UNIVERSITY OF CALIFORNIA, SAN DIEGO

How to encode a tree

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

Sally Picciotto

Committee in charge:

Professor Peter Doyle, Chair
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1999

The dissertation of Sally Picciotto is approved, and it is acceptable in quality and form for publication on microfilm:

Chair

University of California, San Diego

1999

To

Ryan Garibaldi

Jean Isaacs

and Dalit Baum

who have witnessed and greatly assisted in
my growth over the last few years.

*"I'm afraid you misunderstood... I said
I'd like a mango."–G. Larson*

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ABSTRACT OF THE DISSERTATION

How to encode a tree

by

Sally Picciotto
Doctor of Philosophy in Mathematics

University of California San Diego, 1999

Professor Peter Doyle, Chair

We construct bijections giving three "codes" for trees. These codes follow naturally from the Matrix Tree Theorem of Tutte and have many advantages over the one produced by Prüfer in 1918. One algorithm gives explicitly a bijection that is implicit in Orlin's manipulatorial proof of Cayley's formula (the formula was actually found first by Borchardt). Another is based on a proof of Knuth. The third is an implementation of Joyal's pseudo-bijective proof of the formula, and is equivalent to one previously found by Eğecioğlu and Remmel. In each case, we have at least two algorithms, one of which involves hands-on manipulations of the tree while the other involves a combinatorial and linear algebraic manipulation of a matrix.

Chapter 1

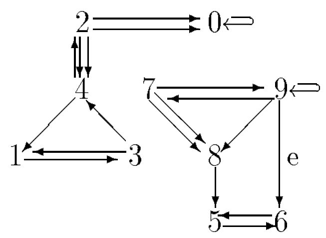
Introduction

This dissertation is a contribution to the history of progressing from algebraic proofs to bijective proofs. In particular, for theorems involving graphs, there is a long history of proofs using matrices. We start with linear algebra, but automatically something is going on beneath the surface that turns out to be a simple bijection.

1.1 Definitions

Definition 1 *A* directed graph *is a quadruple $G = (V, E, \alpha, \omega)$, where the elements of the set V are called* vertices *and the elements of the set E are called* edges*, and α and ω are the boundary maps from E to V. If $e \in E$, then $\alpha(e) \in V$ is the* initial vertex *or* tail *of e and $\omega(e) \in V$ is the* terminal vertex *or* head *of e.*

Note that this definition allows for multiple edges with tail v_1 and head v_2. An edge in a directed graph can be represented by an arrow pointing from the initial vertex to the terminal vertex.
Example:



Here the vertices are labelled with integers. The edge labelled e satisfies $\alpha(\mathrm{e}) = 9$ and $\omega(\mathrm{e}) = 6$. An edge is said to *point from* or *out of* its tail and *point to* or *into* its head. This dissertation deals with directed graphs whose vertices are labelled $0, 1, \ldots, n$. Sometimes the edges have weights associated to them. Sometimes we refer to a directed graph as simply a graph.

Definition 2 *A function $W : E \to S$, where S is any set, defines a* weight *for each edge.*

Definition 3 *The* indegree *of a vertex in a directed graph is the number of edges of which the vertex is the head, and the* outdegree *is the number of edges of which the vertex is the tail.*

Definition 4 *A* path *in a directed graph is an alternating sequence of vertices and edges $v_1, e_1, v_2, e_2, \ldots, e_4, v_{r+1}$ where v_i is the tail of the edge e_i and v_{i+1} is the head of the edge e_i.*

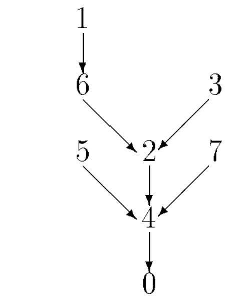
Definition 5 *A* cycle *in a directed graph is a closed path (a path where $v_1 = v_{r+1}$). A cycle with only one edge, $v \to v$, is called a* loop.

Definition 6 *The* complete digraph *(with a given number of vertices) is a directed graph with exactly one edge $v_1 \to v_2$ for each pair of vertices.*

Definition 7 *A* rooted tree *is a digraph with a unique path connecting each vertex to the (unique) vertex with outdegree 0 called the* root. *Any vertex whose indegree is* 0 *is called a* leaf.

Unless otherwise noted, all trees are rooted at 0. Any “free tree” (an undirected connected graph with no cycles) can be transformed uniquely into a tree rooted at 0 by directing all edges toward 0.

Example:



In this tree, the leaves are 1, 3, 5, and 7.

Definition 8 *The* weight *of a tree is the product of the weights of its edges.*

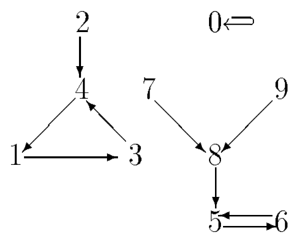
In the example above, if the weight of the edge $i \to j$ is a_{ij} then the weight of the tree is $a_{16}a_{24}a_{32}a_{40}a_{54}a_{62}a_{74}$.

Definition 9 *A* spanning tree *of a graph G is a tree whose vertices are the same as the vertices of G and whose edges are a subset of the edges of G.*

Definition 10 *A* functional digraph *is a directed graph where each vertex is the tail of exactly one edge.*

In a functional digraph, there may be many edges pointing into a vertex but only one pointing out. A functional digraph is a collection of disjoint cycles whose vertices are roots of trees leading into them.

Example: This is a functional digraph:



A functional digraph represents a function $f : \{0, 1, 2, \ldots, n\} \to \{0, 1, 2, \ldots, n\}$, where $f(i) = j$ if and only if the edge $i \to j$ is in the digraph.

Definition 11 *Since each vertex i ($\neq 0$ in a rooted tree) in a functional digraph is the initial vertex for exactly one edge, it makes sense to define* $\texttt{succ}(i) = j$ *to be the terminal vertex of the edge $i \to j$ in the (tree or) functional digraph.*

In the tree example above, $\texttt{succ}(6) = 2$; in the functional digraph, $\texttt{succ}(6) = 5$.

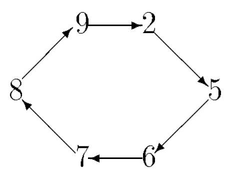
Definition 12 *A* happy functional digraph *is a functional digraph without an edge out of* 0*, and in which* 1 *is in the same connected component as* 0*.*

A happy functional digraph is a collection of trees leading into disjoint cycles, together with a tree rooted at 0 and also containing 1.

Definition 13 *An* ascent *is an edge $i \to j$ where $j > i$.*

Definition 14 *An* Escher cycle *is a cycle in which each edge except one is an ascent.*

Example:



Note that each vertex is smaller than its successor, except for the greatest vertex in the cycle, 9.

Definition 15 *The "naïve code" for a tree is defined to be*

$$\text{naïve} = (\texttt{succ}(1), \texttt{succ}(2), \ldots, \texttt{succ}(n)).$$

The "naïve code" requires no work to find, but not every n-tuple corresponds to a tree. For example, the naïve code (3,2,0,5,4) would correspond to the following graph:

$$\begin{tikzcd} 1 \arrow[d] & 4 \arrow[r, shift right] & 5 \arrow[l, shift right] \\ 3 \arrow[d] & 2 \arrow[loop right] & \\ 0 & & \end{tikzcd}$$

This graph is not a tree because it has a loop and a cycle. It is, however, a happy functional digraph.

We borrow the notation of discrete geometry for some of the proofs in this paper:

Definition 16 *A* signed set $S = S^+ \sqcup S^-$, *where* $\sqcup$ *represents the disjoint union, is an oriented zero-dimensional complex (that is, a collection of distinguishable points that can be partitioned into two subsets, one containing the elements considered "positive" and the other containing the elements considered "negative.").*

Definition 17 *Let* $T = T^+ \sqcup T^-$ *and* $S = S^+ \sqcup S^-$ *be two signed sets. Their* difference *is defined to be the disjoint union of the sets, with the following signs on elements of the union:*

$$(S - T)^+ = S^+ \sqcup T^- \text{ and } (S - T)^- = S^- \sqcup T^+.$$

Example: If $S = \{a, b, c, -d, -e\}$ and $T = \{x, -y, -z\}$, then

$$S - T = \{a, b, c, y, z, -d, -e, -x\}.$$

Definition 18 *The* Kronecker delta *function* δ_{xy} *takes value 1 if* $x = y$ *and 0 otherwise.*

Definition 19 *An* involution $\phi : S \to S$ *is a map on a signed set* S *that satisfies* $\phi \circ \phi(x) = x$ *for all* $x \in S$*.*

Definition 20 *An involution is* sign-reversing *if for any* $x \in S^+$, $\phi(x) \in S^-$ *and for any* $x \in S^-$, $\phi(x) \in S^+$*.*

A sign-reversing involution does not have any fixed points.

1.2 Some History

In 1860, Borchardt [1] discovered through evaluation of a certain determinant (namely, the principal (0,0)-minor of the matrix Tutte used a hundred years later, see §1.4) that the number of labelled trees is $(n+1)^{n-1}$. Cayley [2] independently derived this formula in 1889, and his short paper on the topic alludes to a bijection. However, the invention of a coding algorithm for trees, by Prüfer in 1918, was the first combinatorial proof that this is the formula for the number of trees. His idea was that any tree can be encoded by a vector: an ordered $(n-1)$-tuple of labels chosen from 0 to n. This is done in such a way that the tree can be recovered from the code and vice versa. The number of possible codes (which is of course equal to the number of possible trees) is $(n+1)^{n-1}$.

1.3 The Prüfer Code

In 1918, Prüfer [9] gave the following bijective proof of Cayley's formula.

Given a labelled tree, we suppose that the least leaf is labelled i_1, and that $\texttt{succ}(i_1) = j_1$. Remove i_1 and its edge from the tree, and let i_2 be the least leaf on the new tree, with $\texttt{succ}(i_2) = j_2$. If we repeat this process until there are only two vertices left, the Prüfer code $(j_1, \ldots, j_{n-1})$ uniquely determines the tree.

To recover the tree from any $(n-1)$-tuple, we note that for each vertex except the root, the number of occurrences of that label in the Prüfer code is equal to the indegree of that vertex. The number of occurrences of 0 in the code is one less than the indegree of 0. There must be at least two labels that don't appear in the code, since there are $n+1$ vertices and only $n-1$ entries in the code. Any nonzero vertex not occurring in the code is a leaf in the original tree, so we know that the least one, i_1, has $\texttt{succ}(i_1) = j_1$, the first vertex in the code. We can also tell whether any new leaves were formed when i_1 was removed because we know the indegree of j_1. Step by step, from beginning to end, we can reconstruct each edge of the tree.

Hence, the Prüfer code gives a bijection between trees with $n+1$ vertices and $(n-1)$-tuples of the vertex-labels. Since the number of $(n-1)$-tuples is clearly $(n+1)^{n-1}$, this bijection proves the formula that Borchardt discovered.

However, the algorithm is a bit unnatural. The inverse does not undo the steps in the backwards order; we have to look at the overall code and decipher what had to be true in the tree by starting from the beginning of the code and working our way to the end.

1.3.1 An example

Consider the tree:

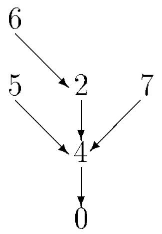
$$\begin{tikzcd}
1 \arrow[d] & & \\
6 \arrow[dr] & & 3 \arrow[dl] \\
5 \arrow[dr] & 2 \arrow[d] & 7 \arrow[dl] \\
& 4 \arrow[d] & \\
& 0 &
\end{tikzcd}$$

with leaves $\{1,5,3,7\}$. Step by step, we build up the code and remove leaves from the tree. First, we see that 1 is the least leaf, so we write down $\texttt{succ}(1)$ and remove 1 from the tree.

$$\begin{tikzcd}
6 \arrow[dr] & & 3 \arrow[dl] \\
5 \arrow[dr] & 2 \arrow[d] & 7 \arrow[dl] \\
& 4 \arrow[d] & \\
& 0 &
\end{tikzcd}$$

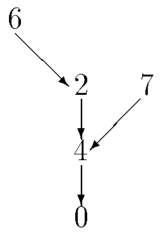
Code so far=(6)

Here, the removal of 1 created a new leaf. Now the leaves are $\{5,6,3,7\}$, so the new least leaf is 3.



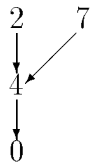
Code so far=(6,2)

The next leaf to fall off of the tree is 5, leaving us with the following tree and code:



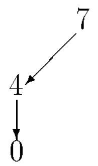
Code so far=(6,2,4)

No new leaves have been created, so the smallest leaf now is 6 and we remove it.



Code so far=(6,2,4,2)

Now that we've removed both 3 and 6, the indegree of 2 is 0. We remove 2 to obtain:



Code so far=(6,2,4,2,4),

and finally:

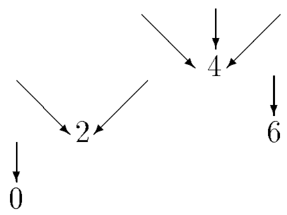
Prüfer Code=(6,2,4,2,4,4).

1.3.2 Finding the tree for a code

To get the other direction of the bijection, we start by counting occurrences of each vertex label in the code to find the list of indegrees. (The indegree of 0 is one greater than the number of occurrences of 0 in the code.) For the code (6,2,4,2,4,4) we have

Vertex	Indegree
0	1
1	0
2	2
3	0
4	3
5	0
6	1
7	0

The four vertices with indegree of 0 are the leaves on the original tree. So far, our knowledge consists of this:

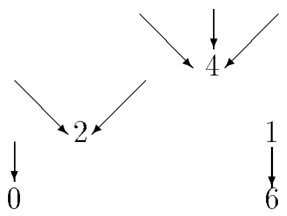


$$P_1 = (6, 2, 4, 2, 4, 4)$$

$$L_1 = \{1, 3, 5, 7\}$$

where P_i is the code at step i and L_i is the Leaf Set at step i. The Leaf Set L_i consists of all the vertices whose labels are not listed in P_i and whose outgoing edges have yet to be determined. It is actually the set of vertices that are leaves after all of the previous "least leaves" have been removed.

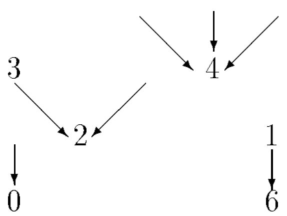
Since the smallest leaf in this example is 1, and we know the code starts with 6, we can see that the edge whose head is 6 must have tail 1. We also see that removing 1 from the tree created a new leaf, 6, so we add 6 to the Leaf Set.



$P_2 = (2,4,2,4,4)$

$L_2 = \{3,5,6,7\}$

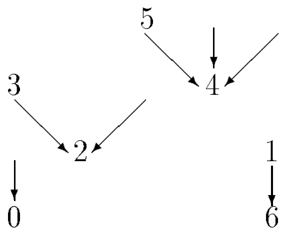
The least leaf of L_2 is 3, so its edge points to 2, the first element in P_2. Removing 3 does not create a new leaf because 2 appears twice in P_2.



$P_3 = (4,2,4,4)$

$L_3 = \{5,6,7\}$

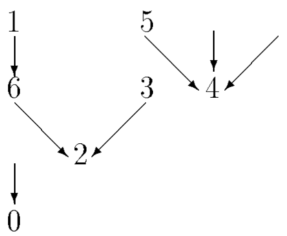
The least leaf of L_3 is 5. It will point at 4, and removing 5 will not create a new leaf.



$P_4 = (2,4,4)$

$L_4 = \{6,7\}$

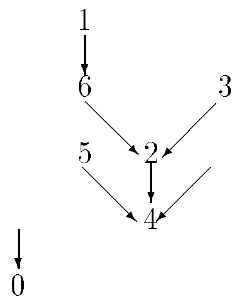
Once we remove 5, the smallest element of L_4 is 6, so there is an edge $6 \to 2$. Also, removing 6 will turn 2 into a leaf, since this is the only occurrence of 2 in P_4.



$P_5 = (4,4)$

$L_5 = \{2,7\}$

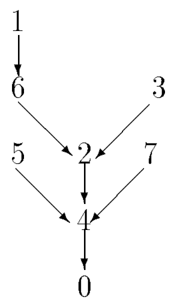
The smallest element of L_5 is 2, so it must point at 4. Since there is still an unaccounted-for edge into 4, removing 2 does not make 4 into a leaf. Thus 7 will be the only leaf left after that step.



$$P_6 = (4)$$

$$L_6 = \{7\}$$

Now, since 7 is the smallest leaf, its edge has head at 4. From that we can also conclude that the edge $4 \to 0$ is the remaining edge in the tree. In general, whichever vertex did not yet have an outgoing edge will have to point to 0 at the end.



Given the code, we were able to reconstruct the tree, and this can be done no matter what $(n-1)$-tuple we are given. It is clear that this algorithm is the inverse of the algorithm given by Prüfer.

1.4 The Matrix Tree Theorem

In 1948, Tutte [13] associated a matrix A_T to the complete loopless directed graph on vertices $\{0, \ldots, n\}$, with edge from i to j of weight a_{ij}. The general matrix is $A_T = (A_{ij})$, with i and j indexed from 0 to n:

$$A_{ij} = \begin{cases} -a_{ij} & i \neq j \\ \displaystyle\sum_{k \neq i, 0 \leq k \leq n} a_{ik} & i = j. \end{cases}$$

The diagonal entry in row i is the sum of the weights of the edges with tail at i. The row sums of such a matrix are zero, so the determinant of the matrix is zero. However, the following result by Tutte is very useful. Denote by A the $n \times n$ submatrix of A_T obtained by crossing out its zeroth row and column.

Theorem 1 (Matrix Tree Theorem) *The determinant of A is the sum of the weights of all spanning trees (rooted at vertex 0) of the graph.*

Zeilberger [14] published a nice bijective proof, also discovered independently by Garsia. A bijective proof of a more general version of the theorem is due to Chaiken [3]. We will think of the entries in our matrices as being indeterminates. When the i,j-entry of the matrix (not on the diagonal) consists of a sum of k indeterminates, the matrix corresponds to a graph with k edges $i \to j$, each having monomial weight. Note that if a_{ij} is an integer, it can represent the number of edges $i \to j$ in a graph (if $a_{ij} = 0$, then there is no edge $i \to j$). Then $\det(A)$ is the number of spanning trees of the graph.

Throughout this dissertation we will be defining signed sets that come from matrices. Each element of a matrix set is an array consisting of exactly one monomial entry from the matrix in each row and each column. Each array comes with the sign corresponding to the array position in the determinant. An element of a matrix set can be thought of as a signed permutation times a diagonal matrix. The matrix set corresponding to a matrix M consists of all possible such arrays.

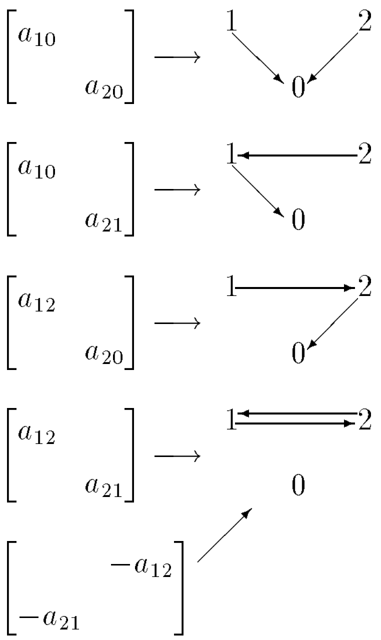
The matrix $\hat{A} = (a_{ij})$ (where i and j are indexed from 0 to n) has the indeterminate weight corresponding to the edge $i \to j$ in its i,j-entry. If we formally subtract this matrix from the diagonal $(n+1)\times(n+1)$ matrix $\hat{D}$ whose i^{th} diagonal entry is $\sum_{j=0}^{n} a_{ij}$, without simplifying, then we obtain a matrix $\hat{D} - \hat{A}$ whose row sums are zero: this matrix corresponds to the complete directed graph *with* loops. It differs from Tutte's matrix only by the presence of $a_{ii} - a_{ii}$ in the i^{th} diagonal entry–essentially we have added zero to each diagonal entry in Tutte's matrix. Obviously this doesn't change the $(0,0)$-minor; loops never appear in trees.

Zeilberger's bijective proof [14] of the Matrix Tree Theorem hinges on the idea that every functional digraph with a cycle corresponds to an array some of whose entries occur both on the diagonal and off the diagonal of Tutte's submatrix A, with opposite signs. In the determinant, these terms would cancel. He effectively introduces a surjective map from the matrix set corresponding to A to the set of digraphs representing functions from $\{1, 2, \ldots, n\}$ to $\{0, 1, \ldots, n\}$ according to the following rule: The entry in row i of the array represents the edge from i, and if this entry is $\pm b_j$, either on or off the diagonal, then the edge is $i \to j$.

Example: For $n = 2$, the submatrix of A is $\left[\begin{smallmatrix} a_{10}+a_{12} & -a_{12} \\ -a_{21} & a_{20}+a_{21} \end{smallmatrix}\right]$. The matrix set is

$$\left\{ \left[\begin{smallmatrix} a_{10} & \\ & a_{20} \end{smallmatrix}\right], \left[\begin{smallmatrix} a_{10} & \\ & a_{21} \end{smallmatrix}\right], \left[\begin{smallmatrix} a_{12} & \\ & a_{20} \end{smallmatrix}\right], \left[\begin{smallmatrix} a_{12} & \\ & a_{21} \end{smallmatrix}\right], \left[\begin{smallmatrix} & -a_{12} \\ -a_{21} & \end{smallmatrix}\right] \right\}$$

and the surjective map is given in the following diagram:

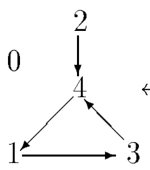


Note that the only graph with a cycle gets mapped to twice. Since we are only interested in counting trees, we can eliminate graphs with cycles and instead map the two preimages to one another.

Using these ideas, Zeilberger constructs what amounts to a sign-reversing involution on the matrix set corresponding to A minus the set of trees.

In our case, we think of A as being morally equal to $\hat{D} - \hat{A}$, and "on the diagonal" as meaning "occurring in $\hat{D}$" and "off the diagonal" as meaning "occurring in the matrix $-\hat{A}$." By defining these terms in this way, we allow for loops. Most of our algorithms for finding codes using a matrix method will require us to know how to "toggle the diagonality" of a cycle. Toggling the diagonality of a cycle in an array in a matrix set simply entails finding the unique array in the same set that satisfies two conditions: (1) the variable corresponding to any edge *not* in the cycle is in the same location as in the original array, and (2) any variable corresponding to an edge that *is* in the cycle occurs within the same row but has the opposite "diagonality" from its location in the original array. Toggling the diagonality of a cycle is a sign-reversing involution on the matrix set's subset corresponding to graphs containing cycles. An off-diagonal cycle will always come with a negative sign because a cycle of odd length has a permutation sign of $+1$, but an odd number of negative terms; a cycle of even length has an even number of negative terms but a negative sign.

Example:

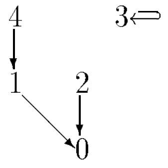


$$\longleftrightarrow \begin{bmatrix} a_{13} & & & \\ & a_{24} & & \\ & & a_{34} & \\ & & & a_{41} \end{bmatrix} \longleftrightarrow \begin{bmatrix} & & -a_{13} & \\ & a_{24} & & \\ & & & -a_{34} \\ -a_{41} & & & \end{bmatrix}.$$

The graph above contains a cycle; the elements of the matrix set corresponding to the (0,0)-minor of $\hat{D} - \hat{A}$ that correspond to this tree are both above: the one on the left consists entirely of entries from $\hat{D}$ while the one on the right has some entries from $-\hat{A}$. a_{24} corresponds to the edge $2 \to 4$ which is not in a cycle, so it appears on the diagonal in both arrays, but the cycle (134) could appear either on or off the diagonal. The sign of the first array is $+1$ because all entries are on the diagonal. The second array turns out to be negative because the 3-cycle has sign $+1$ but there are 3 negative entries.

