Today: Midterm 1 Review

Next: §2.2-2.3: Matrix Inverse

Midterm 1: TONIGHT!

Beginning @ 8pm in GH 242, York 2622 & York 2722

- One double-sided 8.5”x11” note sheet
- No electronic devices
- Write answers on exam booklet. We will provide scratch paper if needed.
- Bring UCSD Student ID.
- Have fun tonight!
(1 point) 0. Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

(6 points) 1. Consider the following system of equations.

\[
\begin{align*}
x_1 - 2x_2 &= 0 \\
3x_1 + hx_2 &= 0 \\
x_1 + 2x_2 &= 4
\end{align*}
\]

(a) Show that there is a unique value of \( h \) for which the system is consistent, and find that value of \( h \).

(b) In the case that the system is consistent, does it have a unique solution, or infinitely many solutions? Justify your answer.
2. (9 points)

The matrix \( A = \begin{bmatrix} 1 & -3 & 3 & 7 \\ -3 & 7 & -1 & -11 \\ 0 & 1 & -4 & -5 \end{bmatrix} \) has reduced row-echelon form \( \begin{bmatrix} 1 & 0 & -9 & -8 \\ 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(a) Describe the general solution of the system of equations whose augmented matrix is \( A \).

(b) Describe the general solution to the vector equation 
\[
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 7 \\ \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

(c) Let \( C \) be the \( 3 \times 3 \) matrix given by the first three columns of \( A \). Is the system \( Cx = b \) consistent for all possible choices of \( b \in \mathbb{R}^3 \)? Briefly explain your answer.
3. Let $A$ be any $7 \times 12$ matrix. For each statement about $A$, circle $T$ if it is always True; circle $F$ if it is ever False. 2 points will be assigned for each correct response, 1 point for each blank non-response, and 0 points for each incorrect response. No justification is required.

(T F) The columns of $A$ span $\mathbb{R}^7$.

(T F) The columns of $A$ are linearly dependent.

(T F) The matrix equation $Ax = 0$ has only the trivial solution $x = 0$. 
(8 points) 4. Let $v_1, v_2, v_3$ be the following three vectors:

$$
v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 1 \\ -3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 8 \\ 5 \\ -1 \end{bmatrix}.$$

(a) Determine if the vectors $v_1, v_2, v_3$ are linearly independent. If they are not, exhibit a non-trivial linear combination of them that yields the 0 vector.

(b) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation satisfying

$$T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = v_1, \quad T \left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = v_2, \quad T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = v_3.$$

Is $T$ one-to-one? Justify your answer.
0. (1 point) Carefully read and complete the instructions at the top of this exam sheet and any additional instructions written on the chalkboard during the exam.

1. (6 points) Consider the following system of linear equations.

\[
\begin{align*}
    x_1 + 2x_2 &= 1 \\
    2x_1 + x_2 &= h \\
    x_1 - x_2 &= 0
\end{align*}
\]

(a) Find all value(s) of \( h \) for which the system is consistent, and describe the corresponding solution set.

\[
\begin{bmatrix}
  1 & 2 & 1 \\
  2 & 1 & 4 \\
  1 & -1 & h
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 2 & 1 \\
  2 & 1 & 4 \\
  0 & -3 & h-1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 2 & 1 \\
  0 & -3 & h-1 \\
  0 & 0 & 0
\end{bmatrix}
\]

If \( h \neq 3 \), last column is pivot, so consistent. If \( h = 3 \), not pivot, so inconsistent.

(b) Find all value(s) of \( h \) for which the system is inconsistent.

\[
h \neq 3
\]

(c) Is the corresponding homogeneous system consistent? If so, describe its solution set.

\textbf{Yes, consistent because} \( x = 0 \) is a solution. \textbf{This is the only solution} because an inhomogeneous version of the system has a unique (particular) solution.
2. (6 points) Let \( A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 2 \\ 3 & 1 & 5 \end{bmatrix} \).

(a) Find the reduced row echelon form of \( A \).

\[
\begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 3 & 1 & 5 
\end{bmatrix} \overset{R_3/3}{\rightarrow} \begin{bmatrix}
1 & -1 & 1 \\ 0 & 2 & 2 \\ 1 & 1/3 & 5/3 
\end{bmatrix} \overset{-3R_1 + R_3}{\rightarrow} \begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 4 & -4 
\end{bmatrix} \overset{R_3 + R_2}{\rightarrow} \begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 
\end{bmatrix} \overset{-R_2 + R_1}{\rightarrow} \begin{bmatrix}
1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 
\end{bmatrix} \overset{-R_3}{\rightarrow} \begin{bmatrix}
1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 
\end{bmatrix} \overset{R_2/2}{\rightarrow} \begin{bmatrix}
1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 
\end{bmatrix} \overset{R_1 + R_2}{\rightarrow} \begin{bmatrix}
1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 
\end{bmatrix}.
\]

(b) Describe the solution set of the homogeneous equation \( Ax = 0 \).

\[
\begin{pmatrix}
0 \\ 0 \\ 0 
\end{pmatrix} \quad \text{(every column is pivotal)}.
\]

(c) Let \( b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \). Is the equation \( Ax = b \) consistent? If it is, describe the solution set.

\[
\begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 3 & 1 & 5 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 4 & -4 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -2 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 
\end{bmatrix}.
\]
3. (6 points) For each \( k \in \mathbb{R} \), let \( S_k \) be the set of vectors in \( \mathbb{R}^3 \) given by \( S_k = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ k \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \\ 1 \\ k \end{pmatrix} \right\} \).

For each of parts (a) - (c), find the value(s) of \( k \) for which \( S_k \) has the indicated property. Be sure to show how you arrived at each answer.

(a) \( S_k \) is linearly independent.

If \( k \neq 3 \), all columns are pivotal.

(b) \( S_k \) is linearly dependent.

If \( k = 3 \), the third column is not pivotal.

(c) \( S_k \) spans \( \mathbb{R}^3 \).

If \( k \neq 3 \), all rows are pivotal.
4. (6 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right) + T \left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) + T \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$ 

Is $T$ one-to-one? Justify your answer.

"$T$ is one-to-one" means that $T(x) = b$ has 0 or 1 solution(s) for any given $b \in \mathbb{R}^3$.

Thus, to show $T$ is not one-to-one, I just need to find some $b$ such that $T(x) = b$ has more than 1 solution. Try $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) + T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$T$ is linear

$$= T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$= T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

So $T(x) = 0$ has solutions $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore T$ is NOT one-to-one.
Eg. \[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 1 \\
\end{bmatrix}
\leftarrow
\text{all rows are pivotal, columns span } \mathbb{R}^2.
\]

\text{not pivotal}