is the problem of proving combinatorially that $OC_n(q,t) = OC_n(t,q)$. By Theorem 5.43, this is equivalent to showing that

$$\sum_{\sigma \in \mathcal{U}_n} q^{coinv(\sigma)} t^{maj(\sigma)} = \sum_{\sigma \in \mathcal{U}_n} q^{maj(\sigma)} t^{coinv(\sigma)} \text{ or}$$
$$\sum_{\sigma \in \mathcal{V}_n} q^{coinv(\sigma)} t^{maj(\sigma)} = \sum_{\sigma \in \mathcal{V}_n} q^{maj(\sigma)} t^{coinv(\sigma)}.$$

Now, Foata and Schützenberger [8, 11] give a bijective proof that the *inv* and *maj* statistics are jointly symmetric on all of S_n. Foata [9] has given a simple modification of this bijection showing that

$$\sum_{\sigma \in S_n} q^{coinv(\sigma)} t^{maj(\sigma)} = \sum_{\sigma \in S_n} q^{maj(\sigma)} t^{coinv(\sigma)}.$$

Unfortunately, this bijection and many variants tried by the present author do not map $\mathcal{U}_n$ onto itself, nor do they map $\mathcal{V}_n$ onto itself. Thus, proving joint symmetry of $OC_n(q,t)$ by this method is still an open question.

Acknowledgement: This material in this chapter is now in preparation for publication in one or more papers by N. Loehr. The dissertation author was the primary investigator and author of this material.

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