For loops, it is a little bit less clear:

Example:



$$\longleftrightarrow \begin{bmatrix} a_{10} & & & \\ & a_{20} & & \\ & & a_{33} & \\ & & & a_{41} \end{bmatrix} \longleftrightarrow \begin{bmatrix} a_{10} & & & \\ & a_{20} & & \\ & & -a_{33} & \\ & & & a_{41} \end{bmatrix}.$$

Here, although the entries are all apparently on the diagonal, we think of the diagonality of the loop at 3 as having changed from the first matrix to the second. The first array consists of entries only from $\hat{D}$ while the $-a_{33}$ in the second one is an entry from $-\hat{A}$.

If a graph has more than one cycle (including loops), we raise the issue of which cycle's diagonality gets toggled. Zeilberger arbitrarily chose to move the cycle with the smallest element in it; we arbitrarily choose to move the cycle with the largest. All choices are equally valid but result in slightly different codes. The choice of the largest element in a cycle is consistent with some tree surgical methods that give the same bijections as our matrix methods.

1.5 Linear Algebra Setup

If we set $a_{ij} = b_j$ for all i, j in Tutte's matrix A_T, we get a matrix with each entry in column $j = -b_j$ except on the diagonal. We can calculate the $(0,0)$-minor using row and column operations.

If we ignore the zeroth row and column, we could find the determinant using the following operations. We start with the submatrix A:

$$\det \begin{bmatrix} b_0+b_2+b_3 & -b_2 & -b_3 \\ -b_1 & b_0+b_1+b_3 & -b_3 \\ -b_1 & -b_2 & b_0+b_1+b_2 \end{bmatrix}.$$

Subtract row 2 from row 3:

$$= \det \begin{bmatrix} b_0+b_2+b_3 & -b_2 & -b_3 \\ -b_1 & b_0+b_1+b_3 & -b_3 \\ 0 & -b_0-b_1-b_2-b_3 & b_0+b_1+b_2+b_3 \end{bmatrix}$$

Add column 3 to column 2:

$$= \det \begin{bmatrix} b_0+b_2+b_3 & -b_2-b_3 & -b_3 \\ -b_1 & b_0+b_1 & -b_3 \\ 0 & 0 & b_0+b_1+b_2+b_3 \end{bmatrix}$$

Subtract row 1 from row 2:

$$= \det \begin{bmatrix} b_0+b_2+b_3 & -b_2-b_3 & -b_3 \\ -b_0-b_1-b_2-b_3 & b_0+b_1+b_2+b_3 & 0 \\ 0 & 0 & b_0+b_1+b_2+b_3 \end{bmatrix}$$

Add column 2 to column 1:

$$= \det \begin{bmatrix} b_0 & -b_2-b_3 & -b_3 \\ 0 & b_0+b_1+b_2+b_3 & 0 \\ 0 & 0 & b_0+b_1+b_2+b_3 \end{bmatrix}$$

(Call this last matrix M.) Now it is evident (since we have an upper-triangular matrix) that $\det M = b_0\,[b_0+b_1+b_2+b_3]^2$. In general, $\det M = b_0\,\left[\sum_{j=0}^{n} b_j\right]^{n-1}$. The number of trees is $(n+1)^{(n-1)}$, and it is clear that this is also the number of terms in $\det M$ (we have an $(n-1)$-fold product of a sum of $n+1$ terms). Let sequences of the b_j as read down the diagonal of the matrix be called "codes." One would like to have a bijection relating these codes to trees. Each array of diagonal entries from M should correspond to a tree.

Note that in a matrix with this much redundancy, there are many different sequences of row and column operations that can lead to an easily calculated determinant.

In the course of this research we found that allowing loops was more natural. Consequently, instead of Tutte's matrix A we use variations on $\hat{D} - \hat{A}$ as defined in §1.4. For our purposes, we will set $a_{ij} = b_j$ in $\hat{A}$ and $a_{ij} = B_j$ in $\hat{D}$. At the end of the long process of row and/or column operations, we set $B_j = b_j$.

Chapter 2

The Happy Code

We can use the Matrix Tree Theorem to find a more "natural" code than the Prüfer code by expanding on Knuth's ideas in [7]. As mentioned in §1.5, we specialize a_{ij} to be b_j in $\hat{A}$ and B_j in $\hat{D}$. Following Knuth, we introduce another indeterminate λ, which will be a placeholder, by putting $\lambda - b_0$ in the $(0,0)$-entry in the matrix, calling this new matrix M_0'. We keep in mind that we are interested in the coefficient of λ in the determinant of M_0', since it is equal to the $(0,0)$-minor of the original Matrix Tree Theorem matrix. We will do row operations to form a series of matrices, all with the same determinant. The coefficient of λ in the final determinant represents the sum of the weights of all the trees, because that was true of the original matrix; the row operations do not affect that. The sequence of matrices is formed by subtracting the zeroth row from each of the other rows, one at a time. (In [7], the row operations are all performed simultaneously.)

Specifically, we begin with the matrix M_0' whose i,j-entry is $-b_j$ when $i \neq j$ and whose i^{th} diagonal entry is $-b_i + \delta_{i0}\lambda + (1-\delta_{i0})\sum_{j=0}^{n} B_j$. (If B_j is set equal to b_j then the row sums are zero for rows 1 through n. Using B_j for the diagonal entries enables us to keep track of loops.) Let $B = \sum_{j=0}^{n} B_j$.

$$M_0' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 & \dots & -b_n \\ -b_0 & B - b_1 & -b_2 & \dots & -b_n \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ -b_0 & -b_1 & -b_2 & \dots & B - b_n \end{bmatrix}.$$

Subtract row 0 from row n, without cancelling anything. Then

$$M_1 = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 & \dots & -b_n \\ -b_0 & B - b_1 & -b_2 & \dots & -b_n \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ -\lambda + b_0 - b_0 & b_1 - b_1 & b_2 - b_2 & \dots & b_n + B - b_n \end{bmatrix}.$$

The next step consists of arithmetic within entries:

$$M_1' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 & \dots & -b_n \\ -b_0 & B - b_1 & -b_2 & \dots & -b_n \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ -\lambda & 0 & 0 & \dots & B \end{bmatrix}.$$

Next, subtract row 0 from row $n-1$, again without cancelling; repeat the process. The i^{th} step is:

$$M_i = \begin{bmatrix} \lambda - b_0 & -b_1 & \dots & -b_{n-i+1} & -b_{n-i+2} & \dots & -b_n \\ -b_0 & B - b_1 & \dots & -b_{n-i+1} & -b_{n-i+2} & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -b_0 - \lambda + b_0 & -b_1 + b_1 & \dots & B - b_{n-i+1} + b_{n-i+1} & -b_{n-i+2} + b_{n-i+2} & \dots & -b_n + b_n \\ -\lambda & 0 & \dots & 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -\lambda & 0 & \dots & 0 & 0 & \dots & B \end{bmatrix},$$

where the complicated row is row $n-i+1$. Remember that the matrix is indexed from 0 to n.

$$M_i' = \begin{bmatrix} \lambda - b_0 & -b_1 & \dots & -b_{n-i+1} & -b_{n-i+2} & \dots & -b_n \\ -b_0 & B - b_1 & \dots & -b_{n-i+1} & -b_{n-i+2} & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ -\lambda & 0 & \dots & B & 0 & \dots & 0 \\ -\lambda & 0 & \dots & 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -\lambda & 0 & \dots & 0 & 0 & \dots & B \end{bmatrix}$$

The last matrix is

$$M_n' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 & \dots & -b_n \\ -\lambda & B & 0 & \dots & 0 \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ -\lambda & 0 & 0 & \dots & B \end{bmatrix}.$$

The coefficient of λ in the determinant of this matrix is

$$S = B^n - b_1 B^{n-1} - B b_2 B^{n-2} - B^2 b_3 B^{n-3} - \dots - B^{n-1} b_n,$$

where we write each term with its factors in the same order in which their columns appeared in the final matrix, M_n'.

2.1 The Sets

We define a sequence of signed sets $A_0, A_0', A_1, A_1', \dots, A_{n+1}, A_{n+1}'$. A_0 is the set of trees on vertices $0, \dots, n$, where each tree comes with a positive sign.

The sets $A_0', A_1, A_1', \dots, A_n'$ are matrix sets as described in §1.4: For $1 \le i \le n$, A_i is the matrix set of arrays corresponding to M_i and for $0 \le i \le n$, A_i' is the matrix set of arrays from M_i'. For example, when $n = 2$, two of the elements of A_2 are: $\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_1 \end{bmatrix}$ and $\begin{bmatrix} & -b_1 & \\ -\lambda & & \\ & & B_2 \end{bmatrix}$. (These arrays with one element in each row and column are understood to come with the sign they would have in the determinant.)

A_{n+1} is the set of signed monomials (written as ordered n-tuples) occurring in S, the coefficient of λ in the determinant of M_n':

$$A_{n+1} = B^n - \left(\{b_1\} \times B^{n-1}\right) - \left(B \times \{b_2\} \times B^{n-2}\right) - \ldots - \left(B^{n-1} \times \{b_n\}\right).$$

Here, we think of B as $B = \{B_0, B_1, \ldots, B_n\}$ and B^k as the k-fold direct product of B with itself. We write the factors in the left-to-right order of the columns in which the entries appeared. The final set, A_{n+1}', is the set of monomials (all positive now) remaining when B_j is set equal to b_j and arithmetic is done on S: $A_{n+1}' = \{b_0\} \times B^{n-1}$. Ignoring the initial b_0, this is isomorphic to the set of codes (the codes are simply the subscripts of these monomials taken in order).

2.2 The involutions

We define a sequence of sign-reversing involutions $\phi_0, \phi_0', \phi_1, \phi_1', \ldots, \phi_n', \phi_{n+1}$ on differences of two consecutive sets. In this set-up, when we write a negative sign in front of an array it implies that the matrix comes from the subtracted set.

Defining ϕ_0

$\phi_0 : A_0 - A_0' \to A_0 - A_0'$ is defined as follows. If t is a tree, then $\phi_0(t)$ is the negative of the array given by the bijective proof of the Matrix Tree Theorem: in the i^{th} diagonal, the B_j term is taken if $\texttt{succ}(i) = j$. If t is an array in the negative matrix set, we look at the graph formed by the edges $i \to j$ for all i, j where an indeterminate with the subscript j is in the i^{th} row of t. If this is a tree, then it is $\phi_0(t)$. If not, then $\phi_0(t)$ can be found by toggling the diagonality of the cycle containing the greatest vertex in a cycle in this graph (see §1.4). In the case where a tree matches an array, this is clearly a sign-reversing involution. For the case of the pairings of two elements of A_0', since we only moved one cycle on or off the diagonal, and we know how to find it, it is clear that repeating the process will get us back where we started. ϕ_0 is sign-reversing, as noted in §1.4

Defining ϕ_i' for $0 \le i \le n-1$

Recall that for $0 \le i \le n-1$, M_{i+1} is obtained from M_i' by row subtraction without cancellation. $\phi_i' : A_i' - A_{i+1} \to A_i' - A_{i+1}$ is defined as follows. If $a \in -A_{i+1}$ and the entry in row $n-i+1$ is $-\lambda$ or $\pm b_j$ for some j, then $\phi_i'(-a) = -a' \in -A_{i+1}$ where a' is obtained from a by interchanging and negating rows 0 and $n-i+1$. (Remember that the matrices are indexed from 0 to n.) Otherwise, $\phi_i'(a) = -a$ (in $-A_{i+1}$ if $a \in A_i$, and vice versa).

An example may help to clarify the method. In the $n = 2$ case, A_0' and A_1 are the sets of arrays in which λ occurs in M_0' and M_1 respectively:

$$M_0' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 \\ -b_0 & B - b_1 & -b_2 \\ -b_0 & -b_1 & B - b_2 \end{bmatrix} \text{ and}$$

$$M_1 = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 \\ -b_0 & B - b_1 & -b_2 \\ -\lambda + b_0 - b_0 & b_1 - b_1 & b_2 + B - b_2 \end{bmatrix}.$$

In the easier situation, where the array does not change, we have:

$$\phi_0'\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in -A_1.$$

Here, we started with an element of A_0' and ended with an element of $-A_1$; the two arrays look identical other than the negative sign outside. Meanwhile, in the more confusing case:

$$\phi_0'\left(-\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & b_2 \end{bmatrix}\right) = -\begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix}.$$

Note that in this example, both arrays appear in the set $-A_1$ but do not exist in A_0', and the actual sign of $\phi_0'(-a)$ is different from that of $-a$. We have switched the rows in which two of the entries appeared, changing their signs but leaving them in their original columns. This is always the procedure for ϕ_i'. Another possibility is:

$$\phi_0'\left(-\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix} \in A_0'.$$

In this example, we started with an element of $-A_1$ and ϕ_0' returned an element of A_0'; ϕ_0' is an involution because if we apply it twice we get back the same element we started with. The involution is sign-reversing because interchanging two rows of a matrix changes the sign of the determinant and changing the signs of two rows has no effect.

Defining ϕ_i for $1 \leq i \leq n$

Since M_i' is obtained from M_i by arithmetic within entries of the matrix, the rest of the involutions for $1 \leq i \leq n$ are of the form $\phi_i : A_i - A_i' \to A_i - A_i'$. If $a \in A_i$ and the entry in the $(n-i+1)^{\text{th}}$ row is $\pm b_j$, then $\phi_i(a) = a' \in A_i$ where a' is obtained from a by changing the sign of the entry in the $(n-i+1)^{\text{th}}$ row. Otherwise, $\phi_i(a) = -a \in A_i'$. If $-a \in -A_i'$, then $\phi_i(-a) = a \in A_i$. Returning to the $n = 2$ example,

$$M_1' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 \\ -b_0 & B - b_1 & -b_2 \\ -\lambda & 0 & B \end{bmatrix}.$$

So we have

$$\phi_1\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & +b_2 \end{bmatrix} \in A_1,$$

and

$$\phi_1\left(-\begin{bmatrix} & -b_1 & \\ & & -b_2 \\ -\lambda & & \end{bmatrix}\right) = \begin{bmatrix} & -b_1 & \\ & & -b_2 \\ -\lambda & & \end{bmatrix} \in A_1.$$

Note that in the first of these two examples, $\phi_1(a)$ and a had opposite signs but were both elements of A_1, whereas in the second example, $-a \in -A_1'$ and $\phi_1(-a) \in A_1$. This is clearly an involution, since there is only one row of M_i' in which entries appear twice with opposite signs.

Defining ϕ_n' and ϕ_{n+1}

The last two involutions are a little bit different. $\phi_n' : A_n' - A_{n+1} \to A_n' - A_{n+1}$ takes an array in the matrix set A_n' and matches it with the product of its non-λ entries in the order of their columns (with the sign the determinant would assign this term), and it takes signed monomials to the location of the corresponding array. There is always a λ in the zeroth column.

For example, in

$$M_2' = \begin{bmatrix} \lambda - b_0 & -b_1 & -b_2 \\ -\lambda & B & 0 \\ -\lambda & 0 & B \end{bmatrix},$$

we have

$$\phi_2'\left(\begin{bmatrix} & & \\ -\lambda & -b_1 & \\ & & B_0 \end{bmatrix}\right) = -b_1 B_0,$$

$$\phi_2'(-B_0 b_2) = \begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix},$$

and

$$\phi_2'\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_0 \end{bmatrix}\right) = B_0 B_0.$$

Again, this is an involution because it matches elements of A_n' (the set of arrays) with monomials, in perfect pairs.

The final involution, $\phi_{n+1} : A_{n+1} - A_{n+1}' \to A_{n+1} - A_{n+1}'$, takes any positive element of A_{n+1} and matches it to another monomial, obtained according to the following formula:

$$\phi_{n+1}\left(\prod_{k=1}^{n} B_{j_k}\right) = \begin{cases} -b_0 \prod_{k=2}^{n} B_{j_k} \in -A'_{n+1} & \text{if } j_1 = 0, \\ -B_{j_{j_1}}\left(\prod_{k=2}^{j_1-1} B_{j_k}\right) b_{j_1}\left(\prod_{k=j_1+1}^{n} B_{j_k}\right) \in A_{n+1} & \text{otherwise.} \end{cases}$$

ϕ_{n+1} applied to any element of $-A'_{n+1}$ gives the same monomial, only with the initial $-b_0$ changed to a positive B_0, in A_{n+1}. If we start with a negative element of A_{n+1}, it must have exactly one b_j in the j^{th} position for some j. When we apply ϕ_{n+1}, we make this b_j upper-case and switch it with the indeterminate in the first position, and change the sign. This is clearly a sign-reversing involution.

The ugliness of the formula belies the simplicity of the process. A few examples with $n = 6$ should help.

$$\phi_7 \left(B_3 B_4 B_6 B_0 B_2 B_0\right) = -B_6 B_4 b_3 B_0 B_2 B_0 \in A_7.$$

All we have done is toggle the capitalization of B_3 (in the first position of the product) and switch this new lower-case entry with the element in the third (its subscript) position (which is B_6). The easiest possible case is:

$$\phi_7 \left(B_0 B_1 B_6 B_2 B_4 B_2\right) = -b_0 B_1 B_6 B_2 B_4 B_2 \in -A'_7.$$

More often some switching is involved, as in the first case and the next one:

$$\phi_7 \left(-B_4 B_3 B_0 B_3 b_5 B_1\right) = B_5 B_3 B_0 B_3 B_4 B_1 \in A_7.$$

(Remember, if there is a lower-case b_j in the product, we switch it with the first element of the product.)

These involutions are a key ingredient in the creation of the Happy Code.

2.3 Garsia and Milne's Involution Principle

Garsia and Milne [5] found an extremely useful method while investigating bijective proofs for the Rogers-Ramanujan identities.

Definition 21 *A pseudo-sign-reversing involution is an involution on a signed set, with the property that any point that is not fixed is sent to a point with the opposite sign.*

Lemma 1 (Scholium: The Involution Principle [5]) *Let A be a finite signed set, $A = A^+ - A^-$, with pseudo-sign-reversing involutions ϕ and ψ whose fixed-point sets are $F(\phi)$ and $F(\psi)$ respectively. Then there is a (fixed-point-free) sign-reversing involution γ on the set $F(\phi) - F(\psi)$. Furthermore, γ can be constructed using the following algorithm:*

```
begin
    if φ(x) = x then
        y ← x
        repeat
            z ← ψ(y)
            y ← φ(z)
        until φ(z) = z or ψ(y) = y
        if φ(z) = z then
            γ(x) ← z
        else
            γ(x) ← y
    else if ψ(x) = x then
        y ← x
        repeat
            z ← φ(y)
            y ← ψ(z)
        until ψ(z) = z or φ(y) = y
        if ψ(z) = z then
            γ(x) ← z
        else
            γ(x) ← y
    else
        {x is not a fixed point of φ or ψ}
end.[1]
```

This Lemma is extremely important because it not only establishes the existence of the involution γ but actually shows how to construct it.

[1] Pseudo-code quoted from [11], pages 141-142

Lemma 2 (The Bread Lemma) *Given two sign-reversing involutions, $\phi : A - B \to A - B$ and $\psi : B - C \to B - C$, there is a sign-reversing involution on $A - C$.*

Proof. Let $\overline{-I_B}$ represent the negative identity map on $B - B$, extended to be the identity on $A - C$.

$$\overline{-I_B}\,(x) = \begin{cases} -x & \text{if } x \in -B + B, \\ x & \text{if } x \in A - C. \end{cases}$$

Let $\phi + \psi : A - B + B - C \to A - B + B - C$ be defined as follows:

$$(\phi + \psi)\,(x) = \begin{cases} \phi\,(x) & \text{if } x \in A - B \\ \psi\,(x) & \text{if } x \in B - C \end{cases}.$$

Then both $\overline{-I_B}$ and $(\phi + \psi)$ are pseudo-sign-reversing involutions, and $F(\overline{-I_B}) = A - C$ and $F(\phi + \psi) = \emptyset$. The algorithm of the Involution Principle provides a sign-reversing involution on $F(\phi + \psi) - F(\overline{-I_B}) = A - C - \emptyset = A - C$. ☺

We call it the Bread Lemma because it can be visualized as a process to remove all of the insides from a B sandwich, leaving the diner with only a couple of slices of bread (the sets A and C).

Lemma 3 *Given any sequence of signed sets $S_0, S_1, \ldots, S_{k+1}$, where S_0 and S_{k+1} contain only positive elements, and sign-reversing involutions $\beta_0, \ldots, \beta_k$ where β_i acts on $S_i - S_{i+1}$, there is a constructible bijection between S_0 and S_{k+1}.*

Proof. By repeated applications of The Bread Lemma (Lemma 2.3), we can "eliminate" all of the in-between sets as follows. Let $A = S_0$, $B = S_1$, $C = S_2$, $\phi = \beta_0$, and $\psi = \beta_1$. The Bread Lemma constructs a sign-reversing involution on $S_0 - S_2$, and this involution still satisfies the hypotheses of the Bread Lemma. Now let $B = S_2$, $C = S_3$, etc. We keep sandwiching in until we arrive at $A = S_0$, $B = S_k$, and $C = S_{k+1}$, where ϕ is the involution achieved by so many applications of the Bread Lemma and ψ is β_{k+1}. One more application, and we have a sign-reversing involution on $S_0 - S_{k+1}$. However, $(S_0 - S_{k+1})^+ = S_0$ and $(S_0 - S_{k+1})^- = -S_{k+1}$ since these two sets contained only positive elements. Hence the only way for the involution to be sign-reversing is for each element of S_0 to be mapped to an element of S_{k+1}. Thus, we have found a bijection between S_0 and S_{k+1}. Note that we have not simply proven the existence of a bijection, but actually provided an algorithm for constructing it. ☺

Theorem 2 *Given the sets* $A_0, A_0', A_1, \ldots, A_{n+1}'$ *and the sign-reversing involutions* $\phi_0, \phi_0', \ldots, \phi_n, \phi_n', \phi_{n+1}$ *defined above, there is a constructible bijection between* A_0 *(the set of trees) and* A_{n+1}' *(the set of codes).*

Proof. The sets $A_0, \ldots, A_{n+1}'$ and the involutions $\phi_0, \ldots, \phi_{n+1}$ satisfy the hypotheses of Lemma 3. Thus we can construct the bijection between the set of trees and the set of codes. ☺

2.4 An example

Consider the case $n = 2$. We will apply the theorem to find the code that corresponds to the tree $1 \to 2 \to 0 \in A_0$.

First we apply ϕ_0 to get an element of $-A_0'$:

$$\phi_0(1 \to 2 \to 0) = -\begin{bmatrix} \lambda & B_2 & \\ & & B_0 \end{bmatrix} \in -A_0'$$

Next we apply $\overline{-I_{A_0'}}$:

$$\overline{-I_{A_0'}}\left(-\begin{bmatrix} \lambda & B_2 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} \lambda & B_2 & \\ & & B_0 \end{bmatrix} \in A_0'$$

We alternate between ϕs and $\overline{-I}$s. Each application of a $\overline{-I}$ merely changes the sign of the element (and of the subset it lies in):

$$\overline{-I_{A_1}} \circ \phi_0'\left(-\begin{bmatrix} \lambda & B_2 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} \lambda & B_2 & \\ & & B_0 \end{bmatrix} \in A_1$$

$$\overline{-I_{A_1'}} \circ \phi_1 \left(\begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \right) = \begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \in A_1'$$

$$\overline{-I_{A_2}} \circ \phi_1' \left(\begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \right) = \begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \in A_2$$

$$\overline{-I_{A_2'}} \circ \phi_2 \left(\begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \right) = \begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \in A_2'.$$

Since $n = 2$, we are in the last matrix set.

$$\overline{-I_{A_3}} \circ \phi_2' \left(\begin{bmatrix} \lambda & & \\ & B_2 & \\ & & B_0 \end{bmatrix} \right) = B_2 B_0 \in A_3$$

This is the exciting part!

$$\phi_3 \left(B_2 B_0 \right) = -B_0 b_2 \in A_3$$

$$\phi_2' \circ \overline{-I_{A_3}} \left(-B_0 b_2 \right) = \begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \in A_2'$$

Now there is nothing to stop us from passing through several sets in a row on our way back up the sequence of sets via the following involutions:

$$\phi_2 \circ \overline{-I_{A_2'}} \left(\begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \right) = \begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \in A_2$$

$$\phi_1' \circ \overline{-I_{A_2}} \left(\begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \right) = \begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \in A_1'$$

$$\phi_1 \circ \overline{-I_{A_1'}} \left(\begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \right) = \begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \in A_1$$

At this point we apply $\phi_0' \circ \overline{-I_{A_1}}$. $\overline{-I_{A_1}}$ takes us to the set $-A_1$, and in this case an application of ϕ_0' maps to another element of $-A_1$:

$$\phi_0' \circ \overline{-I_{A_1}} \left(\begin{bmatrix} & & -b_2 \\ & B_0 & \\ -\lambda & & \end{bmatrix} \right) = - \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & b_2 \end{bmatrix} \in -A_1$$

$$\overline{-I_{A_1}}\left(-\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & b_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & b_2 \end{bmatrix} \in A_1$$

$$\phi_1\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & b_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix} \in A_1$$

$$\overline{-I_{A_0'}} \circ \phi_0' \circ \overline{-I_{A_1}}\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix}\right) = -\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix} \in -A_0'$$

At this point it is the Matrix Tree Theorem that comes to the rescue, in the form of ϕ_0:

$$\phi_0\left(-\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & -b_2 \end{bmatrix}\right) = -\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix} \in -A_0'$$

The involutions now take us directly down the sequence of matrices to the last one.

$$\overline{-I_{A_1'}} \circ \phi_1 \circ \overline{-I_{A_1}} \circ \phi_0' \circ \overline{-I_{A_0'}}\left(-\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix} \in A_1'$$

$$\overline{-I_{A_2'}} \circ \phi_2 \circ \overline{-I_{A_2}} \circ \phi_1'\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix}\right) = \begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix} \in A_2'$$

Coming down the home stretch:

$$\overline{-I_{A_3}} \circ \phi_2'\left(\begin{bmatrix} \lambda & & \\ & B_0 & \\ & & B_2 \end{bmatrix}\right) = B_0B_2 \in A_3$$

And finally:

$$\phi_3\left(B_0B_2\right) = -b_0B_2 \in -A_3'.$$

Thus, the Happy Code for the tree $1 \to 2 \to 0$ is B_2.

To find the tree for a code, we can easily follow the involutions through backwards, undoing the whole process. In this sense, the Happy Code is more natural (hence "happier") than the Prüfer Code.

Computationally, finding the Happy Code for a tree is a slow process. However, later we will see a method for calculating the Happy Code that does not resort to matrices but works directly with the tree.

Chapter 3

The Blob Code

Another code results from a different sequence of sets and involutions, but still using the Involution Principle and the Bread Lemma. We begin with the $n \times n$ submatrix from the Matrix Tree Theorem (obtained by crossing out the zeroth row and column):

$$C'_0 = \begin{bmatrix} B - b_1 & -b_2 & \dots & -b_n \\ -b_1 & B - b_2 & \dots & -b_n \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ -b_1 & -b_2 & \dots & B - b_n \end{bmatrix}$$

The Blob Code is related to the process of alternately performing row operations and column operations on adjacent rows and columns as follows. The first step is to subtract row $n-1$ from row n (without cancellation).

$$R_1 = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-1} & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-1} & -b_n \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -b_1 & -b_2 & \dots & B-b_{n-1} & -b_n \\ -b_1+b_1 & -b_2+b_2 & \dots & -b_{n-1}-B+b_{n-1} & B-b_n+b_n \end{bmatrix}$$

Now we perform arithmetic within entries, but only in row n:

$$R_1' = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-1} & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-1} & -b_n \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -b_1 & -b_2 & \dots & B-b_{n-1} & -b_n \\ 0 & 0 & \dots & -B & B \end{bmatrix}$$

Then we add column n to column $n-1$.

$$C_1 = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-1}-b_n & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-1}-b_n & -b_n \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -b_1 & -b_2 & \dots & B-b_{n-1}-b_n & -b_n \\ 0 & 0 & \dots & -B+B & B \end{bmatrix}$$

And once again, perform arithmetic within entries in row n (*not* column $n-1$):

$$C_1' = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-1}-b_n & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-1}-b_n & -b_n \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -b_1 & -b_2 & \dots & B-b_{n-1}-b_n & -b_n \\ 0 & 0 & \dots & 0 & B \end{bmatrix}$$

All of that was the first step. We work our way up the matrix this way: at the i^{th} step we first subtract row $n-i$ from row $n-i+1$, then add column $n-i+1$ to column $n-i$, cancelling only within row $n-i+1$, until the matrix consists of B on the diagonals and 0 elsewhere, except in the first row. At the end of the i^{th} step the $(n-i)^{\text{th}}$ diagonal entry consists of $B-\sum b_j$ where the sum is over $n-i \le j \le n$. After the last column operation, we set $B_j = b_j$ so that the diagonal entry in row 1 consists only of b_0. At the end of the whole process, the first row consists of b_0 in its first entry and a bunch of garbage in the other entries, but the rest of the matrix is just B on the diagonal. The last 2 matrices are:

$$C_{n-1} = \begin{bmatrix} B-b_1-\sum_{k=2}^{n} b_k & -\sum_{k=2}^{n} b_k & \sum_{k=3}^{n} b_k & \dots & -b_{n-1}-b_n & -b_n \\ -B+B & B & 0 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ . & . & . & \dots & . & . \\ 0 & 0 & . & \dots & B & 0 \\ 0 & 0 & 0 & \dots & 0 & B \end{bmatrix}$$

and

$$C'_{n-1} = \begin{bmatrix} b_0 & -b_2 - b_3 - \cdots - b_n & \ldots & -b_{n-1} - b_n & -b_n \\ 0 & \sum_{j=0}^{n} b_j & \ldots & 0 & 0 \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ . & . & \ldots & . & . \\ 0 & 0 & \ldots & \sum_{j=0}^{n} b_j & 0 \\ 0 & 0 & \ldots & 0 & \sum_{j=0}^{n} b_j \end{bmatrix}.$$

This matrix clearly has determinant equal to $b_0 B^{n-1}$.

3.1 Orlin's ideas

In [8], Orlin introduced the idea of identifying two vertices of a graph. We explain how this notion is used with the matrices in the construction of the Blob Code. Assume we have a weighted directed graph on vertices 0 through n. Loops are allowed. We assume there are no multiple edges, because multiple edges can be subsumed into the weights. The weight of the edge from i to j is a_{ij}.

Definition 22 *In a directed graph D, two vertices i and j are identifiable when $a_{ik} = a_{jk}$ for all k.*

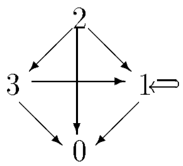
Note that this definition includes $k = i$ and $k = j$; so if there is an edge from i to j then there needs to be a loop on j.

Identifiability is an equivalence relation, so there is some sense in which we can think of two identifiable vertices as being redundant (their outgoing edges have the same heads).

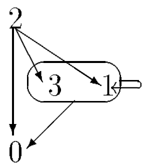
Definition 23 *If we "identify" two identifiable vertices i and j to a generalized vertex, called* `blob`*, and eliminate one set of the duplicate edges, we end up with a new digraph in which there are $a_{ik} (= a_{jk})$ edges* `blob` $\to k$ *for all $k \neq i, j$, and there are $a_{ki} + a_{kj}$ edges $k \to$* `blob`*. There are also $a_{ij} + a_{ji}$ loops on the* `blob`*.*

We take full blame for the naming of the `blob`. We differ from Orlin in our visualization of this process. He considered this "`blob`" to be a new vertex; we think of it as *containing* the original two vertices being identified. Each incoming edge actually points not at the `blob` as a whole but rather at its original terminal vertex within the `blob`.

Here, we set our edge weights to $W(i \to j) = b_j$ for all edges. and look at an example:



Vertices 1 and 3 are identifiable: each has exactly one edge to 0 and one edge to 1. If we identify the two, we obtain the following graph (with weights $w(2 \to 0) = b_0$, $w(2 \to 1) = b_1$, $w(2 \to 3) = b_3$, $w(\texttt{blob} \to 0) = b_0$, and $w(\texttt{blob} \to 1) = b_1$):



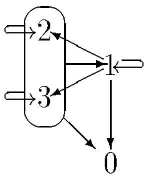
In this graph, 2 and `blob` are not identifiable because 2 has an edge to 3, while there is no loop `blob` $\to 3$.

In the complete digraph with loops, all vertices are identifiable. This was why we altered the matrix to allow for loops. Orlin used this idea to manipulate formulas to get the formula $(n+1)^{n-1}$ for the number of trees. We examine the relationship between this idea and the matrix method.

For the moment we illustrate the process with $n = 3$. We begin with the complete directed graph. The 4×4 matrix corresponding to it, where the edges are weighted by indeterminates indexed by the terminal vertex, is

$$\hat{D} - \hat{A} = \Upsilon_0 = \begin{bmatrix} B - b_0 & -b_1 & -b_2 & -b_3 \\ -b_0 & B - b_1 & -b_2 & -b_3 \\ -b_0 & -b_1 & B - b_2 & -b_3 \\ -b_0 & -b_1 & -b_2 & B - b_3 \end{bmatrix}.$$

Once we identify vertices 2 and 3, the graph looks like this (omitting edges whose initial vertex is 0, since they never appear in a tree and we will not be identifying vertex 0 with any of the others).

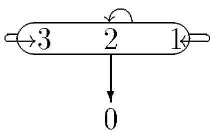


The corresponding matrix is

$$\Upsilon_1 = \begin{bmatrix} B - b_0 & -b_1 & -b_2 - b_3 \\ -b_0 & B - b_1 & -b_2 - b_3 \\ -b_0 & -b_1 & B - b_2 - b_3 \end{bmatrix}.$$

The proper way to think of this is that there are two relevant rows (the zeroth row, representing edges from 0, is not relevant); the "oneth" row represents edges out of 1 and the second represents edges out of `blob`. The zeroth column represents edges into 0; the "oneth" column (not including diagonal entries) represents edges into 1, and the second represents edges into `blob`. An edge $1 \to$ `blob` can be either $1 \to 2$ or $1 \to 3$. If we were to cancel terms in the last row, we would have only $b_0 + b_1$ in the diagonal entry. The positive and negative copies of b_2 and b_3 represent the loops `blob` $\to 2$ and `blob` $\to 3$ respectively.

Once we have identified 1 with `blob` (which we can do because both have edges to 0, 1, 2, and 3), the graph is (with undrawn edges from 0):



The matrix corresponding to this is

$$\Upsilon_2 = \begin{bmatrix} B - b_0 & -b_1 - b_2 - b_3 \\ -b_0 & B - b_1 - b_2 - b_3 \end{bmatrix} = \begin{bmatrix} B - b_0 & -b_1 - b_2 - b_3 \\ -b_0 & b_0 \end{bmatrix}.$$

We will be crossing out the zeroth row and column. Thus, only one entry appears in the part of the matrix in whose determinant we are interested. This is true because now there is only one vertex besides 0 and it only has one non-loop edge.

The determinants of Υ_0, Υ_1, and Υ_2 are related as follows: $b_0 B^2 = \det(\Upsilon_0) = \det(\Upsilon_1) \times B = \det(\Upsilon_2) \times B^2$.

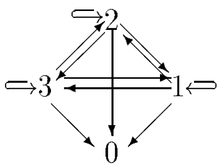
3.2 The sets

Much as we did with the Happy Code, we use the matrices in the definition of a sequence of signed sets, but now we insert some of Orlin's ideas as well. The sets are $G_0, G'_0, S_1, S'_1, T_1, T'_1, G_1, \ldots, T'_{n-1}, G_{n-1}, G'_{n-1}, S_n$.

In this sequence, the set G_0 is the set of trees, and G'_0 is the matrix set of arrays defined by C'_0. There are more matrices than we had for the Happy Code, and extra sets in between. For $1 \leq i \leq n-1$, G_i is the set of ordered pairs (τ, γ) where τ is a spanning tree (rooted at 0) on a directed graph D_i (described below) and γ is an ordered i-tuple of b_j's. D_i is defined to be the complete digraph with $n - i$ vertices, where vertex $n - i$ is actually `blob` which contains $n - i, n - i + 1, \ldots, n$. The labels in `blob` are terminal vertices to edges, but they all share the same outgoing edges; in any tree, `blob` has only one outgoing edge.

For $1 \leq i \leq n$, S_i and S'_i are the sets of arrays from R_i and R'_i respectively, and T_i denotes the set of arrays from C_i. Finally, we use both T'_i *and* G'_i to denote the set of arrays from C'_i, for $0 \leq i \leq n - 1$. Arrays are signed, as they were in the Happy Code. The final set is $S_n = \{b_0\} \times B^{n-1}$ where, in the set notation, B is understood to stand for the set $B = \{b_0, \ldots, b_n\}$ and B^{n-1} stands for the $(n - 1)$-fold direct product $B \times B \times \ldots \times B$.

As an example we list the sets for the case $n = 3$. Matrices are thought of as sets of arrays. (In the graphs, edges with initial vertex 0 have been omitted from the pictures, since they never appear in a spanning tree and the zeroth row of the matrix has already been ignored.)

G_0=the set of rooted spanning trees of 

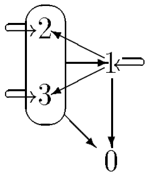
$$G_0' \leftrightarrow C_0' = \begin{bmatrix} B-b_1 & -b_2 & -b_3 \\ -b_1 & B-b_2 & -b_3 \\ -b_1 & -b_2 & B-b_3 \end{bmatrix}$$

$$S_1 \leftrightarrow R_1 = \begin{bmatrix} B-b_1 & -b_2 & -b_3 \\ -b_1 & B-b_2 & -b_3 \\ -b_1+b_1 & -b_2-B+b_2 & B-b_3+b_3 \end{bmatrix}$$

$$S_1' \leftrightarrow R_1' = \begin{bmatrix} B-b_1 & -b_2 & -b_3 \\ -b_1 & B-b_2 & -b_3 \\ 0 & -B & B \end{bmatrix}$$

$$T_1 \leftrightarrow C_1 = \begin{bmatrix} B-b_1 & -b_2-b_3 & -b_3 \\ -b_1 & B-b_2-b_3 & -b_3 \\ 0 & -B+B & B \end{bmatrix}$$

$$T_1' \leftrightarrow C_1' = \begin{bmatrix} B-b_1 & -b_2-b_3 & -b_3 \\ -b_1 & B-b_2-b_3 & -b_3 \\ 0 & 0 & B \end{bmatrix}$$

$G_1 =$ the set of rooted spanning trees of  $\times B$

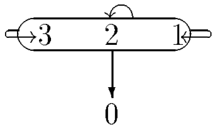
$$G_1' \leftrightarrow C_1' = \begin{bmatrix} B - b_1 & -b_2 - b_3 & -b_3 \\ -b_1 & B - b_2 - b_3 & -b_3 \\ 0 & 0 & B \end{bmatrix}$$

$$S_2 \leftrightarrow R_2 = \begin{bmatrix} B - b_1 & -b_2 - b_3 & -b_3 \\ -b_1 - B + b_1 & B - b_2 - b_3 + b_2 + b_3 & -b_3 + b_3 \\ 0 & 0 & B \end{bmatrix}$$

$$S_2' \leftrightarrow R_2' = \begin{bmatrix} B - b_1 & -b_2 - b_3 & -b_3 \\ -B & B & 0 \\ 0 & 0 & B \end{bmatrix}$$

$$T_2 \leftrightarrow C_2 = \begin{bmatrix} B - b_1 - b_2 - b_3 & -b_2 - b_3 & -b_3 \\ -B + B & B & 0 \\ 0 & 0 & B \end{bmatrix}$$

$$T_2' \leftrightarrow C_2' = \begin{bmatrix} b_0 & -b_2 - b_3 & -b_3 \\ 0 & \sum_{j=0}^{3} b_j & 0 \\ 0 & 0 & \sum_{j=0}^{3} b_j \end{bmatrix}$$

$G_2 =$ the set of rooted spanning trees of  $\times B^2$

$$G'_2 \leftrightarrow C'_2 = \begin{bmatrix} b_0 & -b_2 - b_3 & -b_3 \\ 0 & \sum_{j=0}^{3} b_j & 0 \\ 0 & 0 & \sum_{j=0}^{3} b_j \end{bmatrix}$$

The final set is $S_3 = \{b_0\} \times B \times B$, where $B = \{b_0, b_1, \ldots, b_n\}$ by abuse of notation.

3.3 The involutions

Some of the involutions are defined similarly to the involutions we used for the Happy Code, but there are many more of them.

Defining μ'_0

$\mu'_0 : G_0 - G'_0 \to G_0 - G'_0$ maps a tree to an array from $-G'_0$ by taking the b_j in the i^{th} diagonal entry for each edge $i \to j$. The remaining elements of $-C'_0$ are matched in pairs (by toggling the diagonality of the cycle with the largest element) according to the bijective proof of the Matrix Tree Theorem, just as they were for the Happy Code.

Defining ρ_i for $1 \leq i \leq n-1$

For $1 \leq i \leq n-1$, $\rho_i : G'_{i-1} - S_i \to G'_{i-1} - S_i$ maps corresponding arrays in G'_{i-1} and $-S_i$ to each other, and then takes the extra elements of $-S_i$ and matches them up according to the row operation that took C'_{i-1} to R_i. If $-a \in -S_i$ and the entry in row $n-i+1$ is $+b_j$ or $-B_j$, then $\rho_i(-a) = -a' \in -S_i$ where a' is obtained from a by interchanging and negating rows $n-i$ and $n-i+1$. Otherwise,

$\rho_i(a) = -a$ (in G'_{i-1} if $a \in -S_i$ and vice versa). Consider what happens if we begin with an element of G'_1:

$$\rho_2\left(\begin{bmatrix} B_0 & & \\ & B_1 & \\ & & B_2 \end{bmatrix}\right) = -\begin{bmatrix} B_0 & & \\ & B_1 & \\ & & B_2 \end{bmatrix} \in -S_2.$$

If we start with an element of G'_1, there is always a corresponding element of $-S_2$: the same array but with a negative sign outside it. We could also start with an element of $-S_2$:

$$\rho_2\left(-\begin{bmatrix} B_0 & & \\ & B_0 & \\ & & B_1 \end{bmatrix}\right) = \begin{bmatrix} B_0 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in G'_1.$$

In that example, there was a corresponding element of G'_1. Sometimes, there isn't:

$$\rho_2\left(-\begin{bmatrix} & -b_2 & \\ -B_3 & & \\ & & B_0 \end{bmatrix}\right) = -\begin{bmatrix} B_3 & & \\ & +b_2 & \\ & & B_0 \end{bmatrix} \in -S_2.$$

Here we switched the rows (and signs!) of the entries in the first and second rows without changing the columns of these entries. Note that the resulting element of $-S_2$ does not have a corresponding element in G'_1 either (because the b_2 on that diagonal is not the one from B). It is clear that ρ_i is a sign-reversing involution.

Defining ρ'_i for $1 \le i \le n-1$

$\rho'_i : S_i - S'_i \to S_i - S'_i$, for $1 \le i \le n-1$, is defined as follows: If $a \in S_i$ and the entry in row $n-i+1$ is $\pm b_j$, then $\rho'_i(a) = a' \in S_i$ where a' is obtained by changing the sign of the entry in row $n-i+1$ of a and all other entries remain unchanged. Otherwise, $\rho'_i(a) = -a$ (in $-S'_i$ if $a \in S_i$ and vice versa). For example, if $n = 3$ and we start with an element of S_1,

$$\rho'_1\left(\begin{bmatrix} & & -b_3 \\ & B_0 & \\ -b_1 & & \end{bmatrix}\right) = \begin{bmatrix} & & -b_3 \\ & B_0 & \\ +b_1 & & \end{bmatrix} \in S_1.$$

Other elements of S_1 get mapped to elements of $-S'_1$ (and all elements of $-S'_1$ get mapped to elements of S_1):

$$\rho'_1\left(\begin{bmatrix} B_2 & & \\ & B_0 & \\ & & B_0 \end{bmatrix}\right) = -\begin{bmatrix} B_2 & & \\ & B_0 & \\ & & B_0 \end{bmatrix} \in -S'_1$$

and

$$\rho'_1\left(\begin{bmatrix} B_2 & & \\ & & -b_3 \\ & -B_0 & \end{bmatrix}\right) = -\begin{bmatrix} B_2 & & \\ & & -b_3 \\ & -B_0 & \end{bmatrix} \in -S'_1.$$

Defining κ_i for $1 \leq i \leq n-1$

For $1 \leq i \leq n-1$, $\kappa_i : S_i' - T_i \to S_i' - T_i$ works similarly to ρ_i. The difference is that now the column operation is the key. If we begin with an element of S_1', we get an element of $-T_1$:

$$\kappa_1 \left(\begin{bmatrix} B_2 & & \\ & B_0 & \\ & & B_2 \end{bmatrix} \right) = - \begin{bmatrix} B_2 & & \\ & B_0 & \\ & & B_2 \end{bmatrix} \in -T_1.$$

In fact, κ_i applied to an element of S_i' always gives the corresponding element of T_i: the same array, but with a negative sign. The reverse sometimes happens if we apply κ_i to an element of T_i:

$$\kappa_1 \left(- \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in S_1'.$$

However, if there is no corresponding element in S_i', we switch the columns of the entries in columns $n-i+1$ and $n-i$ (but not the signs this time, since the column operations were addition). For example,

$$\kappa_1 \left(- \begin{bmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & B_0 & \end{bmatrix} \in -T_1.$$

Here we switched the entries in columns 2 and 3, leaving them in their original rows. In general, we have: If $a \in -T_i$ and the entry in column $n-i$ is B_j in row $n-i+1$ or $-b_j$ where $j \geq n-i+1$, then $\kappa_i(a) = a' \in -T_i$ where a' is obtained from a by interchanging columns $n-i$ and $n-i+1$. For all other a, $\kappa_i(a) = -a$ (in S_i if $a \in -T_i$ and vice versa).

Defining κ_i' for $1 \leq i \leq n-2$

$\kappa_i' : T_i - T_i' \to T_i - T_i'$ works similarly to ρ_i' for $1 \leq i \leq n-1$. If $a \in T_i$ and the entry in row $n-i+1$ is in column $n-i$, then $\kappa_i'(a) = a' \in T_i$ where a' is obtained from a by changing the sign of the entry in the $(n-i+1, n-i)$ position. Otherwise, $\kappa_i(a) = -a$ (in T_i if $a \in -T_i'$ and vice versa). For example, starting with an element of T_1:

$$\kappa_1' \left(\begin{bmatrix} B_0 & & \\ & B_2 & \\ & & B_2 \end{bmatrix} \right) = - \begin{bmatrix} B_0 & & \\ & B_2 & \\ & & B_2 \end{bmatrix} \in -T_1'.$$

If we start with an element of T_1, there are two possibilities: either there is a corresponding array in T_1' as above, or else the array cancels via arithmetic within an entry, as in the next example:

$$\kappa_1'\left(\begin{bmatrix} & & {\scriptstyle -b_3} \\ {\scriptstyle -b_1} & & \\ & {\scriptstyle B_0} & \end{bmatrix}\right) = \begin{bmatrix} & & {\scriptstyle -b_3} \\ {\scriptstyle -b_1} & & \\ & {\scriptstyle -B_0} & \end{bmatrix} \in T_1.$$

Defining κ_{n-1}'

$\kappa_{n-1}' : T_{n-1} - T_{n-1}' \to T_{n-1} - T_{n-1}'$ is essentially the same as the previous κ_i's, except that now we set $B_j = b_j$ and cancel in row 1. If $a \in T_{n-1}$ and the entry in column 1 is anything other than B_0 in the upper-left corner of the matrix, then $\kappa_{n-1}'(a) = a' \in T_{n-1}$ where a' is obtained from a by changing the sign of the entry in column 1 and making all B_j lower-case. For all other a, $\kappa_{n-1}'(a) = -a' \in T_{n-1}'$ obtained by leaving all entries the same but making B_j lower-case.

$$\kappa_2'\left(\begin{bmatrix} {\scriptstyle B_3} & & \\ & {\scriptstyle B_0} & \\ & & {\scriptstyle B_2} \end{bmatrix}\right) = \begin{bmatrix} {\scriptstyle -b_3} & & \\ & {\scriptstyle b_0} & \\ & & {\scriptstyle b_2} \end{bmatrix} \in T_2,$$

$$\kappa_2'\left(\begin{bmatrix} {\scriptstyle -b_2} & & \\ & {\scriptstyle B_0} & \\ & & {\scriptstyle B_0} \end{bmatrix}\right) = \begin{bmatrix} {\scriptstyle b_2} & & \\ & {\scriptstyle b_0} & \\ & & {\scriptstyle b_0} \end{bmatrix} \in T_2, \text{ and}$$

$$\kappa_2'\left(\begin{bmatrix} & {\scriptstyle -b_2} & \\ {\scriptstyle -B_0} & & \\ & & {\scriptstyle B_0} \end{bmatrix}\right) = \begin{bmatrix} & {\scriptstyle -b_2} & \\ {\scriptstyle b_0} & & \\ & & {\scriptstyle b_0} \end{bmatrix} \in T_2.$$

Defining μ_i and μ_i' for $1 \leq i \leq n-1$

$\mu_i : T_i' - G_i \to T_i' - G_i$ reads the entries of the matrix and translates them into the digraph. The upper-left $(n-i) \times (n-i)$ corner represents (by the Matrix Tree Theorem) the spanning trees of D_i (in fact, these submatrices are obtained from the matrices Υ_i from the example in §3.1 by crossing out the zeroth row and column). When μ_i is applied to an element x in T_i', one of two things happens. Case 1: if edges are drawn from k to j for each b_j appearing in row $k \leq n-i$ in x (remembering that it is okay for j to be inside `blob`), and the resulting graph is a tree, then $\mu_i(x)$ is the pair whose first element is that tree, and whose second element is the i-tuple found by reading down the diagonal starting at row $n-i+1$.

Case 2: if those edges do not form a tree, then there is at least one cycle, and by moving the cycle with the largest element onto or off of the diagonal (according to where it already was), we find the element of T_i' that is $\mu_i(x)$. (Actually, there can only be one cycle, so we don't have to worry about which cycle to move). It is clear that for elements $x \in T_i'$ such that $\mu_i(x) \in T_i'$, μ_i acts as an involution. Define $\mu_i((\tau,\gamma))$ to be the element of T_i' found by putting b_j in the diagonal entry in row k whenever there is an edge in the tree $k \to j$, and filling in the rest of the diagonal entries from left to right by taking them from the code. Then it is clear that μ_i acts as an involution in the rest of the cases too.

If $\mu_i(x) \in -G_i$ (in other words, it is a $-(\tau,\gamma)$ pair), then step i of the overall procedure is finished. For example,

$$\mu_1\left(\left[\begin{smallmatrix} B_2 & & \\ & B_0 & \\ & & B_3 \end{smallmatrix}\right]\right) = -\left(1 \longrightarrow \boxed{\substack{3\\2}} \longrightarrow 0, (b_3)\right) \in -G_2,$$

which signifies that step 1 is finished. On the other hand,

$$\mu_1\left(\left[\begin{smallmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{smallmatrix}\right],$$

indicating that we will have to apply several more involutions before step 1 is done.

$\mu_i' : G_i - G_i' \to G_i - G_i'$ is essentially the negative of μ_i, since the sets T_i' and G_i' are identical. Moving from G_i into G_i' is the beginning of the $(i+1)^{\text{th}}$ step.

Defining ρ_n

The final involution, $\rho_n : G_{n-1}' - S_n \to G_{n-1}' - S_n$ matches arrays from C_{n-1}' to negative monomials that consist of the entries in order from left to right, similarly to ϕ_n' in the Happy Code.

$$\rho_n\left(\left[\begin{smallmatrix} b_0 & & & \\ & b_{k_2} & & \\ & & \ddots & \\ & & & b_{k_n} \end{smallmatrix}\right]\right) = b_0 b_{k_2} \ldots b_{k_n}.$$

For example,

$$\rho_3\left(\left[\begin{smallmatrix} b_0 & & \\ & b_3 & \\ & & b_0 \end{smallmatrix}\right]\right) = -b_0 b_3 b_0,$$

$$\rho_3\left(\begin{bmatrix} b_0 & & \\ & b_2 & \\ & & b_2 \end{bmatrix}\right) = -b_0 b_2^2, \text{ and}$$

$$\rho_3\left(-b_0 b_1 b_3\right) = \begin{bmatrix} b_0 & & \\ & b_1 & \\ & & b_3 \end{bmatrix}.$$

In fact, since all elements of G'_{n-1} are positive, and so are all elements of S_n, ρ_n is a simple bijection between the elements of G'_{n-1} and the elements of $-S_n$.

3.4 How to Find the Blob Code

We use these involutions the same way we did for the Happy Code.

Theorem 3 *Given the sets $G_0, G'_0, S_1, S'_1, T_1, T'_1, G_1, \ldots, G'_{n-1}, S_n$ and the sign-reversing involutions $\mu'_0, \rho_1, \rho'_1, \kappa_1, \kappa'_1, \mu_1, \mu'_1, \ldots, \kappa'_{n-1}, \mu_{n-1}, \mu'_{n-1}, \rho_n$, there is a bijection between G_0 (the set of trees) and S_n (the set of codes).*

Proof. The sets and involutions satisfy the conditions of Lemma 3. Thus we can construct the bijection between the set of trees and the set of codes. ☺

3.4.1 An example

To clarify the method, we use the matrix method to construct the Blob Code for the tree $2 \to 1 \to 3 \to 0 \in G_0$. It will help to remember that elements in each signed set can only have one of the involutions of types $\rho_i, \rho'_i, \kappa_i, \kappa'_i, \mu_i, \mu'_i$ applied to them, and we always alternate between negative identity maps and our defined involutions (μ, κ, and ρ, with indices and with or without primes).

Involution	Acts on
ρ_i	$G'_{i-1} - S_i$
ρ'_i	$S_i - S'_i$
κ_i	$S'_i - T_i$
κ'_i	$T_i - T'_i$
μ_i	$T'_i - G_i$
μ'_i	$G_i - G'_i$

This particular example is a sort of "worst case scenario" for a small graph, but for larger n such a long process would be more likely. Coding trees with no inversions is much easier, as is coding any tree that satisfies the condition that for every i whose path to 0 goes through some $j > i$, it also holds that $\mathtt{succ}(i) > i$.

Step 1

First we apply μ_0' to get an element of $-G_0'$:

$$\mu_0'(2 \to 1 \to 3 \to 0) = -\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in -G_0'.$$

Next we apply $\overline{I_{G_0'}}$:

$$\overline{-I_{G_0'}}\left(-\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in G_0'$$

As in the example for the Happy Code, we alternate between the involutions we defined, and the negative identity maps.

$$\overline{-I_{S_1}} \circ \rho_1\left(\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in S_1$$

$$\overline{-I_{S_1'}} \circ \rho_1'\left(\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in S_1'$$

$$\overline{-I_{T_1}} \circ \kappa_1\left(\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in T_1$$

$$\overline{-I_{T_1'}} \circ \kappa_1'\left(\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix} \in T_1'$$

Here is the first time the involutions do anything interesting:

$$\overline{-I_{T_1'}} \circ \mu_1\left(\begin{bmatrix} B_3 & & \\ & B_1 & \\ & & B_0 \end{bmatrix}\right) = -\begin{bmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{bmatrix} \in -T_1'$$

Since μ_1 doesn't move us out of T_1', the negative identity map results in a move to $-T_1'$. Note that any time we are in a negative set, we are moving "up" the sequence of matrices (or stalled where we are). Because the array did not correspond to a tree in the graph where 2 and 3 are identified, the basic effect of μ_1 at that step was to find the array with off-diagonal entries that cancels it in the matrix. ¿From $-T_1'$, we apply κ_i'.

$$\overline{-I_{T_1}} \circ \kappa_1' \left(- \begin{bmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{bmatrix} \right) = - \begin{bmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{bmatrix} \in -T_1$$

The next few involutions have the effect of switching columns:

$$\overline{-I_{T_1}} \circ \kappa_1 \left(- \begin{bmatrix} & -b_3 & \\ -b_1 & & \\ & & B_0 \end{bmatrix} \right) = \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & B_0 & \end{bmatrix} \in T_1$$

$$\overline{-I_{T_1}} \circ \kappa_1' \left(\begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & B_0 & \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \in -T_1$$

$$\overline{-I_{S_1'}} \circ \kappa_1 \left(- \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \in -S_1'$$

$$\overline{-I_{S_1}} \circ \rho_1' \left(- \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \in -S_1$$

And switching rows:

$$\overline{-I_{S_1}} \circ \rho_1 \left(- \begin{bmatrix} & & -b_3 \\ -b_1 & & \\ & -B_0 & \end{bmatrix} \right) = \begin{bmatrix} & & -b_3 \\ & B_0 & \\ b_1 & & \end{bmatrix} \in S_1$$

$$\overline{-I_{S_1}} \circ \rho_1 \left(\begin{bmatrix} & & -b_3 \\ & B_0 & \\ b_1 & & \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ & B_0 & \\ -b_1 & & \end{bmatrix} \in -S_1$$

$$\overline{-I_{G_0'}} \circ \rho_1 \left(- \begin{bmatrix} & & -b_3 \\ & B_0 & \\ -b_1 & & \end{bmatrix} \right) = - \begin{bmatrix} & & -b_3 \\ & B_0 & \\ -b_1 & & \end{bmatrix} \in -G_0'$$

We have defined the involutions in such a way that there is no passing the set G_i when moving up; we apply the Matrix Tree Theorem again (the effect, in this case, of μ_0'):

$$\overline{-I_{G_0'}} \circ \mu_0' \left(- \begin{bmatrix} & & -b_3 \\ & B_0 & \\ -b_1 & & \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in G_0'$$

And from here on, it's easy for the rest of the step:

$$\overline{-I_{S_1}} \circ \rho_1 \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in S_1$$

$$\overline{-I_{S_1'}} \circ \rho_1' \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in S_1'$$

$$\overline{-I_{T_1}} \circ \kappa_1 \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in T_1$$

$$\overline{-I_{T_1'}} \circ \kappa_1' \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in T_1'$$

$$\overline{-I_{G_1}} \circ \mu_1 \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \left(1 \longrightarrow \boxed{\begin{matrix} 2 \\ 3 \end{matrix}} \longrightarrow 0, (1) \right) \in G_1$$

Since we've gotten to G_1 and have a tree and a partial code, we are done with this step.

Step 2

Starting where we left off,

$$\overline{-I_{G_1'}} \circ \mu_1' \left(1 \longrightarrow \boxed{\begin{matrix} 2 \\ 3 \end{matrix}} \longrightarrow 0, (1) \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in G_1'$$

$$\overline{-I_{S_2}} \circ \rho_2 \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in S_2$$

$$\overline{-I_{S_2'}} \circ \rho_2' \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in S_2'$$

$$\overline{-I_{T_2}} \circ \kappa_2 \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in T_2$$

Now is the first time in Step 2 that we cannot move on to the next set, because for each of the above applications of involutions there was a corresponding element in the next set. The elements of T_2 can only be acted on by κ_2'.

$$\kappa_2' \left(\begin{bmatrix} B_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} -b_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in T_2.$$

$$\overline{-I_{T_2}}\left(\begin{bmatrix} -b_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} -b_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix} \in -T_2.$$

¿From $-T_2$, the involution κ_2 will either take us to S_2 or else leave us in $-T_2$ (in this case, the latter):

$$\kappa_2\left(-\begin{bmatrix} -b_3 & & \\ & B_0 & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} & -b_3 & \\ B_0 & & \\ & & B_1 \end{bmatrix} \in -T_2.$$

Another application of a negative identity map is now required as part of the algorithm of the Involution Principle.

$$\overline{-I_{T_2}}\left(-\begin{bmatrix} & -b_3 & \\ B_0 & & \\ & & B_1 \end{bmatrix}\right) = \begin{bmatrix} & -b_3 & \\ B_0 & & \\ & & B_1 \end{bmatrix} \in T_2.$$

Now we go back to the appropriate involution, κ_2' in this case:

$$\kappa_2'\left(\begin{bmatrix} & -b_3 & \\ B_0 & & \\ & & B_1 \end{bmatrix}\right) = \begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix} \in T_2.$$

$$\overline{-I_{T_2}}\left(\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix} \in -T_2.$$

$$\overline{-I_{S_2'}} \circ \kappa_2\left(-\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix} \in -S_2'.$$

$$\overline{-I_{S_2}} \circ \rho_2'\left(-\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix} \in -S_2.$$

Again we get stuck at a set. The involution ρ_2 should either send us to T_1' or leave us where we are, and it is the latter that occurs.

$$\rho_2\left(-\begin{bmatrix} & -b_3 & \\ -B_0 & & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} B_0 & & \\ & b_3 & \\ & & B_1 \end{bmatrix} \in -S_2$$

It is time for another negative identity map:

$$\overline{-I_{S_2}}\left(-\begin{bmatrix} B_0 & & \\ & b_3 & \\ & & B_1 \end{bmatrix}\right) = \begin{bmatrix} B_0 & & \\ & b_3 & \\ & & B_1 \end{bmatrix} \in S_2.$$

Since we are back in S_2, we apply ρ_2' followed by a negative identity map:

$$\overline{-I_{S_2'}} \circ \rho_2'\left(\begin{bmatrix} B_0 & & \\ & b_3 & \\ & & B_1 \end{bmatrix}\right) = -\begin{bmatrix} B_0 & & \\ & -b_3 & \\ & & B_1 \end{bmatrix} \in -S_2$$

$$\overline{-I_{G_1'}} \circ \rho_2 \left(- \begin{bmatrix} B_0 & & \\ & -b_3 & \\ & & B_1 \end{bmatrix} \right) = - \begin{bmatrix} B_0 & & \\ & -b_3 & \\ & & B_1 \end{bmatrix} \in -G_1'$$

This array does not correspond to a tree because there is a loop `blob` $\rightarrow 3$. So we toggle the diagonality of the cycle.

$$\overline{-I_{G_1'}} \circ \mu_1' \left(- \begin{bmatrix} B_0 & & \\ & -b_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in G_1'$$

Now we are all set to go through to the end of the step:

$$\overline{-I_{S_2}} \circ \rho_2 \left(\begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in S_2$$

$$\overline{-I_{S_2'}} \circ \rho_2' \left(\begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in S_2'$$

$$\overline{-I_{T_2}} \circ \kappa_2 \left(\begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in T_2$$

And we continue:

$$\overline{-I_{T_2'}} \circ \kappa_2' \left(\begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \in T_2'$$

$$\overline{-I_{G_2}} \circ \mu_2 \left(\begin{bmatrix} B_0 & & \\ & B_3 & \\ & & B_1 \end{bmatrix} \right) = \left(\begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix} \longrightarrow 0, (3,1) \right) \in G_2.$$

We are almost done, because the last step is always considerably shorter.

Step 3

¿From here we have

$$\overline{-I_{G_2'}} \circ \mu_2' \left(\begin{smallmatrix} 3 \\ 1\,2 \end{smallmatrix} \longrightarrow 0, (3,1) \right) = \begin{bmatrix} b_0 & & \\ & b_3 & \\ & & b_1 \end{bmatrix} \in G_2',$$

and finally,

$$\rho_3 \left(\begin{bmatrix} b_0 & & \\ & b_3 & \\ & & b_1 \end{bmatrix} \right) = -(b_0, b_3, b_1) \in -S_3.$$

Thus, the Blob Code for the tree $2 \rightarrow 1 \rightarrow 3 \rightarrow 0$ is (3,1).

Notice how the Blob Code differs from the Happy Code: we are constantly referring back to the altered graph. It turns out we need not use matrices at all.

Chapter 4

Tree Surgery for the Blob Code

A related algorithm for finding the Blob Code for a tree involves progressively identifying vertices, starting at n and ending with a blob-vertex consisting of all the vertices from 1 to n. As the blob grows, so does the code; meanwhile, the number of edges shrinks. The idea, as in the matrix method, is that if we consider our tree to be a spanning tree within the complete directed graph (with loops), every pair of vertices is identifiable. We keep track of the tree in the new graph that would correspond to our original tree. The difference is that now we ignore the matrices.

4.1 Tree Surgery Algorithm

The algorithm takes as its input a rooted tree (as a set of edges) whose vertices are the labels $\{0, 1, \ldots, n\}$. The algorithm uses a function $\texttt{path}(x)$ that finds the path (an ordered list of vertices) from x to 0, that is,

$$\texttt{path}(x) = (x, \texttt{succ}(x), \texttt{succ}(\texttt{succ}(x)), \ldots, 0).$$

Other procedures used are "remove edge" and "add edge."

Tree Surgery algorithm for the Blob Code

begin

$\texttt{blob} \leftarrow \{n\}$

$\texttt{code} \leftarrow ()$

$i \leftarrow 1$

repeat

if $\texttt{path}(n-i) \cap \texttt{blob} \neq \emptyset$ **then**

$\texttt{code} \leftarrow (\texttt{succ}(n-i), \texttt{code})$

remove edge $(n-i) \rightarrow \texttt{succ}(n-i)$

$\texttt{blob} \leftarrow \texttt{blob} \cup \{n-i\}$

else

$\texttt{code} \leftarrow (\texttt{succ}(\texttt{blob}), \texttt{code})$

remove edge $\texttt{blob} \rightarrow \texttt{succ}(\texttt{blob})$

add edge $\texttt{blob} \rightarrow \texttt{succ}(n-i)$

remove edge $(n-i) \rightarrow \texttt{succ}(n-i)$

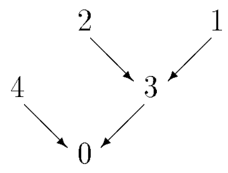
$\texttt{blob} \leftarrow \texttt{blob} \cup \{n-i\}$

$i \leftarrow i+1$

until $i = n$

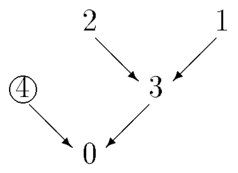
end.

Example:

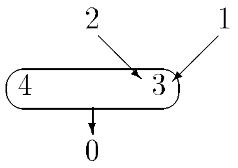


Beginning with this tree, we create a `blob` containing a single vertex (the one with the largest label).

Step 1

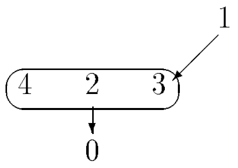


The blob contains only the vertex 4; $n - i = 3$ and `code` $= ()$. Does the path from 3 to 0 go through the blob? No. So we follow the **then** instructions. We take `succ(blob)`, which is 0, and put it at the beginning of the code, then delete that edge and add an edge from `blob` to `succ(3)` (which is 0). Then we delete the edge from 3 to 0 and put 3 into the blob. The new tree is:



Step 2

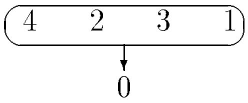
$n - i = 2$ and `code` $= (0)$. Since $i < n$, we continue. Does the path from 2 to 0 go through the blob? Yes. We follow the **else** in the algorithm. Put `succ(2)`, which is 3, at the beginning of the code, get rid of that edge and put 2 in the blob.



Step 3

Now $n - i = 1$ and `code` $= (3, 0)$. Since $i > 0$, we continue. Does the path from 1 to 0 go through the blob? Yes. Prepend `succ(1)`, which is 3 again, to the code, get rid of that edge and put 1 in the blob.

Now we are done. $i = n$ and `code` $= (3, 3, 0)$, and we stop. Here is the new tree:



To see what the tree algorithm (which doesn't even refer to matrices at all) has to do with the matrix method, we note that the i^{th} row of the initial matrix C_0' represents the possible edges out of i. Thus, a row operation that cancels most of the entries of that row obliterates the information of what the edge out of i was. This resembles the placing of i into the blob–since there is only one edge leaving the blob, we no longer know where the individual vertex i was pointing. However, the information is not entirely lost because the code-in-progress is still in the matrix. In fact, the row operation followed by the column operation corresponds directly to the blobbing of vertices and adding to the code.

More specifically, the relationship between the tree method and the matrix method is as follows: At the end of step i, we are in the set G_i. The matrix C_i' represents the graph with vertices $n-i, \ldots, n$ in the blob. The upper-left corner with $n-i$ rows and columns is the Matrix Tree Theorem matrix for that graph, and the i rows with nothing except B on the diagonal represent the set of possible codes-in-progress. If the path from $n-i$ to 0 does not pass through the `blob`, we follow the **else** at step i in the tree surgery algorithm, which corresponds to getting to pass through matrices easily from G_{i-1}' to G_i. If it does (ie, we follow the **then** at step i in the tree surgery algorithm), the matrix method will involve several bounces up and down within the matrices between S_i and T_i'.

4.2 Tree Surgery Is A Bijection

The tree surgery method is reversible. The inverse algorithm takes a code $(c_1, c_2, \ldots c_{n-1})$ and finds the corresponding tree:

Algorithm to go from Blob Code to Tree
begin

```
i ← 0
blob = {1, ..., n}
edges = {blob → 0}
repeat
        i ← i + 1
        blob ← blob \ {i}
        if path(c_1) ∩ blob ≠ ∅ then
           add edge i → c_1
        else
           add edge i → succ(blob)
           remove edge blob → succ(blob)
           add edge blob → c_1
        behead code
until i = n − 1
end.
```

It is easy to check that this algorithm undoes the Blob Code algorithm, one step at a time.

4.3 The Two Methods Give the Same Blob Code

Theorem 4 *The matrix method and the tree surgery method give the same Blob Code.*

Proof. We assume constant n and proceed by induction on the number of steps i taken so far. The base case is $i = 0$, the zeroth step. Before we do anything (using either method), we have a tree and an empty code. We consider the vertex n to be a `blob` containing only one label (n). At the end of the 0^{th} step, both methods have the same code-in-progress (namely, an empty code) and the same tree.

Now we assume that at the end of the $(i-1)^{\text{th}}$ step, the two methods result in the same tree and code-in-progress.

At the beginning of step i, each method has a pair consisting of a tree with a `blob` as one of the vertices and a partial code of length $(i-1)$. The `blob` contains $n-i+1, n-i+2, \dots, n$, so its size is i.

The matrix method requires following the involutions through sets of arrays. Rows $n-i$ through $n-i+1$ look like this in the sequence of matrices:

$$C'_{i-1} =$$

$$\begin{array}{c|c}
 & \begin{array}{ccccccc} 1 & \dots & n-i & n-i+1 & n-i+2 & \dots & n \end{array} \\ \hline
\begin{array}{c} \vdots \\ n-i \\ n-i+1 \\ n-i+2 \\ \vdots \end{array} &
\begin{bmatrix}
\vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\
-b_1 & \dots & B-b_{n-i} & -\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \dots & -b_n \\
-b_1 & \dots & -b_{n-i} & B-\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \dots & -b_n \\
0 & \dots & 0 & 0 & B & \dots & 0 \\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots
\end{bmatrix}
\end{array}$$

$$R_i =$$

$$\begin{bmatrix}
 & \vdots & \vdots & \vdots & & \vdots \\
\dots & B-b_{n-i} & -\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \dots & -b_n \\
\dots & -b_{n-i}-B+b_{n-i} & B-\sum_{n-i+1}^{n} b_k+\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k+\sum_{n-i+2}^{n} b_k & \dots & -b_n+b_n \\
\dots & 0 & 0 & B & \dots & 0 \\
 & \vdots & \vdots & \vdots & & \vdots
\end{bmatrix}$$

$$R'_i = \begin{bmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -b_1 & \ldots & B - b_{n-i} & -\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \ldots & -b_n \\ 0 & \ldots & -B & B & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & B & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

$$C_i = \begin{bmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -b_1 & \ldots & B - b_{n-i} - \sum_{n-i+1}^{n} b_k & -\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \ldots & -b_n \\ 0 & \ldots & -B + B & B & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & B & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

$$C'_i = \begin{bmatrix} \vdots & & \vdots & \vdots & \vdots & & \vdots \\ -b_1 & \ldots & B - \sum_{n-i}^{n} b_k & -\sum_{n-i+1}^{n} b_k & -\sum_{n-i+2}^{n} b_k & \ldots & -b_n \\ 0 & \ldots & 0 & B & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & B & \ldots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \end{bmatrix}$$

At step i in the matrix method, we are dealing with the sets G'_{i-1} (the set of trees (with a `blob` containing i labels) and partial codes of length $i-1$), S_i, S'_i, T_i, T'_i (the sets of arrays in the matrices above, respectively), and G_i (the set of trees with a `blob` containing $i+1$ labels together with partial codes of length i).

Suppose we are at the start of step i. This means that no matter which method we are using, we have a tree and a partial code. Let $\texttt{succ}(\texttt{blob}) = l$ and $\texttt{succ}(n-i) = k$. Also suppose that the first element in the partial code is b_m. Note that since we have a tree, $l \leq n-i$ because all vertices with labels greater than $n-i$ are in the `blob`.

An application of $\overline{-I_{S_i}} \circ \rho_i$ leaves us with $\begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in S_i$.

$$\overline{-I_{S_i'}} \circ \rho_i' \left(\begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in S_i'.$$

$$\overline{-I_{T_i}} \circ \kappa_i \left(\begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in T_i.$$

Note that these positive capitalized entries on the diagonal do not disappear from the matrices.

$$\overline{-I_{T_i'}} \circ \kappa_i' \left(\begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in T_i'.$$

Next we will be applying $\overline{-I_{G_i}} \circ \mu_i$, and there are two possible outcomes.

Case 1 Consider the case where the path from $n-i$ to 0 does not go through the `blob` (that is, k is not inverted in the original tree). If the path from k to 0 does not pass through the `blob`, then $\overline{-I_{G_i}} \circ \mu_i = (\tau, \gamma)$ where γ is the code from G_{i-1} with b_l prepended to it and τ is a tree containing the same edges as the tree from G_{i-1} with the following exceptions: $n-i \to k$ has been deleted, $n-i$ has been added to the `blob`, and the edge `blob` $\to l$ has been replaced by the edge `blob` $\to k$. This is a tree because if the path from $n-i$ to 0 does not pass through the `blob`, then moving the `blob` to the position where $n-i$ was does not create a cycle.

Note that the effect is exactly the same as the result of the tree surgery method. Tree surgery would have removed and added exactly those same edges, and prepended the same label to the code.

Case 2 This is the more complicated case. Here, when we apply μ_i, we don't get a tree because a cycle would be created (the path from $n-i$ to 0 goes through the `blob`, but now $n-i$ should be in the `blob` with `succ`(`blob`) $= k$. Hence there is a cycle containing `blob` and other vertices all of whose labels are less than $n-i$). Thus, $\overline{-I_{T_i'}} \circ \mu_i \left(\begin{bmatrix} \ddots & & & & \\ & B_k & & & \\ & & B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right)$ is a negative element in T_i' with all entries that correspond to edges in the cycle moved off the diagonal. In this matrix, row $n-i$ contains $-b_k$ in the k^{th} column; the rest of the off-diagonal entries are higher up in the matrix, including some unique entry in the $n-i$ column (say b_r, where $r \geq n-i$; this corresponds to an edge into `blob`). If $k > n-i$ (that is, `succ`$(n-i) \in$ `blob`), then the matrix will look a little different than the one below; we will deal with that case later.

Case 2a If $k < n-i$ ($k \neq n-i$ because then we would have a loop in the tree at the start of the step), we have

$$\overline{-I_{T_i'}} \circ \mu_i \left(\begin{bmatrix} \ddots & & & & \\ & B_k & & & \\ & & B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & & \\ & & & -b_r & & \\ & & \ddots & & & \\ & -b_k & & & & \\ & & & B_l & & \\ & & & & B_m & \\ & & & & & \ddots \end{bmatrix} \in -T_i'$$

Note that this $-b_r$ represents an edge into the `blob` and thus r can be any label greater than or equal to $n-i$. Also, there may be many vertices in the cycle that is now off the diagonal.

$\overline{-I_{T_i}} \circ \kappa_i'$ of this gives the same array in $-T_i$. However, $\overline{-I_{T_i}} \circ \kappa_i$ of *that* switches the entries in columns $n-i$ and $n-i+1$, leaving us with an element of T_i because R_i' only has $-b_{n-i}$ above the diagonal in column $n-i$. In some row above $n-i$, our array in T_i has $-b_r$ in column $n-i+1$; it also has B_l in the $(n-i+1, n-i)$

position; nothing else has moved (the $(n-i,k)$ position contains $-b_k$).

$$\overline{-I_{T_i}} \circ \kappa_i \left(- \begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & & & B_l & \\ & & & & & & B_m \\ & & & & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & B_l & & & \\ & & & & & B_m & \\ & & & & & & \ddots \end{bmatrix}$$

$\overline{-I_{T_i}} \circ \kappa_i'$ changes the sign of the B_l in row $n-i+1$, leaving us in $-T_i$. This new array appears in $-S_i'$ and $-S_i$ too: $\overline{-I_{S_i'}} \circ \kappa_i$ takes us to $-S_i'$ and $\overline{-I_{S_i}} \circ \rho_i'$ takes us to $-S_i$.

$$\overline{-I_{T_i}} \circ \kappa_i' \left(\begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & B_l & & & \\ & & & & & B_m & \\ & & & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & -B_l & & & \\ & & & & & B_m & \\ & & & & & & \ddots \end{bmatrix} \in -T_i;$$

$$\overline{-I_{S_i'}} \circ \kappa_i \left(- \begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & -B_l & & & \\ & & & & & B_m & \\ & & & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & & & \\ & & & & -b_r & & \\ & & & \ddots & & & \\ & -b_k & & & & & \\ & & & -B_l & & & \\ & & & & & B_m & \\ & & & & & & \ddots \end{bmatrix} \in -S_i';$$

and

$$\overline{-I_{S_i}} \circ \rho_i' \left(- \begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & -b_k & & & \\ & & -B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) =$$

$$- \begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & -b_k & & & \\ & & -B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -S_i.$$

Now we will end up switching the entries in rows $n-i$ and $n-i+1$: $\overline{-I_{S_i}} \circ \rho_i$ has this effect, with the result that our new array in S_i has B_l in the $(n-1)^{\text{th}}$ diagonal entry and b_k in the $(n-i+1,k)$ position.

$$\overline{-I_{S_i}} \circ \rho_i \left(- \begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & -b_k & & & \\ & & -B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & & B_l & & \\ & b_k & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in S_i.$$

An application of $\overline{-I_{S_i}} \circ \rho_i'$ changes the sign of the b_k in row $n-i+1$, putting us in $-S_i$:

$$\overline{-I_{S_i}} \circ \rho_i' \left(\begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & & B_l & & \\ & b_k & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & & & -b_r & \\ & & \ddots & & \\ & & B_l & & \\ & -b_k & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -S_i.$$

This same array appears in $-G'_{i-1}$ and is what we get by applying $\overline{-I_{G'_{i-1}}} \circ \rho_i$. Now when we apply $\overline{-I_{G'_{i-1}}} \circ \mu'_{i-1}$ we have a different cycle. Here, the graph in question has edges `blob` $\to k$ and $(n-i) \to l$ instead of vice versa. The off-diagonal entries must correspond to a cycle, so we move the cycle back onto the diagonal, landing in G'_{i-1}.

$$\overline{-I_{G'_{i-1}}} \circ \mu'_{i-1}\left(-\begin{bmatrix} \ddots & & & & & \\ & & & -b_r & & \\ & & \ddots\ B_l & & & \\ & -b_k & & & & \\ & & & & B_m & \\ & & & & & \ddots \end{bmatrix}\right) = \begin{bmatrix} \ddots & & & \\ & B_l & & \\ & & B_k & \\ & & & B_m \\ & & & & \ddots \end{bmatrix}.$$

Note that the only way this array differs from the one we started with at the very beginning of step i is that the entries in rows $n-i$ and $n-i+1$ have been interchanged.

Now when we apply $\rho_i, \rho'_i, \kappa_i, \kappa'_i$ with the appropriate negative identity maps in between, we eventually reach

$$\begin{bmatrix} \ddots & & & \\ & B_l & & \\ & & B_k & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in T'_i,$$

and then

$$\overline{-I_{G_i}} \circ \mu_i\left(\begin{bmatrix} \ddots & & & \\ & B_l & & \\ & & B_k & \\ & & & B_m \\ & & & & \ddots \end{bmatrix}\right),$$

which is a tree with edge `blob` $\to l$ (where $n-i$ is now in the `blob`) together with a code beginning with $(b_k, b_m, \dots)$. Since the tree surgery method would have deleted the edge from $n-i$ to k, placed $n-i$ in the `blob`, prepended b_k to the code and left the edge `blob` $\to l$, the matrix method had exactly the same effect.

<u>Case 2b</u> Here we treat separately the case where `succ`$(n-i) \in$ `blob`. In this case, we have

$$\overline{-I_{T'_i}} \circ \mu_i\left(\begin{bmatrix} \ddots & & & \\ & B_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix}\right) = -\begin{bmatrix} \ddots & & & \\ & -b_k & & \\ & & B_l & \\ & & & B_m \\ & & & & \ddots \end{bmatrix} \in -T'_i.$$

$$\overline{-I_{T_i}} \circ \kappa_i' \left(- \begin{bmatrix} \ddots & & & & \\ & -b_k & & & \\ & & B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & -b_k & & & \\ & & B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -T_i.$$

Now κ_i will switch the columns of two of the entries.

$$\overline{-I_{T_i}} \circ \kappa_i \left(- \begin{bmatrix} \ddots & & & & \\ & -b_k & & & \\ & & B_l & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in T_i.$$

$$\overline{-I_{T_i}} \circ \kappa_i' \left(\begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & -B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -T_i.$$

κ_i' is defined to change the sign of the entry B_l in row $n-i+1$, but nothing else in the array changes. This element also occurs in the sets $-S_i'$ and $-S_i$, so

$$\overline{-I_{S_i}} \circ \rho_i' \overline{-I_{S_i'}} \circ \kappa_i \left(- \begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & -B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & -B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -S_i.$$

In Case 2a we actually made it all the way up to the set G_{i-1}', but this time we do not; the next thing that happens is that the entries in rows $n-i$ and $n-i+1$ are interchanged, with the requisite sign changes:

$$\overline{-I_{S_i}} \circ \rho_i \left(- \begin{bmatrix} \ddots & & & & \\ & & -b_k & & \\ & -B_l & & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = \begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in S_i.$$

$$\overline{-I_{S_i}} \circ \rho_i' \left(\begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & -b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -S_i.$$

$$\overline{-I_{G_{i-1}'}} \circ \rho_i \left(- \begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & -b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \right) = - \begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & -b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in -G_{i-1}'$$

This array does not correspond to a tree because there is a loop `blob` $\to k$

$$\overline{-I_{G'_{i-1}}} \circ \mu'_{i-1}\left(-\begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & -b_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix}\right) = \begin{bmatrix} \ddots & & & & \\ & B_l & & & \\ & & B_k & & \\ & & & B_m & \\ & & & & \ddots \end{bmatrix} \in G'_{i-1}$$

Now we can go ahead and apply (with the obvious negative identity maps in between) $\rho_i, \rho'_i, \kappa_i$, and κ'_i, eventually ending up with this same array in the set T'_i. All of this had exactly the same effect that the manipulations in Case 2a did–namely, we interchanged the two entries on the diagonal, switching b_k with b_l. The same argument we used above shows that this had the same effect as the tree surgery method.

Since we have accounted for all possible cases, we conclude that these two methods give the same code at step i. Thus at step n the effect of the two methods is the same, so by induction the Blob Code can be found using either method. ☺

Chapter 5

Tree Surgery for the Happy Code

Considering that the matrix method did not refer back to the graph at each step, it is surprising that there is a purely bijective method for finding the Happy Code. In fact, we do have another form of tree surgery for the Happy Code, so we can avoid resorting to matrices and involutions.

5.1 Tree Surgery Algorithm

Begin by finding the path from 1 to 0. The method consists of deleting $\texttt{succ}(1)$ from the path and moving it to a separate connected component of the graph, and forming a cycle with it, then repeating the process. The algorithm corresponds directly to the matrix/involution algorithm of chapter 2. The algorithm below takes as its input a tree in the form of a set of edges.

The Tree Surgery Algorithm for Happy Code

begin

    $J \leftarrow \texttt{succ}(1)$

    if $J \neq 0$ **then**

        repeat

            $j \leftarrow \texttt{succ}(1)$

remove edge $1 \to j$

add edge $1 \to \texttt{succ}(j)$

if $j \geq J$ **then**

add edge $j \to j$

$J \leftarrow j$

else

add edge $j \to \texttt{succ}(J)$

remove edge $J \to \texttt{succ}(J)$

add edge $J \to j$

until $\texttt{succ}(1) = 0$

else

{the Happy Code is practically the same as the naïve code}

$\texttt{code} \leftarrow (\texttt{succ}(2), \texttt{succ}(3), \ldots, \texttt{succ}(n))$

end.

This algorithm turns out to be essentially equivalent to the matrix method shown in Chapter 2.

5.2 An Example

Consider the tree $1 \to 3 \to 2 \to 4 \to 0$. Step 1: pull $\texttt{succ}(1) = 3$ out of the path from 1 to 0 and put it in a cycle.

$$\begin{tikzcd} 1 \arrow[d] & \\ 2 \arrow[d] & 3 \arrow[loop right] \\ 4 \arrow[d] & \\ 0 & \end{tikzcd}$$

One nice thing about the Happy Code is that we don't have to keep track of the code as we go; we just read it off at the end. Step 2: pull 2 (the new $\texttt{succ}(1)$) out of the path from 1 to 0 and put it in a cycle. Since it is not the largest vertex in a cycle, we insert it after the largest (which is 3).

$$\begin{tikzcd} 1 \arrow[d] & 3 \arrow[r, shift left] & 2 \arrow[l, shift left] \\ 4 \arrow[d] & & \\ 0 & & \end{tikzcd}$$

The last step is to pull 4 out of the path from 1 to 0; it gets a loop because it is the largest element of the cycles.

$$\begin{tikzcd} & 3 \arrow[r, shift left] & 2 \arrow[l, shift left] \\ 1 \arrow[d] & & \\ 0 & 4 \arrow[loop right] & \end{tikzcd}$$

Now we can write down, in order, the successors of $2, 3, 4$ to find the code: $(3, 2, 4)$. Notice how much faster the tree surgery procedure is! Also, it is nice to know that it would be even faster if the path from 1 to 0 were shorter. Another nice feature of this method is that we no longer have to keep track of the code as we go; instead, we find it directly once we have finished performing surgery on the tree. The weight of the happy functional digraph at the end of the process is equal to the weight of the original tree.

If the tree were branchier, the method would not be any more complicated. Edges that are not part of the path from 1 to 0 are not affected by tree surgery; at the end of the surgical procedures the code is the list of the respective successors of all vertices ≥ 2.

This tree surgery method is related to Joyal's proof that there are $(n+1)^{n-1}$ trees. See §7.2 for a discussion.

5.3 Tree surgery is a bijection

Again, there is a simple inverse for the Happy Code tree surgery. We assume that we have a procedure that figures out which vertices are in cycles. The input is a code $(c_1, c_2, \ldots, c_{n-1})$.

Algorithm to go from Happy Code to Tree

begin

  $\texttt{edges} = \{1 \to 0\}$

  for $i = 2$ **to** n **do**

    add edge $i \to c_{i-1}$

  while $\texttt{cycles} \neq \emptyset$ **do**

    $J \leftarrow \max_{j \in \texttt{cycles}} j$

    $k \leftarrow \texttt{succ}(J)$

    add edge $J \to \texttt{succ}(k)$

    remove edge $J \to k$

    add edge $k \to \texttt{succ}(1)$

    remove edge $1 \to \texttt{succ}(1)$

    add edge $1 \to k$

end.

It is clear that this algorithm undoes the Happy Code tree surgery, one step at a time.

5.4 The Two Methods Give the Same Happy Code

5.4.1 A Lemma

In order to prove that the tree surgery method gives the same code as the matrix method, we will need the following lemma. The notion of a cycle being "active" or "inactive" is content-free. A cycle is "active" if we label it as active, and inactive otherwise. Actually we will see later that "active" corresponds to appearing off the diagonal in the matrix, and "inactive" corresponds to being on the diagonal.

Lemma 4 *The input for the following algorithm is an active loop at vertex L and an active cycle (which may also be a loop) containing at least one vertex greater than L. Let J be the largest element in the cycle. Then the output is the original cycle, now inactive, with L inserted between J and* $\texttt{succ}(J)$.

begin
    repeat
        $p \leftarrow$ largest vertex in an active cycle
        $q \leftarrow$ second-largest vertex in an active cycle
        $m \leftarrow \texttt{succ}(q)$
        add edge $q \rightarrow \texttt{succ}(p)$
        remove edge $p \rightarrow \texttt{succ}(p)$
        remove edge $q \rightarrow m$
        add edge $p \rightarrow m$
        toggle "activity" of the cycle containing J
    until there are no active cycles.
end.

Proof. We begin by noting that for a cycle of length c, the worst-case scenario is that each edge (other than $J \rightarrow \texttt{succ}(J)$) is an ascent and all vertices are larger than the one in the loop. For such a cycle, the algorithm terminates after $2^c - 1$ iterations. In fact, in this situation the iterative algorithm above is actually equivalent to a recursive algorithm. This is proven by induction.

The base case is that the cycle is a loop at J. This is an Escher cycle of length $c = 1$. The algorithm sets $p = J$, $q =$ the vertex of the loop, and $m = q$. It removes the loops and adds edges $J \rightarrow q$ and $q \rightarrow J$, then toggles the activity of the cycle containing J. There are no more active cycles and the algorithm has inserted q directly after J in its cycle. Furthermore it has taken $2^1 - 1 = 1$ step.

The induction hypothesis is that it takes $2^{c-1}-1$ steps to complete the algorithm if the cycle is an Escher cycle of length $c-1$, and that the result is that of inserting the loop vertex after J in the cycle.

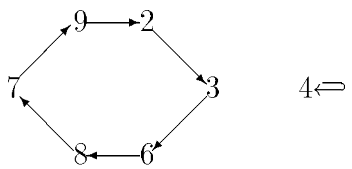
Now we consider an Escher cycle of length c containing only vertices larger than the loop vertex. Since each vertex of the cycle is larger than the loop vertex, the only way to be able to change the edge from the loop vertex (call it L) is to make all but one of the vertices in the cycle inactive.

This is a slow process. The first step of the algorithm removes J from the cycle, forming a loop which becomes inactive. Next, the second-largest vertex is removed, and J becomes active again. The following step will form a 2-cycle with these two vertices and make it inactive. The procedure continues until only the smallest vertex from the cycle (the original $\texttt{succ}(J)$) is left in a loop, with J and the rest of the vertices in an inactive cycle. By the induction hypothesis, this takes $2^{c-1} - 1$ steps because it is precisely the reverse of adding that smallest vertex to the cycle. The next step of the iterative algorithm switches the successors of L and the old $\texttt{succ}(J)$ and makes the rest of the vertices active again. The remaining steps merely undo all of the previous steps, with the exception that L has been inserted before the old $\texttt{succ}(J)$ in all the cycles containing it. The number of steps before we finish is thus $2 \cdot (2^{c-1} - 1) + 1 = 2^c - 1$. Furthermore, since L has been inserted before $\texttt{succ}(J)$, in the final cycle it appears right after J.

Thus we have the result in the case where the cycle is an Escher cycle all of whose vertices are greater than L. However, in fact any cycle reduces to an Escher cycle of vertices greater than L in the following way: any vertices smaller than L will never be affected by the edge switching, because L is active until the bitter end and is never the largest active vertex. So these vertices can be considered to be chained to their successors and thus do not effect the length of time the algorithm takes nor its effect. Furthermore, any vertices that fall in between a vertex and its nearest greater neighbor are also chained to their successors. ☺

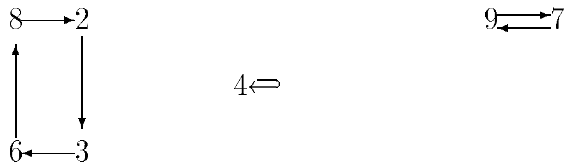
Example:

Step 0 <u>Active</u> <u>Inactive</u>



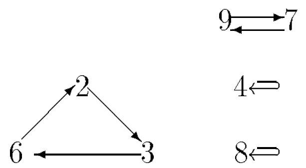
When we switch the successors of the two largest vertices, we replace the edges $9 \to 2$ and $8 \to 7$ by the edges $9 \to 7$ and $8 \to 2$. This breaks our cycle into two cycles, one of which is inactive:

Step 1 <u>Active</u> <u>Inactive</u>



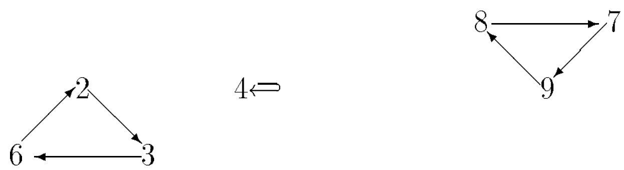
We repeat. We replace the edges $8 \to 2$ and $6 \to 8$ by the edges $6 \to 2$ and $8 \to 8$ (a loop) and reactivate the cycle containing 9.

Step 2 <u>Active</u> <u>Inactive</u>



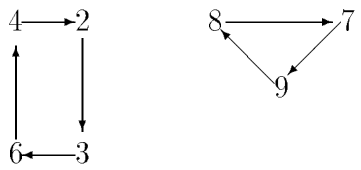
The two largest active vertices are 8 and 9, so 8 is inserted into 9's cycle.

Step 3 <u>Active</u> <u>Inactive</u>



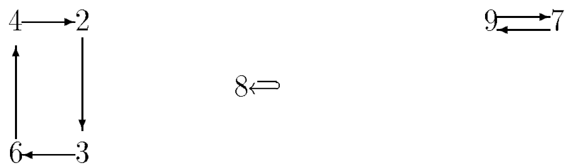
Now that the loop vertex, 4, is the second-largest active vertex, it gets inserted into the other active cycle. Note that it ends up inserted just before `succ`(9). This marks the approximate halfway point of the process. From now on we basically undo everything we did.

Step 4 Active Inactive



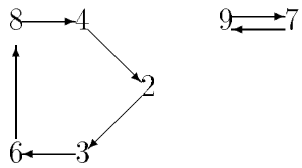
Now we switch the edges from 8 and 9, which has the effect of removing 8 from 9's cycle. Step 5 corresponds to Step 2, only with 4 inserted before 2 and the activity of 9's cycle toggled.

Step 5 Active Inactive



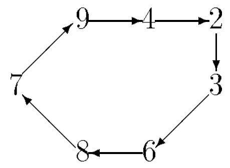
Now 8 will get inserted into the larger cycle, and 9's cycle is reactivated. Step 6 corresponds to Step 1, except that 4 has been inserted before 2 and the cycle containing 9 has the opposite activity.

Step 6 Active Inactive

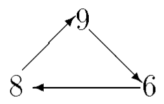


Step 7 corresponds to Step 0.

Step 7 Active Inactive



The end result is that of inserting 4 into the cycle, right after 9. It took, in this case, $7 = 2^3 - 1$ steps, because the cycle we started with is "equivalent" to the following Escher cycle with vertices larger than 4:



Note that 2 and 3 (the two vertices less than 4, our loop vertex) are "chained" together and to 6, and their outgoing edges never change. Meanwhile, 7 is "chained" to its successor, 9, because the edge into 7 is not an ascent.

5.4.2 The proof

Theorem 5 *The tree surgery method gives the same Happy Code as the matrix method.*

Proof. We assume constant n and proceed by induction on the length of the path from 1 to 0. The base case is the case where the tree includes the edge $1 \to 0$. In that case, the tree surgery method doesn't have to go through the **repeat** loop at all and the code is given by $(\mathtt{succ}(2), \mathtt{succ}(3), \ldots, \mathtt{succ}(n))$. The matrix method goes as follows for the base case: first, an application of $\overline{-I_{A_0'}} \circ \phi_0$ gives us an array with $B_{\mathtt{succ}(i)}$ in the i^{th} diagonal position. Let $j_i = \mathtt{succ}(i)$. Next,

$$\overline{-I_{A_1}} \circ \phi_0' \left(\begin{bmatrix} \lambda & & & & \\ & B_0 & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \right) = \begin{bmatrix} \lambda & & & & \\ & B_0 & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \in A_1.$$

Since the involutions have been defined in such a way that none of these diagonal entries ever get cancelled by a matrix operation, we have after many similar applications

$$\overline{-I_{A'_n}} \circ \phi_n \left(\begin{bmatrix} \lambda & & & & \\ & B_0 & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \right) = \begin{bmatrix} \lambda & & & & \\ & B_0 & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \in A'_n.$$

Now we have

$$\overline{-I_{A_{n+1}}} \circ \phi'_n \left(\begin{bmatrix} \lambda & & & & \\ & B_0 & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \right) = B_0 B_{j_2} \ldots B_{j_n} \in A_{n+1}.$$

$$\phi_{n+1} \left(B_0 B_{j_2} \ldots B_{j_n} \right) = -b_0 B_{j_2} \ldots B_{j_n} \in -A'_{n+1}.$$

Here, since $j_i = \texttt{succ}(i)$, we end up with the same code we got by tree surgery. Thus the base case is true.

Our induction hypothesis is that the two methods give the same code for all happy functional digraphs where the path from 1 to 0 is of length $i-1$. We show that if we start with a functional digraph whose path from 1 to 0 is of length i, both methods will manipulate the graph into one with a shorter path from 1 to 0.

The length of the path from 1 to 0 is i. As we start, we have an array with all entries on the diagonal. We will automatically (as in the base case) make it down to A_{n+1} by a sequence of involutions with no complications, because none of these diagonal entries get cancelled in the row operation arithmetic. Let $\texttt{succ}(i) = j_i$

and $j_1 = r$. Then

$$\overline{-I_{A_{n+1}}} \circ \phi'_n \circ \cdots \circ \overline{-I_{A'_0}} \circ \phi_0 \left(\begin{bmatrix} \lambda & & & & \\ & B_r & & & \\ & & B_{j_2} & & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \right) = B_r B_{j_2} \ldots B_{j_n} \in A_{n+1}.$$

Now since $r \neq 0$,

$$\phi_{n+1}(B_r B_{j_2} \ldots B_{j_n}) = -B_{j_r} B_{j_2} \ldots b_r \ldots B_{j_n} \in A_{n+1},$$

and

$$\overline{-I'_{A_n}} \circ \phi'_n \circ \overline{-I_{A_{n+1}}} (-B_{j_r} B_{j_2} \ldots b_r \ldots B_{j_n}) = - \begin{bmatrix} & & & -b_r & & \\ & B_{j_r} & & & & \\ & & \ddots & & & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{j_n} \end{bmatrix} \in -A'_n.$$

There will be no problem in applying involutions and we will move swiftly through the sequence of sets $-A'_n, -A_n, -A'_{n-1}, -A_{n-1}, \ldots$ until we reach the one where the λ first appears in this (the r^{th}) row.

$$\overline{-I_{A_{n-r+1}}} \circ \phi'_{n-r} \left(- \begin{bmatrix} & B_{j_r} & & -b_r & \\ & & \ddots & & \\ -\lambda & & & & \\ & & & \ddots & B_{j_n} \end{bmatrix} \right) = \begin{bmatrix} \lambda & & & & \\ & B_{j_r} & & & \\ & & \ddots & +b_r & \\ & & & \ddots & \\ & & & & B_{j_n} \end{bmatrix} \in A_{n-r+1}.$$

Now that we are in A_{n-r+1}, we apply ϕ_{n-r+1}:

$$\overline{-I_{A_{n-r+1}}} \circ \phi_{n-r+1}\left(\begin{bmatrix} \lambda & B_{jr} & & & \\ & & \ddots & & \\ & & & {}_{+b_r} & \\ & & & & \ddots \\ & & & & B_{jn} \end{bmatrix}\right) = \\ - \begin{bmatrix} \lambda & B_{jr} & & & \\ & & \ddots & & \\ & & & {}_{-b_r} & \\ & & & & \ddots \\ & & & & B_{jn} \end{bmatrix} \in -A_{n-r+1}.$$

This array appears in all of the previous matrices, so we get all the way back up to A_0'. ϕ_0 toggles the diagonality of the cycle with the largest element. Note that so far, what has happened is that we have switched the successors for 1 and r. In other words, we have removed r from the path from 1 to 0, and created a loop at r; 1 now points directly at what used to be after r on the path to 0.

Case 1 If r is the largest vertex in a cycle,

$$\overline{-I_{A_0'}} \circ \phi_0\left(-\begin{bmatrix} \lambda & B_{jr} & & & \\ & & \ddots & & \\ & & & {}_{-b_r} & \\ & & & & \ddots \\ & & & & B_{jn} \end{bmatrix}\right) = \begin{bmatrix} \lambda & B_{jr} & & & \\ & & \ddots & & \\ & & & B_r & \\ & & & & \ddots \\ & & & & B_{jn} \end{bmatrix} \in A_0'.$$

We are done because the effect of the tree surgery method would have been exactly the same: we would have removed r from the path from 1 to 0, and created a loop on it. By the induction hypothesis, the two methods will give the same code because the path from 1 to 0 now has length $i-1$.

Otherwise, we probably still have a long way to go.

Case 2 If the largest element in a cycle is not r, then tree surgery has the effect of inserting r after the largest element in a cycle, J.

In this case, the application of ϕ_0 will move another cycle off the diagonal. Our new element of A'_0 looks like this:

$$\begin{bmatrix} \lambda & & & & & & & \\ & B_{j_r} & & & & & & \\ & & \ddots & & & & & \\ & & & -b_r & & & & \\ & & & & \ddots & & & \\ & & -b_k & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix},$$

where k is one of the vertices in the new off-diagonal cycle, and $k = \texttt{succ}(J)$ where J is the largest vertex in a cycle. ($-b_k$ is in row J.) Nothing interesting happens with the involutions until we reach A_{n-J+1}:

$$\overline{-I_{A_{n-J+1}}} \circ \phi_{n-J+1}\left(\begin{bmatrix} \lambda & & & & & & \\ & B_{j_r} & & & & & \\ & & \ddots & & & & \\ & & & -b_r & & & \\ & & & & \ddots & & \\ & & -b_k & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$- \begin{bmatrix} \lambda & & & & & & \\ & B_{j_r} & & & & & \\ & & \ddots & & & & \\ & & & -b_r & & & \\ & & & & \ddots & & \\ & & +b_k & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in -A_{n-J+1}.$$

$$\overline{-I_{A_{n-J+1}}} \circ \phi'_{n-J}\left(\begin{bmatrix} \lambda & & & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & & -b_r & & \\ & & & & \ddots & \\ & +b_k & & & & \\ & & & & \ddots & \\ & & & & & B_{jn} \end{bmatrix}\right) =$$

$$\begin{bmatrix} & & -b_k & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_r & & & \\ & & & & \ddots & & \\ -\lambda & & & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in A_{n-J+1}.$$

¿From here we can move through $A_{n-J+1}, A'_{n-J+1}, \ldots$ until we reach the next set where an off- diagonal entry disappears. The next time it happens depends on how far down in the matrix M_{n_J+1} the row with the next off-diagonal entry appears.

<u>Case 2a</u> If r is the second-largest vertex in an off-diagonal cycle, then we're in business.

$$\overline{-I_{A_{n-r+1}}} \circ \phi_{n-r+1}\left(\begin{bmatrix} & & -b_k & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_r & & & \\ & & & & \ddots & & \\ -\lambda & & & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$-\begin{bmatrix} & & -b_k & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & +b_r & & & \\ & & & & \ddots & & \\ -\lambda & & & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in -A_{n-r+1}.$$

Now applications of involutions will switch the entries in row 0 and row r:

$$\overline{-I_{A_{n-r+1}}} \circ \phi'_{n-r}\left(-\begin{bmatrix} & & -b_k & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & & +b_r & & \\ & & & & \ddots & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{jn}\end{bmatrix}\right) =$$

$$\begin{bmatrix} & & -b_r & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & +b_k & & & \\ & & & & \ddots & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{jn}\end{bmatrix} \in A_{n-r+1}.$$

But we still don't get to move on to a matrix set with a larger subscript:

$$\overline{-I_{A_{n-r+1}}} \circ \phi_{n-r+1}\left(\begin{bmatrix} & & -b_r & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & +b_k & & & \\ & & & & \ddots & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{jn}\end{bmatrix}\right) =$$

$$-\begin{bmatrix} & & -b_r & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & -b_k & & & \\ & & & & \ddots & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{jn}\end{bmatrix} \in -A_{n-r+1}$$

Note that now we are headed up (toward A'_0) again. The next interesting involution occurs when we again have an element of the set where $-\lambda$ first appears in the J^{th}

row.

$$\overline{-I_{A_{n-J+1}}} \circ \phi'_{n-J}\left(- \begin{bmatrix} & & -b_r & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & -b_k & & & \\ & & & \ddots & & \\ -\lambda & & & & & \\ & & & & \ddots & \\ & & & & & B_{jn} \end{bmatrix} \right) =$$

$$\begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_k & & & \\ & & & & \ddots & & \\ & & +b_r & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in A_{n-J+1}.$$

Now b_r is in the J^{th} row. After changing its sign we will continuing applying involutions whose images are in sets with decreasing subscripts:

$$\overline{-I_{A_{n-J+1}}} \circ \phi_{n-J+1}\left(\begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_k & & & \\ & & & & \ddots & & \\ & & +b_r & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \right) =$$

$$- \begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_k & & & \\ & & & & \ddots & & \\ & & -b_r & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in -A_{n-J+1}$$

and now we'll make it all the way back up to A_0 without interruption. When we get there, we note that now the only difference in our graph is that $\texttt{succ}(J) = r$ instead of k, and k is now in row r so $\texttt{succ}(r) = k$. In fact, we have inserted r after the largest vertex in a cycle without changing anything else about the graph–exactly what would've happened in the tree surgery method. Since all off-diagonal entries are now in the same cycle with J, we have

$$\overline{-I_{A_0'}} \circ \phi_0 \left(- \begin{bmatrix} \lambda & & & & & \\ & B_{jr} & & & & \\ & & \ddots & & & \\ & & -b_k & & & \\ & & & \ddots & & \\ & & -b_r & & & \\ & & & & \ddots & \\ & & & & & B_{jn} \end{bmatrix} \right) =$$

$$\begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & B_k & & & \\ & & & & \ddots & & \\ & & & & & B_r & \\ & & & & & & \ddots \\ & & & & & & & B_{jn} \end{bmatrix} \in A_0'.$$

By the induction hypothesis, from here (a graph where the path from 1 to 0 is of length $(i-1)$) we know that the two methods give the same code.

Case 2b However, if r is not the second-largest vertex in a cycle, the procedure is a bit longer. In general, Case 2 started with

$$\begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & & & \\ & & & -b_r & & & \\ & & & & \ddots & & \\ & & -b_k & & & & \\ & & & & & \ddots & \\ & & & & & & B_{jn} \end{bmatrix} \in A_0'.$$

The result of applying the first bunch of involutions before we end up back at A_0' again is to switch the rows of the lowest (meaning their row indices are largest) two off-diagonal entries. Let l be the second largest vertex in the cycle containing J, and let $m = \texttt{succ}(l)$. So our starting matrix actually looks something like this (although it is possible that $\texttt{succ}(l) = m = J$):

$$
\begin{bmatrix}
\lambda & & & & & & & \\
& B_{j_r} & & & & & & \\
& & \ddots & & & -b_l & & \\
& & & -b_r & & & & \\
& & & & \ddots & & -b_J & \\
& & -b_m & & & & & \\
& & & & & \ddots & & \\
& & & & -b_k & & & \\
& & & & & & \ddots & \\
& & & & & & & B_{j_n}
\end{bmatrix} \in A_0'.
$$

As before, we can get down to A_{n-J+1} uneventfully, but then interesting things happen:

$$
\overline{-I_{A_{n-J+1}}} \circ \phi_{n-J+1}\left(\begin{bmatrix}
\lambda & & & & & & & \\
& B_{j_r} & & & & & & \\
& & \ddots & & & -b_l & & \\
& & & -b_r & & & & \\
& & & & \ddots & & -b_J & \\
& & -b_m & & & & & \\
& & & & & \ddots & & \\
& & & & -b_k & & & \\
& & & & & & \ddots & \\
& & & & & & & B_{j_n}
\end{bmatrix}\right) =
$$

$$
- \begin{bmatrix}
\lambda & & & & & & & \\
& B_{j_r} & & & & & & \\
& & \ddots & & & -b_l & & \\
& & & -b_r & & & & \\
& & & & \ddots & & -b_J & \\
& & -b_m & & & & & \\
& & & & & \ddots & & \\
& & & & +b_k & & & \\
& & & & & & \ddots & \\
& & & & & & & B_{j_n}
\end{bmatrix} \in -A_{n-J+1}.
$$

$$\overline{-I_{A_{n-J+1}}} \circ \phi'_{n-J}\left(\begin{bmatrix} \lambda & & & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & -b_m & & & & & \\ & & & & & \ddots & & \\ & & & & +b_k & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$\begin{bmatrix} & & & & -b_k & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & -b_m & & & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in A_{n-J+1}.$$

Now we apply ϕ_{n-J+1}, and have no further interruptions until we reach A_{n-l+1}:

$$\overline{-I_{A_{n-l+1}}} \circ \phi_{n-l+1}\left(\begin{bmatrix} & & & & -b_k & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & -b_m & & & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$-\begin{bmatrix} & & & & -b_k & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & +b_m & & & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in -A_{n-l+1}.$$

Here, the lowest off-diagonal entry in the matrix was $-b_m$ in row l, so it changed sign; now we apply ϕ'_{n-l}.

$$\overline{-I_{A_{n-l+1}}} \circ \phi'_{n-l}\left(-\begin{bmatrix} & & & & -b_k & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & +b_m & & & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \right) =$$

$$\begin{bmatrix} & & -b_m & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & +b_k & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in A_{n-l+1}.$$

Positive off-diagonal entries never survive. We apply ϕ_{n-l+1}:

$$\overline{-I_{A_{n-l+1}}} \circ \phi_{n-l+1}\left(\begin{bmatrix} & & -b_m & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & +b_k & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \right) =$$

$$-\begin{bmatrix} & & -b_m & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & -b_k & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in -A_{n-l+1}.$$

This array will take us back up to A_{n-J+1} (this should remind you of what happened in Case 2a).

$$\overline{-I_{A_{n-J+1}}} \circ \phi'_{n-J}\left(- \begin{bmatrix} & & -b_m & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & -b_k & & & \\ & & & & & \ddots & & \\ -\lambda & & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$\begin{bmatrix} \lambda & & & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & -b_k & & & \\ & & & & & \ddots & & \\ & +b_m & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in A_{n-J+1}.$$

And our last little side trip:

$$\overline{-I_{A_{n-J+1}}} \circ \phi_{n-J+1}\left(\begin{bmatrix} \lambda & & & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & -b_k & & & \\ & & & & & \ddots & & \\ & +b_m & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix}\right) =$$

$$- \begin{bmatrix} \lambda & & & & & & & \\ & B_{jr} & & & & & & \\ & & \ddots & & & -b_l & & \\ & & & -b_r & & & & \\ & & & & \ddots & & -b_J & \\ & & & & -b_k & & & \\ & & & & & \ddots & & \\ & -b_m & & & & & & \\ & & & & & & \ddots & \\ & & & & & & & B_{jn} \end{bmatrix} \in -A_{n-J+1}.$$

This last matrix appears in all of the previous sets. Thus we pass through a number of sets, finally arriving at

$$
-\begin{bmatrix}
\lambda & & & & & & & & \\
& B_{j_r} & & & & & & & \\
& & \ddots & & & -b_l & & & \\
& & & -b_r & & & & & \\
& & & & \ddots & & -b_J & & \\
& & & & -b_k & & & & \\
& & & & & \ddots & & & \\
& & -b_m & & & & & & \\
& & & & & & \ddots & & \\
& & & & & & & & B_{j_n}
\end{bmatrix} \in -A_0'.
$$

This is where we apply ϕ_0. Unfortunately, this time we are not as lucky as in Case 2a, where everything moved back on diagonal. Note that this new matrix corresponds to a graph that differs from the one at the start of Case 2 by only 2 edges–namely, we have switched the successors of J and l, the two largest vertices in the cycle containing J. Necessarily we now have three cycles; everything in between J and l has been shorted out and forms its own cycle. Now when we apply ϕ_0, we move the cycle containing J onto the diagonal. This corresponds to the cycle containing J being considered inactive.

$$\overline{-I_{A_0'}} \circ \phi_0 \left(- \begin{bmatrix} \lambda & & & & & & \\ & B_{jr} & & & & & \\ & & \ddots & & -b_l & & \\ & & & -b_r & & & \\ & & & & \ddots & -b_J & \\ & & & & -b_k & & \\ & & & & & \ddots & \\ & & -b_m & & & & \\ & & & & & & \ddots \\ & & & & & & & B_{jn} \end{bmatrix} \right) =$$

$$\begin{bmatrix} \lambda & & & & & & & & & \\ & B_{jr} & & & & & & & & \\ & & \ddots & & & -b_l & & & & \\ & & & -b_r & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & B_J & & & & \\ & & & & & & \ddots & & & \\ & & & & -b_k & & & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & B_m & \\ & & & & & & & & & \ddots \\ & & & & & & & & & & B_{jn} \end{bmatrix} \in A_0'$$

The cycle containing J is no longer off the diagonal, so nothing will happen to it as we move down the sequence of matrices (hence the notion of it being inactive). If we let p be the largest vertex in a cycle that appears off the diagonal at this stage, and q be the second-largest, then essentially the same procedure we just finished will be duplicated, only with p as the lowest row with an off-diagonal entry. Each time we do this, the lowest two off-diagonal entries in the array are switched, and we toggle the diagonality ("activity") of the cycle containing J (which will change as we go). But the effect of this switching in the matrix is the trading of successors for p and q at each stage, and thus this matrix process is equivalent to the graph surgery from Lemma 4

Now we appeal to the lemma. The effect of this huge process is to insert r after J in the cycle containing J. Furthermore, since r has been removed from the path joining 1 to 0, the new happy functional digraph has a shorter path. Thus, by induction, the tree surgery method and the matrix method give the same code. ☺

Chapter 6

The Dandelion Code

The method for this code is sort of a mélange of the methods of the Happy Code and the Blob Code. As we did for the Blob Code, we consider the $n \times n$ submatrix obtained from $\hat{D} - \hat{A}$ by crossing out the zeroth row and column, and we apply the Matrix Tree Theorem at every possible opportunity. However, following the method of the Happy Code, we only do row operations and we always subtract the top row. We will again use B to denote $\sum_0^n B_j$.

The matrix we start with, with rows and columns indexed from 1 to n, is

$$N_0' = \begin{bmatrix} B - b_1 & -b_2 & \dots & -b_n \\ -b_1 & B - b_2 & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots \\ -b_1 & -b_2 & \dots & B - b_n \end{bmatrix}.$$

We will subtract the first row from each of the other rows in turn, in the usual way: from the bottom up, with cancellation being a separate step.

$$N_1 = \begin{bmatrix} B - b_1 & -b_2 & \dots & -b_n \\ -b_1 & B - b_2 & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots \\ -b_1 - B + b_1 & -b_2 + b_2 & \dots & B - b_n + b_n \end{bmatrix},$$

and we cancel only terms in the n^{th} row at this point.

$$N_1' = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_n \\ -b_1 & B-b_2 & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots \\ -B & 0 & \dots & B \end{bmatrix}.$$

Next we subtract the first row from the $(n-1)^{\text{th}}$ row, and we continue; at step i, we subtract row 1 from row $(n-i+1)$, until we reach the last matrix:

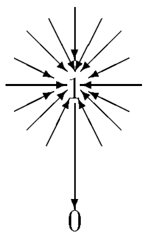
$$N_{n-1}' = \begin{bmatrix} B-b_1 & -b_2 & -b_3 & \dots & -b_n \\ -B & B & 0 & \dots & 0 \\ -B & 0 & B & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -B & 0 & 0 & \dots & B \end{bmatrix}.$$

It may not be clear by inspection what $\det N_{n-1}'$ is, but we already know the answer because of the Matrix Tree Theorem.

6.1 The Sets

The sequence of sets is $F_0, F_0', D_1, D_1', F_1, \dots, F_{n-1}'$. F_0 is the set of trees in the original graph; F_0' is the set of arrays from N_0'. For $1 \le i \le n-1$, the set F_i is the set of spanning trees in an altered graph. The altered graph at step i has the same edges out of $1, 2, \dots, n-i$ as the original graph, and each of the vertices $n-i+1, \dots, n$ has multiple edges pointing to 1 with certain weights, but no edges to any other vertex. Specifically, at step i, we replace the edge $n-i+1 \to j$ in the graph at step i by an edge $n-i+1 \to 1$ with weight B_j, for each j. After all, if we apply the Matrix Tree Theorem to N_1' (for example), we see that row n represents edges $n \to j$ and has mostly zeroes, implying that there is no edge from n to any vertex besides 1. If an off-diagonal entry is a sum, it corresponds to multiple edges, each with monomial weight. So for $1 \le i \le n-1$, F_i is the set of spanning trees in the altered graph corresponding to the matrix at step N_i'. D_i is the set of arrays from N_i and D_i' is the set of arrays from N_i'. For $1 \leq i \leq n-2$, $F_i' = D_i'$. Finally, F_{n-1}' is the set of codes.

Note that the last graph, whose spanning trees make up F_{n-1}, has $n+1$ monomial-weighted edges of the form $k \rightarrow 1$ for each $k = 2, \ldots, n$, and one edge $1 \rightarrow k$ for each $k = 0, 1, 2, \ldots, n$. However, since any spanning tree rooted at 0 must contain an edge into 0, we know that the only possible edge out of 1 that can occur in a spanning tree is the edge $1 \rightarrow 0$, so that F_{n-1} can also be thought of as the set of spanning trees of the graph below.



This picture should enlighten the reader as to the name for this Code.

6.2 The Involutions

The involutions are defined very similarly to the ones for the Blob Code.

As usual, the first involution, $\mu_0' : F_0 - F_0' \rightarrow F_0 - F_0'$, takes each tree in the original graph to the corresponding array in N_0', and pairs up the extra arrays according to toggling the diagonality of the cycle containing the greatest element.

For $1 \leq i \leq n-1$, $\mu_i : D_i' - F_i \rightarrow D_i' - F_i$ is the involution of the bijective proof of the Matrix Tree Theorem, which matches each positive array from N_i' (that is, each element of $(D_i')^+$) to a negative tree in $-F_i$. Meanwhile, for $0 \leq i \leq n-2$, $\mu_i' : F_i - F_i' \rightarrow F_i - F_i'$ is essentially the negative of map μ_i; it matches trees and arrays in the same way but with opposite signs.

For $1 \leq i \leq n-1$, $\xi_i : F_{i-1}' - D_i \rightarrow F_{i-1}' - D_i$ is the involution that matches arrays to one another according to the row operation. Thus if $a \in -D_i$ and the entry in row $n-i+1$ is $+b_j$ or $-B_j$, then $\xi_i(a) = a' \in -D_i$ where a' is obtained by interchanging and negating rows 1 and $n-i+1$. For all other $a \in F'_{i-1} - D_i$, $\xi_i(a) = -a$ (in $-D_i$ if $a \in F'_{n-1}$ and vice versa).

For $1 \leq i \leq n-1$, $\xi'_i : D_i - D'_i \to D_i - D'_i$ is the involution that matches up arrays according to the arithmetic within entries in row $n-i+1$. If $a \in D_i$ and the entry in row $n-i+1$ is $\pm b_j$, then $\xi'_i(a) = a' \in D_i$ where a' is obtained from a by changing the sign of the entry in row $n-i+1$; for all other a, $\xi'_i(a) = -a$ (in D_i if $a \in -D'_i$ and vice versa).

The final involution $\hat{\mu}_{n-1} : F_{n-1} - F'_{n-1} \to F_{n-1} - F'_{n-1}$ matches trees to codes. The code for a tree is given by the weights of the outgoing edges from vertices $2, 3, \ldots, n$ in order. Thus if the weight of the edge $i \to 1$ in the tree τ is w_i for each $i = 2, 3, \ldots n$, then $\hat{\mu}_{n-1}(\tau) = (w_2, w_3, \ldots, w_n)$.

6.3 How to Find the Dandelion Code

Theorem 6 *Given the sets $F_0, F'_0, D_1, D'_1, F_1, \ldots, F'_{n-1}$ and the sign-reversing involutions $\mu'_0, \xi_1, \xi'_1, \mu_1, \mu'_1, \ldots, \xi'_{n-1}, \hat{\mu}_{n-1}$, we can construct the bijection between F_0 (trees in our original graph) and D_n (codes).*

Proof. Again, our sets and involutions satisfy the hypotheses of Lemma 3, so we can construct the bijection. ☺

6.3.1 An example

For $n = 4$, consider the tree $1 \to 3 \to 4 \to 2 \to 0 \in F_0$. First, the Matrix Tree Theorem tells us what array corresponds to this tree.

$$\mu'_0(1 \to 3 \to 4 \to 2 \to 0) = -\begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix} \in -F'_0.$$

And the obligatory negative identity map:

$$\overline{-I_{F'_0}}\left(-\begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix} \in F'_0.$$

By now this is child's play.

$$\overline{-I_{D_1}} \circ \xi_1\left(\begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix} \in D_1.$$

$$\overline{-I_{D_1'}} \circ \xi_1'\left(\begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix}\right) = \begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix} \in D_1'.$$

Now it gets slightly tricky. This array does not correspond to a tree in the graph where 4 only has edges pointing at 1, because it represents the following functional digraph:

$$\begin{tikzcd} & & 1 \arrow[dl] & \\ 2 \arrow[d] & 3 \arrow[rr] & & 4 \arrow[ul, "B_2"'] \\ 0 & & & \end{tikzcd}$$

The next step is to apply μ_1 to the array; the cycle gets moved off the diagonal:

$$\overline{-I_{D_1'}} \circ \mu_1\left(\begin{bmatrix} B_3 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_2 \end{bmatrix}\right) = -\begin{bmatrix} & & -b_3 & \\ & B_0 & & \\ & & & -b_4 \\ -B_2 & & & \end{bmatrix} \in -D_1'$$

$$\overline{-I_{D_1}} \circ \xi_1'\left(-\begin{bmatrix} & & -b_3 & \\ & B_0 & & \\ & & & -b_4 \\ -B_2 & & & \end{bmatrix}\right) = -\begin{bmatrix} & & -b_3 & \\ & B_0 & & \\ & & & -b_4 \\ -B_2 & & & \end{bmatrix} \in -D_1$$

$$\overline{-I_{D_1}} \circ \xi_1\left(-\begin{bmatrix} & & -b_3 & \\ & B_0 & & \\ & & & -b_4 \\ -B_2 & & & \end{bmatrix}\right) = \begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & +b_3 & \end{bmatrix} \in D_1$$

$$\overline{-I_{D_1}} \circ \xi_1'\left(\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & +b_3 & \end{bmatrix}\right) = -\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & -b_3 & \end{bmatrix} \in -D_1$$

$$\overline{-I_{F_0'}} \circ \xi_1\left(-\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & -b_3 & \end{bmatrix}\right) = -\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & -b_3 & \end{bmatrix} \in -F_0'.$$

Since we still have a cycle, the effect of μ_0' will be to put it back on the diagonal.

$$\overline{-I_{F_0'}} \circ \mu_0'\left(-\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & & -b_4 \\ & & -b_3 & \end{bmatrix}\right) = \begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in F_0'$$

$$\overline{-I_{D_1}} \circ \xi_1 \left(- \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \in D_1$$

$$\overline{-I_{D_1'}} \circ \xi_1' \left(- \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \in D_1'$$

Now μ_1 will give us a tree:

$$\overline{-I_{F_1}} \circ \mu_1 \left(\begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{tikzcd} 3 \arrow[r] & 4 \arrow[r, "B_3"] & 1 \arrow[d] \\ & & 2 \arrow[d] \\ & & 0 \end{tikzcd}$$

When we apply $\overline{-I_{F_1'}} \circ \mu_1'$ to this, we get back the same array we left in D_1', only now we are in F_1'. We continue the same process.

$$\overline{-I_{D_2}} \circ \xi_2 \left(\begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \in D_2;$$

$$\overline{-I_{D_2'}} \circ \xi_2' \left(\begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \in D_2'.$$

Once again we use the Matrix Tree Theorem, this time in the form of μ_2, to find out if we have a tree in the digraph where 3 and 4 have multiple outgoing edges to 1.

$$\overline{-I_{F_2}} \circ \mu_2 \left(\begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{tikzcd} & 3 \arrow[d, "B_4"] \\ 4 \arrow[r, "B_3"'] & 1 \arrow[d] \\ & 2 \arrow[d] \\ & 0 \end{tikzcd}$$

We do, so we move on. μ_2' followed by $\overline{-I_{F_2'}}$ gives us the same array we left behind before reaching that tree.

$$\overline{-I_{D_3}} \circ \xi_3 \left(\begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 &&& \\ & B_0 && \\ && B_4 & \\ &&& B_3 \end{bmatrix} \in D_3;$$

$$\overline{-I_{D_3'}} \circ \xi_3' \left(\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = \begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in D_3'.$$

Now we run into trouble again. This array corresponds to a graph with a cycle between 1 and 2, so μ_3 has the following effect:

$$\overline{-I_{D_3'}} \circ \mu_3 \left(\begin{bmatrix} B_2 & & & \\ & B_0 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = - \begin{bmatrix} & -b_2 & & \\ -B_0 & & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in -D_3'.$$

$$\overline{-I_{D_3}} \circ \xi_3' \left(- \begin{bmatrix} & -b_2 & & \\ -B_0 & & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = - \begin{bmatrix} & -b_2 & & \\ -B_0 & & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in -D_3$$

$$\overline{-I_{D_3}} \circ \xi_3 \left(- \begin{bmatrix} & -b_2 & & \\ -B_0 & & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & & \\ & +b_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in D_3$$

$$\overline{-I_{D_3}} \circ \xi_3' \left(\begin{bmatrix} B_0 & & & \\ & +b_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = - \begin{bmatrix} B_0 & & & \\ & -b_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in -D_3$$

This does not correspond to a tree in the graph where 3 and 4 have edges only to one, because there is a loop at the vertex 2.

$$\overline{-I_{D_3}} \circ \xi_3 \left(- \begin{bmatrix} B_0 & & & \\ & -b_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & & \\ & B_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in D_3,$$

$$\overline{-I_{D_3'}} \circ \xi_3' \left(\begin{bmatrix} B_0 & & & \\ & B_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = \begin{bmatrix} B_0 & & & \\ & B_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \in D_3',$$

and finally

$$\overline{-I_{F_3}} \circ \mu_3 \left(\begin{bmatrix} B_0 & & & \\ & B_2 & & \\ & & B_4 & \\ & & & B_3 \end{bmatrix} \right) = \begin{tikzcd} & 3 \arrow[d, "B_4"] & \\ 4 \arrow[r, "B_3"'] & 1 \arrow[d] & 2 \arrow[l, "B_2"] \\ & 0 & \end{tikzcd}$$

At this point we can read the code off from the picture by looking at the weights of the edges coming out of vertices $2, 3, 4$ in order. $\overline{-I_{F'_{n-1}}} \circ \hat{\mu}_{n-1}$ of this tree is its dandelion code: $(B_2, B_4, B_3) \equiv (2, 4, 3)$. Note that although our ending tree looks different from the original tree, its total weight is equal to the weight of the original tree.

6.4 Tree Surgery Method

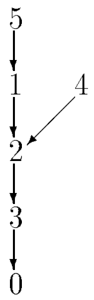
The same code can be found by skipping the matrix steps in between, since we can predict their effect.

The plan is this: We take the tree, and at step i we remove the edge $n-i+1 \to \texttt{succ}(n-i+1)$ and instead put in an edge $n-i+1 \to 1$ with weight $B_{\texttt{succ}(n-i+1)}$. If no cycle is created in the process, then we move on to the next step. If there is a cycle, we have to do something about it: we remove the edges $1 \to \texttt{succ}(1)$ and $n-i+1 \to 1$ and replace them by edges $1 \to \texttt{succ}(n-i+1)$ and $n-i+1 \to 1$, this last edge having weight $B_{\texttt{succ}(1)}$. At the end, we read off a version of the naïve code from the vertices $2, \ldots, n$ (instead of the successors of each vertex (since each points at 1 now), we look at the weights of these edges). The algorithm takes as its input a tree as a set of edges.

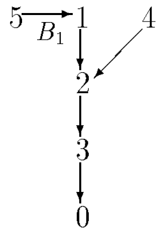
Tree Surgery Method for Dandelion Code
begin
  for $i = 1$ **to** $n-1$ **do**
    $m \leftarrow \texttt{succ}(n-i+1)$
    $k \leftarrow \texttt{succ}(1)$
    remove edge $(n-i+1) \to m$
    add edge $(n-i+1) \to 1$ with weight B_m
    if a cycle has been created **then**
      remove edge $1 \to k$
      remove edge $(n-i+1) \to 1$
      add edge $1 \to m$
      add edge $(n-i+1) \to 1$ with weight B_k
  for $j = 2$ **to** n **do**
    $w_j \leftarrow$ the weight of the edge $j \to 1$
  `code` $\leftarrow (w_2, w_3, \ldots, w_n)$
end.

In section §7.2 we will discuss the relationship between the Dandelion Code and Joyal's proof of Cayley's formula in [6] as well as the bijection in [4]. In fact this algorithm turns out to differ only in notation from one given by Eğecioğlu and Remmel in [4]. What is beautiful is the fact that the matrix method and the tree surgery method result in this same bijection. Using our method, we can see the underlying relationship of the tree surgical bijection with linear algebra and the Matrix Tree Theorem.

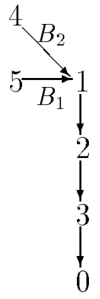
Example:



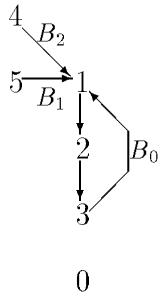
The first step is to remove the edge $5 \rightarrow 1$ and replace it by an edge $5 \rightarrow 1$ of weight B_1. This is a bit redundant. The point is that whatever the successor of 5 is becomes the subscript of the weight of the edge $5 \rightarrow 1$.



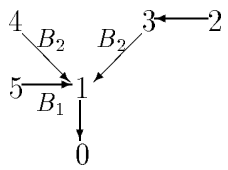
The next step removes the edge $4 \rightarrow 2$ and replaces it by an edge $4 \rightarrow 1$ with weight B_2. This does not create a cycle, so this is another quick step.



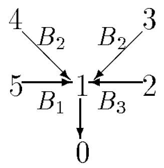
Now we remove the edge $3 \to 0$ and replace it by an edge $3 \to 1$ of weight B_0.



We have created a cycle, so we'd better fix it. We remove the edge $1 \to 2$ and replace it by the edge $1 \to 0$, and replace the edge $3 \to 1$ with weight B_0 by an edge $3 \to 1$ with weight B_2.



The last step is to replace the edge $2 \to 3$ by an edge $2 \to 1$ of weight B_3.



Now we look at the weights. The code is (B_3, B_2, B_2, B_1).

The inverse algorithm is fairly self-explanatory. It takes a code $(c_1, c_2, \ldots, c_n)$ and finds the corresponding tree.

Algorithm to go from Dandelion Code to Tree

```
begin
    edges ← {1 → 0}
    for i = 2 to n do
        add edge i → 1 of weight c_{i-1}
    for i = 2 to n do
        k ← the subscript of the weight of the edge i → 1
        remove edge i → 1
        add edge i → k
        if cycles ≠ ∅ then
            m ← succ(1)
            remove edge i → k
            add edge 1 → k
            remove edge 1 → m
            add edge i → m
end.
```

6.5 The Two Methods Give the Dandelion Code

Theorem 7 *The tree surgery method gives the same Dandelion Code as the matrix method.*

Proof. Again, we assume constant n and proceed by induction on step i. The base case is $i = 0$. At the start of the zeroth step, using either method, we have a tree in this original graph.

At the end of the i^{th} step, which is the start of the $(i+1)^{\text{th}}$ step, we assume that both methods have led to the same tree in which all vertices $j \geq n - i + 1$ have weighted edges with heads at 1.

The matrices are

$$N_i' = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ -b_1 & -b_2 & \dots & B-b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ -B & 0 & \dots & 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -B & 0 & \dots & 0 & 0 & \dots & B \end{bmatrix},$$

$$N_{i+1} = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ -b_1-B+b_1 & -b_2+b_2 & \dots & B-b_{n-i}+b_{n-i} & -b_{n-i+1}+b_{n-i+1} & \dots & -b_n+b_n \\ -B & 0 & \dots & 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -B & 0 & \dots & 0 & 0 & \dots & B \end{bmatrix},$$

and

$$N_{i+1}' = \begin{bmatrix} B-b_1 & -b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ -b_1 & B-b_2 & \dots & -b_{n-i} & -b_{n-i+1} & \dots & -b_n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ -B & 0 & \dots & B & 0 & \dots & 0 \\ -B & 0 & \dots & 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ -B & 0 & \dots & 0 & 0 & \dots & B \end{bmatrix}.$$

Let w_j represent the weight of the edge $j \to 1$ for these vertices (remember that for each j, $w_j = b_r$ for some r), and let $m_k = \texttt{succ}(k)$ for $1 \le k \le n-i$. In the matrix method, the tree is an element of F_i. When we apply $\overline{-I_{F_i'}} \circ \mu'_{i+1}$, we get

$$\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in F_i'.$$

Now we proceed as usual for the matrix method:

$$\overline{-I_{D_{i+1}}} \circ \xi_{i+1} \left(\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \right) =$$

$$\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in D_{i+1},$$

and

$$\overline{-I_{D'_{i+1}}} \circ \xi'_{i+1} \left(\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \right) =$$

$$\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in D'_{i+1}.$$

Now we note that the next step depends on the status of our tree in the new graph.

<u>Case 1</u> Suppose that

$$\overline{-I_{F_{i+1}}} \circ \mu_{i+1} \left(\begin{bmatrix} B_{m_1} & & & & & \\ & \ddots & & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \right)$$

is a tree in F_{i+1}. We have removed the edge $(n-i) \to m_{n-i}$, and added an edge $(n-i) \to 1$ with weight $B_{m_{n-i}}$. We set $w_{n-i} = B_{m_{n-i}}$, and are finished with this step. Clearly we have the same tree we would have if we had used the tree surgery method.

<u>Case 2</u> Suppose that

$$\mu_{i+1}\left(\begin{bmatrix} B_{m_1} & & & & \\ & \ddots & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix}\right)$$

is another array in D'_{i+1}. The only way for this to happen is if this array does not correspond to a tree in the graph where all of $(n-i)$'s edges point to 1. Since we started at a tree where all the vertices greater than $n-i$ point at 1, the only possibility is that there is a cycle including both $(n-i)$ and 1. None of the vertices $j \geq n-i+1$ can appear in this cycle since it includes only the vertices on the path from 1 to $(n-i)$ and these vertices all point to 1; hence, the bottom portion of the matrix is not affected. Thus,

$$\overline{-I_{D'_{i+1}}} \circ \mu_{i+1}\left(\begin{bmatrix} B_{m_1} & & & & \\ & \ddots & & & \\ & & B_{m_{n-i}} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix}\right) =$$

$$- \begin{bmatrix} & -b_{m_1} & & & \\ & & \ddots & & \\ -B_{m_{n-i}} & & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in -D'_{i+1},$$

with as many off-diagonal entries above row $n-i+1$ as there are vertices in the cycle being moved off the diagonal. These entries appear in all (i,j) positions satisfying the condition that $i \to j$ is an edge in the cycle.

$$\overline{-I_{D_{i+1}}} \circ \xi'_{i+1}\left(-\begin{bmatrix} & -b_{m_1} & & & \\ & \ddots & & & \\ -B_{m_{n-i}} & & & & \\ & & w_{n-i+1} & & \\ & & & \ddots & \\ & & & & w_n \end{bmatrix}\right) =$$

$$-\begin{bmatrix} & -b_{m_1} & & & \\ & \ddots & & & \\ -B_{m_{n-i}} & & & & \\ & & w_{n-i+1} & & \\ & & & \ddots & \\ & & & & w_n \end{bmatrix} \in -D_{i+1},$$

since all entries in N'_{i+1} appear also in N_{i+1}. However, next we switch entries in rows $n-i$ and 1:

$$\overline{-I_{D_{i+1}}} \circ \xi_{i+1}\left(-\begin{bmatrix} & -b_{m_1} & & & \\ & \ddots & & & \\ -B_{m_{n-i}} & & & & \\ & & w_{n-i+1} & & \\ & & & \ddots & \\ & & & & w_n \end{bmatrix}\right) =$$

$$\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & +b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in D_{i+1}.$$

Note that this will not take everything back to the diagonal (b_{m_1} is not on the diagonal in these next few arrays).

$$\overline{-I_{D_{i+1}}} \circ \xi'_{i+1}\left(\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & +b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix}\right) =$$

$$-\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & -b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in -D_{i+1}$$

$$\overline{-I_{F_i'}} \circ \xi_{i+1}\left(-\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & -b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix}\right) =$$

$$-\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & -b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in -F_i'.$$

Things finally get straightened out in the next step; all the off-diagonal entries are returned to the diagonal because there is still only one cycle:

$$\overline{-I_{F_i'}} \circ \mu'_{i+1}\left(-\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & \cdots & & \\ & & -b_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix}\right) =$$

$$\begin{bmatrix} B_{m_{n-i}} & & & & & \\ & \ddots & & & & \\ & & B_{m_1} & & & \\ & & & w_{n-i+1} & & \\ & & & & \ddots & \\ & & & & & w_n \end{bmatrix} \in F_i',$$

where now all entries are on the diagonal. This holds because the result of switching the entries in rows $n-i$ and 1 is to get rid of the edges from those two vertices and replace them by the edges $1 \to m_{n-i}$ and $(n-i) \to m_1$. Since there was a cycle containing these vertices before (the cycle was $1 \to m_1 \to \cdots \to (n-i) \to 1$, where the last edge had weight $B_{m_{n-i}}$), what we have done is to remove 1 from the cycle and pull the cycle out of the tree; every vertex that was in the cycle has been removed from the path joining 1 to 0, and 1 is in the component of the graph that is still a tree. This lone cycle has to be returned to the diagonal. Now we can follow the involutions joyfully back down the sequence of matrices:

$$\overline{-I_{D'_{i+1}}} \circ \xi'_{i+1} \circ \overline{-I_{D_{i+1}}} \circ \xi_{i+1} \left(\begin{bmatrix} B_{m_{n-i}} & & & & \\ & \ddots & & & \\ & & B_{m_1} & w_{n-i+1} & \\ & & & & \ddots \\ & & & & & w_n \end{bmatrix} \right) =$$

$$\begin{bmatrix} B_{m_{n-i}} & & & & \\ & \ddots & & & \\ & & B_{m_1} & w_{n-i+1} & \\ & & & & \ddots \\ & & & & & w_n \end{bmatrix} \in D'_{i+1}.$$

(There are no interesting steps in between.) When we apply μ_{i+1} to this matrix, we get a tree in the graph where $n-i$ has edges only to 1. This is simply because $n-i$ is no longer on the path from 1 to 0. We note that the weight of the new edge $(n-i) \to 1$ is B_{m_1}, so we set $w_{n-i} = B_{m_1}$ and are finished with this step. This is exactly the same as the tree surgery result.

Having accounted for all the cases, we see that at the end of step $i+1$, the tree surgery method and the matrix method give the same weights of edges for vertices $j \geq n-i$. By induction, we conclude that at the end of step $n-1$, both methods give the same weights and that consequently, the Dandelion code found will be the same using each method. ☺

It is interesting to note that although the Dandelion matrix method seems more closely related to the Blob matrix method than the Happy one, the tree surgery algorithm is closer to the Happy Code.

Chapter 7

Permutations of the naïve code

7.1 The Happy Code: an easier method

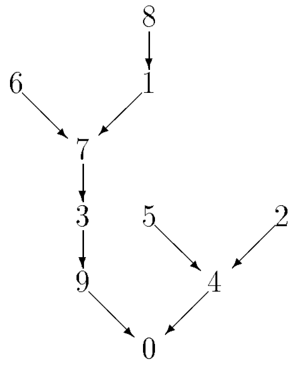
We have a third method for the Happy Code, that depends only on taking the naïve code (the input is in the form $p = (p_1, p_2, \ldots p_n) = (B_{j_1}, B_{j_2}, \ldots, B_{j_n})$) and permuting it according to the following algorithm:

Fast algorithm for Happy Code
begin
  while $p_1 \neq B_0$ **do**
    $a \leftarrow$ subscript of p_1
    $t \leftarrow p_a$
    $p_a \leftarrow b_a$
    $p_1 \leftarrow t$
    $k \leftarrow n$
    while $k > a$ **and** $\forall j,\ p_k \neq b_j$ **do**
      $k \leftarrow k - 1$
    $t \leftarrow p_a$
    $p_a \leftarrow p_k$
    $p_k \leftarrow t$

$\texttt{happycode} \leftarrow$ the subscripts of $(p_2, p_3, \ldots, p_n)$, in order

end.

For example, if we start with the tree



then the procedure goes as follows: first we note that the naïve code is $B_7B_4B_9B_0B_4B_7B_3B_1B_0$.

$$\begin{tikzcd}
B_7B_4B_9B_0B_4B_7B_3B_1B_0 \arrow[d] & \\
B_3B_4B_9B_0B_4B_7b_7B_1B_0 \arrow[r] & B_9B_4b_3B_0B_4B_7b_7B_1B_0 \arrow[d] \\
& B_9B_4b_7B_0B_4B_7b_3B_1B_0 \arrow[d] \\
& B_0B_4b_7B_0B_4B_7b_3B_1b_9
\end{tikzcd}$$

We find the code by looking at the subscripts after the initial B_0: the Happy Code for the tree shown above is (4,7,0,4,7,3,1,9).

Theorem 8 *The algorithm above gives the Happy Code as defined in previous sections.*

Proof. At any stage in this algorithm, a lower-case entry indicates membership in a cycle. We think of the i^{th} entry as having $\texttt{succ}(i)$ as its subscript. All this method does at each step is to change $\texttt{succ}(1)$ (the first entry) to $\texttt{succ}(\texttt{succ}(1))$

and insert `succ`(1) after the largest vertex in a cycle, since the largest vertex will be the furthest lower-case entry to the right. This is exactly what tree surgery accomplishes, so this is essentially a shorthand notation for tree surgery. Note that this also proves that the algorithm terminates. ☺

7.2 The Dandelion Code: an easier method

An even faster method exists for the Dandelion Code. The algorithm has as its input a tree as a set of edges. It uses the previously mentioned function $\texttt{path}(x)$ which finds the path from x to 0, returning a list of vertices $(x, \texttt{succ}(x), \ldots, 0)$.

Fast algorithm for Dandelion Code

begin

  $p \leftarrow \texttt{path}(1)$

  $m \leftarrow$ length of p

  $p \leftarrow (p_2, \ldots, p_{m-1})$

  $m \leftarrow m - 2$

  repeat

    $a \leftarrow$ the position of the maximum element of p

    $(p_1, p_2, \ldots, p_a)$ becomes a cycle

    $p \leftarrow (p_{a+1}, \ldots, p_m)$

  until $p = \emptyset$

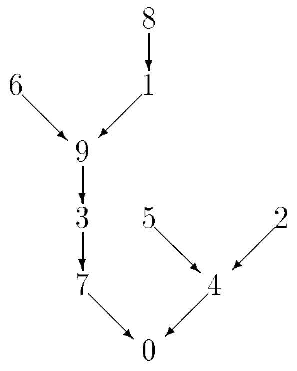
  rewrite the resulting collection of cycles as a permutation in 2-line notation

  this permutation gives the new `succ` function on the vertices on the path

  $\texttt{code} \leftarrow (\texttt{succ}(2), \texttt{succ}(3), \ldots, \texttt{succ}(n))$

end.

Example: We begin with the following tree:



and the procedure goes as follows:

First we note that the path from 1 to 0 is (9,3,7). We want to write this as cycles according to the algorithm. 9 is the largest thing on the path, so we end a cycle after it. Then 3 is not the largest remaining label on the path, so we don't end a cycle after it, but 7 is, so we do. Then we have to include the successors of the other vertices:

$$(9,3,7) \longrightarrow (9)(37) \longrightarrow \begin{pmatrix} 379 \\ 739 \end{pmatrix} \longrightarrow \begin{pmatrix} 23456789 \\ 47049319 \end{pmatrix}$$

The code is given by the bottom line: (4,7,0,4,9,3,1,9).

Another example: If we start with the tree

$$1 \rightarrow 6 \rightarrow 4 \rightarrow 9 \rightarrow 8 \rightarrow 3 \rightarrow 2 \rightarrow 5 \rightarrow 7 \rightarrow 0,$$

then the procedure is as follows:

$$(6,4,9,8,3,2,5,7) \longrightarrow (649)(8)(3257) \longrightarrow \begin{pmatrix} 23456789 \\ 52974386 \end{pmatrix} \longrightarrow \begin{pmatrix} 23456789 \\ 52974386 \end{pmatrix}$$

(Here, the path consisted of all the other vertices in the graph, so the last 2 steps look identical.) So the Dandelion Code for this tree is (5,2,9,7,4,3,8,6).

At first glance it may not be clear that this algorithm is even a bijection. However, it is. We will need the following:

Definition 24 *If S is a set of disjoint cycles, let $\preceq$ be the partial ordering on S defined by $C_1 \preceq C_2$ if and only if the largest vertex in C_1 is less than the largest vertex in C_2.*

Theorem 9 *The fast algorithm for the Dandelion Code has as its inverse the following algorithm:*

Fast Algorithm to go from Dandelion Code to Tree
begin
    `edges` $\leftarrow \{1 \to 0\}$
    for $i = 2$ **to** n **do**
        add edge $i \to c_{i-1}$
    write cycles as permutations in cycle notation
    write them in descending order according to $\preceq$
    within each cycle, cyclically reorder so that the largest element appears last
    $s \leftarrow$ the permutation with the parentheses ignored, as a list
    prepend 1 to s
    append 0 to s
    for $j = 1$ **to** $|s| - 1$ **do**
        remove edge $s_j \to$ `succ`(s_j)
        add edge $s_j \to s_{j+1}$
end.

This is clearly the inverse of the Fast Algorithm for the Dandelion Code. In slightly different form, these algorithms were previously discovered by Eğecioğlu and Remmel [4], apparently using some version of the Involution Principle [10]. Their bijection θ_{n+1} is isomorphic to our bijection as follows. Starting with a tree whose vertices are labelled $\{1, 2, \ldots, n+1\}$, we subtract from $n+1$ the labels of all vertices besides 1 on the path from 1 to $n+1$. Then we apply the fast algorithm for the Dandelion Code, and then subtract from $n+1$ the labels of all vertices in cycles. The result is the same functional digraph that Eğecioğlu and Remmel produced, except that we also have an edge $1 \to 0$ and the vertex $n+1$ has been relabelled with 0.

The Dandelion Code is reminiscent of Joyal's proof of the formula for the number of labelled trees [6]. His argument rested on the fact that the number of linear orderings of a set is the same as the number of collections of cycles from that set. An undirected tree, two of whose vertices are "special," should correspond to a functional digraph found by taking the linear ordering of the vertices between the two "special vertices" and using the corresponding collection of disjoint cycles.

The obvious bijection between linear orderings (of the vertices on the path from one special vertex to the other) and collections of cycles is to consider the linear ordering to be the second line of the 2-line notation for permutations, and the collection of cycles to be the permutation. Although this is probably what Joyal had in mind, it is somewhat unnatural in that it usually preserves very few of the original edges in the tree.

The relationship between Joyal's proof and the Dandelion Code is that for our purposes, the "special" vertices are always 1 and 0, and we are specific about the bijection between linear orderings and collections of disjoint cycles. The bijection we choose (namely, the one where 1 and 0 are ignored and the path between them is broken into cycles according to the algorithm above) is more natural than the obvious one because it preserves nearly all of the original edges of the tree. Eğecioğlu and Remmel phrase this in terms of a weight-preserving property of the bijection.

Essentially, the Dandelion Code is an implementation of Joyal's argument, where we consider only functional digraphs where there is a loop at 0 and a loop at 1, 1 is considered to be the largest vertex, and we use the algorithm of the fast Dandelion Code and its inverse as the bijection between linear orderings and collections of cycles. In Chapter 8 we discuss the relationship between the Dandelion Code and the Happy Code, which indicates that the Happy Code is a different implementation of Joyal's argument.

Theorem 10 *The fast algorithm above gives the Dandelion Code as defined in Chapter 6.*

Proof. We note that the tree surgery algorithm has $n-1$ steps, whereas the fast algorithm has an unclear number that is usually less than $n-1$. However, in step i of the tree surgery method, if vertex $n-i+1$ is not on the path from 1 to 0, then performing the tree surgery of pointing its edge at 1 does not create a cycle. Thus, $w_{n-i+1} = b_{\mathtt{succ}(n-i+1)}$ will be the $(n-i)^{\text{th}}$ entry in the code. This matches the effect of the fast algorithm, which essentially starts with the naïve code and then changes the entries only of vertices on the path from 1 to 0.

For vertices that lie between 1 and 0 and do not have any inversions, the tree surgery algorithm notes the cycle that has appeared and does the equivalent of reverting to the original tree from the start of step i and switching the successors of $n-i+1$ and 1. Then, to get rid of the smaller cycle that this graph has, it removes the new edge $(n-i+1) \to \mathtt{succ}(1)$ and adds an edge $(n-i+1) \to 1$ with weight $\mathtt{succ}(1)$. All of this amounts to exactly what the fast algorithm does. Because the tree surgery algorithm changes the edges of the vertices from n down to 2, the largest vertex on the current path from 1 to 0 is the one whose edge will point initially back at 1 and create a cycle, though in fact what will happen is that whatever was the current $\mathtt{succ}(1)$ is what gets put as the weight of the edge. This corresponds to a cycle in the sense of permutations, and since this cycle is removed from the path, the rest of the vertices in that cycle now keep their original successors for the final code. ☺

Chapter 8

Relationship between Codes

There is actually a close relationship between the Happy Code and the Dandelion Code.

Example: If we start with the tree $1 \to 6 \to 4 \to 9 \to 8 \to 3 \to 2 \to 5 \to 7 \to 0$, we found in §7.2 that its Dandelion Code was (5,2,9,7,4,3,8,6). If we reverse the order of the vertices between 1 and 0, the new tree is

$$1 \to 7 \to 5 \to 2 \to 3 \to 8 \to 9 \to 4 \to 6 \to 0,$$

Note that the Dandelion Code of the tree with the reversed path from 1 to 0 is *not* the reverse of the Dandelion Code of the original tree:

$$(7,5,2,3,8,9,4,6) \longrightarrow (752389)(46) \longrightarrow \begin{pmatrix} 23456789 \\ 38624597 \end{pmatrix} \longrightarrow \begin{pmatrix} 23456789 \\ 38624597 \end{pmatrix}$$

However, the Happy Code for this new tree yields the Dandelion Code for the original tree. The naïve code is $B_7B_3B_8B_6B_2B_0B_5B_9B_4$. The fast Happy Code algorithm goes as follows:

$$
\begin{tikzcd}
B_7B_3B_8B_6B_2B_0B_5B_9B_4 \arrow[d] & \\
B_5B_3B_8B_6B_2B_0b_7B_9B_4 \arrow[d] & \\
B_2B_3B_8B_6b_5B_0b_7B_9B_4 \arrow[r] & B_2B_3B_8B_6b_7B_0b_5B_9B_4 \arrow[d] \\
B_3b_5B_8B_6b_7B_0b_2B_9B_4 \arrow[d] & B_3b_2B_8B_6b_7B_0b_5B_9B_4 \arrow[l] \\
B_8b_5b_3B_6b_7B_0b_2B_9B_4 \arrow[r] & B_8b_5b_2B_6b_7B_0b_3B_9B_4 \arrow[d] \\
 & B_9b_5b_2B_6b_7B_0b_3b_8B_4 \arrow[d] \\
 & B_4b_5b_2B_6b_7B_0b_3b_8b_9 \arrow[d] \\
B_6b_5b_2b_9b_7B_0b_3b_8b_4 \arrow[d] & B_6b_5b_2b_4b_7B_0b_3b_8b_9 \arrow[l] \\
B_0b_5b_2b_9b_7b_6b_3b_8b_4 \arrow[r] & B_0b_5b_2b_9b_7b_4b_3b_8b_6
\end{tikzcd}
$$

The subscripts give the Happy Code: (5,2,9,7,4,3,8,6), which is the same as the Dandelion Code of the tree where the path from 1 to 0 was in the other order.

Theorem 11 *If the order of the vertices on the path from 1 to 0 is reversed, the Happy Code of the new tree will be the same as the Dandelion Code of the original tree (and vice versa).*

Proof. To understand how the Happy Code and the Dandelion Code are so closely related, we note that both depend on the path from 1 to 0. Vertices occurring elsewhere in the tree have the same effect on the code using either algorithm;

if i is such a vertex then $\texttt{succ}(i)$ will appear in the $(i-1)^{\text{th}}$ position of both the Dandelion Code and the Happy Code for both the tree and its path-reversed modification.

Recalling the tree surgery method for the Happy Code, we construct a similar method to the faster algorithm of the Dandelion Code. First, we write out only the path from 1 to 0. We know that as we move from left to right (from 1 to 0) along it, each vertex gets placed in a cycle, immediately following the largest vertex already in a cycle. Thus we are comparing each vertex with the vertices to its left in the path. We end a cycle just *before* a new largest vertex (among the labels to its left). This is the reverse of the fast Dandelion Code, which ends a cycle just *after* the largest vertex among the labels to its right.

Meanwhile, within the cycles, each new label is inserted after the largest label in the cycle for the Happy Code. But the first element in the cycle is the largest (by virtue of how we have split path into cycles), and the vertices are added to it one at a time from left to right–always inserted after this largest label. The effect is that of reversing the path order of the remaining vertices in the cycle. The resulting cycle is exactly the cycle that arises from the Dandelion Code of the path-reversed modification of the original tree. ☺

The Happy Code is another implementation of Joyal's almost-bijection. This time the choice of bijection between linear orderings and sets of cycles is not as natural because it changes more of the edges of the tree.

Chapter 9

Conclusion

9.1 The codes are distinct

An example suffices to prove that these codes are different from one another and from the Prüfer Code.

Example: Consider the tree

$$\begin{tikzcd}
1 \arrow[d] & & \\
6 \arrow[dr] & & 3 \arrow[dl] \\
5 \arrow[dr] & 2 \arrow[d] & 7 \arrow[dl] \\
& 4 \arrow[d] & \\
& 0 &
\end{tikzcd}$$

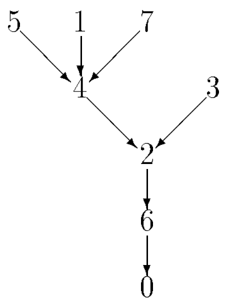
whose Prüfer Code we calculated in §1.3.1 to be (6,2,4,2,4,4).

The Dandelion Code for the tree is found as follows:

$$(6,2,4) \longrightarrow (6)(24) \longrightarrow \binom{246}{426} \longrightarrow \binom{234567}{422464}$$

So the Dandelion Code for the tree is (4,2,2,4,6,4).

The Happy Code can be found by reversing the order of the path from 1 to 0 and finding the Dandelion Code of the altered tree:

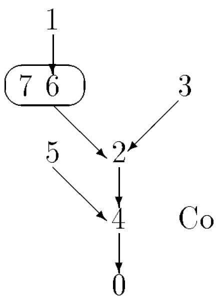


This tree's Dandelion Code is (6,2,2,4,4,4):

$$(4,2,6) \longrightarrow (426) \longrightarrow \begin{pmatrix}246\\624\end{pmatrix} \longrightarrow \begin{pmatrix}234567\\622444\end{pmatrix}.$$

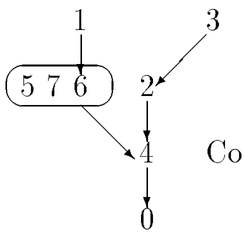
Thus the Happy Code of our main example tree is (6,2,2,4,4,4).

The Blob Code takes a little more work:

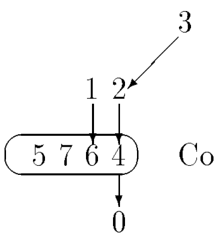


Code so far = (4)

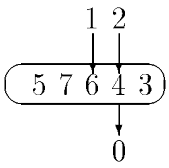
It'll take a few more steps.



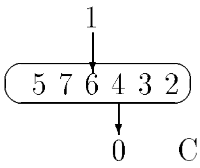
Code so far = (2,4)



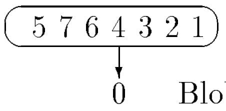
Code so far = (4,2,4)



Code so far = (2,4,2,4)



Code so far = (4,2,4,2,4)



Blob Code = (6,4,2,4,2,4)

This is different from the other codes.

So, to review, the tree we started with has the following codes:

Method	Code
Blob	(6,4,2,4,2,4)
Happy	(6,2,2,4,4,4)
Dandelion	(4,2,2,4,6,4)
Prüfer	(6,2,4,2,4,4)

Thus, we conclude that the various codes are all distinct.

9.2 Clever weighting of edges

In [4], Eğecioğlu and Remmel use a six-variable weighted version of Cayley's formula instead of the $(n+1)$-variable version we have been using. They were able to produce a bijection that counts descents and ascents. Specifically, where we have given the edge $i \to j$ the weight b_j, they have given it the weight xq^it^j if the edge is a descent and yp^is^j if it is an ascent or loop.

It is possible to extend both their results and ours by clever weighting of edges.

We examine the result of weighting edges as follows:

$$W(i \to j) = \begin{cases} b_j & \text{if } i \to j \text{ is not an ascent} \\ a_{ij} & \text{if } i \to j \text{ is an ascent} \end{cases}$$

Here, loops are considered not to be ascents.

Using these weights, we show an example for $n = 4$:

$$U_0' = \begin{bmatrix} b_0+b_1+a_{12}+a_{13}+a_{14}-b_1 & -a_{12} & -a_{13} & -a_{14} \\ -b_1 & b_0+b_1+b_2+a_{23}+a_{24}-b_2 & -a_{23} & -a_{24} \\ -b_1 & -b_2 & b_0+b_1+b_2+b_3+a_{34}-b_3 & -a_{34} \\ -b_1 & -b_2 & -b_3 & b_0+b_1+b_2+b_3+b_4-b_4 \end{bmatrix}$$

First we subtract row 3 from row 4, obtaining

$$U_1 = \begin{bmatrix} b_0+b_1+a_{12}+a_{13}+a_{14}-b_1 & -a_{12} & -a_{13} & -a_{14} \\ -b_1 & b_0+b_1+b_2+a_{23}+a_{24}-b_2 & -a_{23} & -a_{24} \\ -b_1 & -b_2 & b_0+b_1+b_2+b_3+a_{34}-b_3 & -a_{34} \\ 0 & 0 & -b_0-b_1-b_2-b_3-a_{34} & b_0+b_1+b_2+b_3+a_{34} \end{bmatrix}.$$

Next we add column 4 to column 3.

$$U_1' = \begin{bmatrix} b_0+b_1+a_{12}+a_{13}+a_{14}-b_1 & -a_{12} & -a_{13}-a_{14} & -a_{14} \\ -b_1 & b_0+b_1+b_2+a_{23}+a_{24}-b_2 & -a_{23}-a_{24} & -a_{24} \\ -b_1 & -b_2 & b_0+b_1+b_2+b_3+a_{34}-b_3-a_{34} & -a_{34} \\ 0 & 0 & 0 & b_0+b_1+b_2+b_3+a_{34} \end{bmatrix}.$$

The method is parallel to that of the Blob Code, only our weights are slightly different. We continue until we reach the final matrix, an upper-triangular matrix whose i^{th} diagonal entry is $\sum_{k=0}^{i-1} b_k + \sum_{k=i}^{n} a_{i-1,k}$, except in row 1 where the diagonal entry is b_0. This yields both algebraic and bijective proofs of a generalized version of Cayley's formula, which we call the UCSD formula (since the inspiration for it came from methods of Eğecioğlu and Remmel).

The UCSD formula for the sum of the weights of all possible trees is

$$\sum_{\tau} W(\tau) = \det(U_{n-1}') = b_0 \prod_{i=2}^{n} \left[\sum_{k=0}^{i-1} b_k + \sum_{j=i}^{n} a_{i-1,j} \right].$$

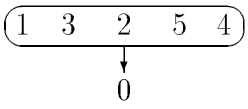
A code is a term from this product, kept in the order of the columns from which it came. Specifically, the set of codes is $\{(x_1, x_2 \ldots, x_n) | x_1 = b_0,$ and for $2 \le i \le n$, $x_i = a_{i-1,j}$ for some $j > i-1$ or $x_i = b_j$ for some $j \le i-1\}$.

The advantage of this new weighting system is that the code reveals all ascents and descents to and from each vertex. The ascending edges can be read directly from the subscripts of the a weights, while the descending indegree of any vertex j is given by the number of occurrences of b_j. Let $I_A(j)$ denote the ascending indegree of j, $I_D(j)$ the descending indegree, $O_A(j)$ the ascending outdegree and $O_D(j)$ the descending outdegree. The total indegree of j is the number of occurrences of b_j plus the number of times that j occurs as the second subscript of an a. The descending outdegree of a vertex $j \neq 0$ is simply $O_D(j) = 1 - O_A(j)$.

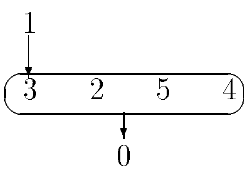
Example: If a tree turns out to have code $(b_0, a_{13}, b_2, b_0, a_{45})$, then we know the following:

Vertex	I_A	I_D	O_A	O_D
0	0	2	0	0
1	0	0	1	0
2	0	1	0	1
3	1	0	0	1
4	0	0	1	0
5	1	0	0	1

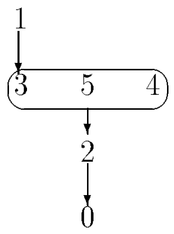
¿From the code we can thus also conclude not only that the edges $1 \to 3$ and $4 \to 5$ are in the tree, but that so is the edge $2 \to 0$ (because 2 has descending outdegree of 1, and 1 has indegree of zero; 2 must point at something less than 2 but it can't be 1). All that remains is to figure out the edges from 3 and 5. One must point at 2 and the other at 0 to use up all of our indegrees. By defining involutions as we did for the original Blob Code, we could find it using the matrix method with the above matrices. We can also use the inverse tree surgery algorithm from §4.2.



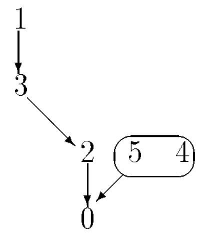
The initial b_0 in the code tells us that the `blob` points at 0. The next element in the code, a_{13}, indicates that when we remove 1 from the `blob`, its edge points at 3 and the `blob` stays where it is.



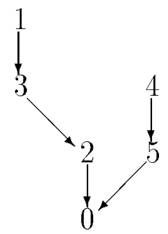
The next part of the code is b_2. If 2 is removed from the `blob`, then the (nonexistent) path from 2 to 0 does not pass through the `blob`, so we remove the edge `blob` $\to 0$ and add edges `blob` $\to 2$ and $2 \to 0$.



The next weight in the code is b_0. We remove 3 from the `blob`. The path from 0 to 0 does not go through the `blob`, so we remove the edge `blob` $\to 2$ and add edges $3 \to 2$ and `blob` $\to 0$.



The final piece of information from the code is a_{45}. This automatically tells us what the final edge is.



This weighted version of the Blob Code is just as easily calculated as the original Blob Code, but displays more information.

We can also use the Dandelion Code to verify directly that the UCSD formula holds. The right side of the equation, a product of sums of monomials, expands out to a sum of terms of degree n. Each term is the weight of a happy functional digraph consisting of the edges $i \to j$ whenever the i^{th} indeterminate in the sequence is b_j or

a_{kj} for some k. The Dandelion Code gives a weight-preserving bijection between this set of happy functional digraphs and the set of trees. Thus, the right side of the equation must equal the left side. This is essentially the same proof that Eğecioğlu and Remmel use for their six-variable version of the Cayley formula.

Using the Dandelion Code matrix method for an algebraic proof of this formula is less straightforward than using the Blob Code matrix method.

9.3 Applications and more questions

All of our simple (non-weighted) codes have interesting features. The Happy Code is less natural than the other two and probably can not be generalized to display more information than the Prüfer Code. It is the hardest of the three codes to get a mental handle on, because in the matrix method, we only apply the bijective proof of the Matrix Tree Theorem to the original matrix. This drastically complicates the proof that the matrix method and tree surgery method for this code are equivalent. However, Lemma 4, required in that proof, is easy to state, beautiful, and surprising.

The Dandelion Code is very efficiently calculated, and allows us an easier way to find the Happy Code. It implements Joyal's almost-bijective proof of Cayley's formula in a beautiful and natural way. If thought of in the way suggested by the fast algorithm for it (as Eğecioğlu and Remmel did), it preserves most of the edges of the original tree. Furthermore, this bijection provides a direct proof of the UCSD formula.

The simple Blob Code is interesting in that it elaborates on some of Orlin's ideas and provides a bijection behind his manipulatoric proof of the formula for the number of trees. Furthermore, it doesn't single out vertex 1 as being more special than the others, whereas the other two codes require one (rather arbitrarily) to examine the path from 1 to 0. Best of all, the matrix and tree surgery methods both generalize easily to a weighted code that keeps track of all ascents and descents in the tree.

All of these codes share the property that they are consistent with the Matrix Tree Theorem. They are natural in that we can undo them one step at a time, in reverse order from the way they were found, simply by following the involutions through in the other order. They also can be found by simpler, tree-surgical bijections similar to that of Prüfer, yet the inverses of these methods are the simple inverse operations of the formations of the codes. Meanwhile, there does not seem to be any way to "matrixify" the Prüfer Code, and its inverse is decidedly unnatural. In addition, our three codes lose none of the information encapsulated in the Prüfer Code (the indegrees of each vertex; which vertices are leaves).

Furthermore, the Dandelion Code generalizes to forests (collections of rooted trees) very nicely [10],[12]. The number of forests with $n+1$ vertices, k of which are roots, is $k \cdot (n+1)^{n-k}$. Since the Matrix Tree Theorem also generalizes to forests of k rooted trees (where $k < n+1$) using minors obtained by crossing out k rows and columns, it is possible that the bijective proof by Chaiken [3] can lead to extensions of the Blob and Happy Codes to forests as well. However, this is not necessary. We can easily extend any of the three codes to forests of k trees with roots $-1, -2, \ldots, -k$ and non-root vertices $k+1, k+2, \ldots, n+1$ by replacing b_0 with $b_{-1} + b_{-2} + \cdots + b_{-k}$ (and B_0 with $B_{-1} + \cdots + B_{-k}$) whenever they appear in the matrices. The result is immediate using the exact same methods, and the tree surgery methods are not affected substantially by the change.

The codes themselves may not be useful for much yet. Although each of them has some relationship with the idea of inversions, none actually count inversions. (An inversion occurs whenever a vertex $j > i$ appears on the path from i to 0.) A future direction for research might be to attempt to find a code that is both consistent with the Matrix Tree Theorem *and* able to enumerate the inversions of the tree, because the total number of inversions in a tree, $inv(\tau)$, is of interest to algebraic combinatorists. The Hilbert series of the space of diagonal harmonics (when restricted to t=1), $Hilb_n(t,q)|_{t=1}$, is conjectured to be $\sum q^{inv(\tau)}$ where τ ranges over all trees with vertices $0, \ldots, n$. Thus a statistic on one of the codes that has the same distribution as $inv(\tau)$ might assist algebraic combinatorists in finding a basis for the space of diagonal harmonics. Unfortunately, such a code is elusive.

Another possible direction for future research is to examine the method of the Happy Code when applied to the Blob Code's row and column operations. Namely, if we use a placeholder λ in the (0,0) position and only apply the Matrix Tree Theorem to the original matrix, setting $B_j = b_j$ at the end, do we get a different code? If so, does it have any advantages over the codes we have already found?

As noted in §1.5, there are many sequences of row and column operations that can lead to an easily calculated determinant. Since the matrix involution method is quite general, any of these should give a coding algorithm for trees. We know that not all coding algorithms based on the structure of the tree correspond to matrix methods. Naturally we are led to wonder whether there are always simple tree surgical methods that correspond to the codes we find through matrices. The true beauty of these results is that each code was defined through row operations on the matrix before the corresponding tree surgical methods were discovered. Thus, linear algebra gave birth to bijections who grew up and became independent proofs in their own right.

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