Today: §4.6 Rank & §4.4 Coordinates

Next: §3.1-3.2: Determinants

Homework:
MyMathLab Homework #5: Due Tuesday
MATLAB Assignment #4: Due February 23
Midterm #2: 2 weeks from tonight.
The dimension of a vector space $V$ is the number of vectors in any basis. (Makes sense, since all bases have the same number of vectors.)

E.g., $V = \left\{ \begin{bmatrix} a+b \\ a-b \\ \frac{2a}{a} \end{bmatrix} : a, b \in \mathbb{R} \right\}$
**Definition:** Given a matrix $A$,

$$\text{rank}(A) := \dim(\text{Col}(A))$$

$$\text{nullity}(A) := \dim(\text{Null}(A))$$

$$A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$
**Theorem**: The pivotal columns of $A$ form a basis for $\text{Col}(A)$.

The non-pivotal columns of $A$ correspond to a basis for $\text{Null}(A)$.

\[ \therefore \text{rank}(A) = \]

\& \text{nullity}(A) =

\[ \therefore \text{rank}(A) + \text{nullity}(A) = \]

One more important subspace:

**Definition**: The row space of $A$, $\text{Row}(A)$, is the span of the rows of $A$. 
Eg. \( \mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -1 & 2 \\ 3 & 3 & 0 & -3 & 6 \\ 1 & 2 & 3 & 0 & 6 \end{bmatrix} \)

\[
\text{Row} (\mathbf{A}) = \text{Span}\left\{ \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & 0 & -3 & 6 \\ 1 & 2 & 3 & 0 & 6 \end{bmatrix} \right\}
\]

**Theorem:** \( \text{Row} (\mathbf{A}) \quad \text{dim} (\text{Row} (\mathbf{A})) \quad \text{Row} (\text{rref} (\mathbf{A})) \)
What's so great about a basis, anyway?

**Theorem:** If \( B = \{ b_1, \ldots, b_n \} \) is a basis for \( V \), then each vector \( v \in V \) has a unique expansion

\[
  v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n
\]

for some unique scalars \( x_1, \ldots, x_n \in \mathbb{R} \).
This means, if $V$ has a basis $B=\{b_1, \ldots, b_n\}$ (so $\dim(V)=n$) then we can identify $V$ with $\mathbb{R}^n$ by identifying each vector $v$ with its $B$-coordinate vector

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \mapsto [v]_B = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

Eg. If $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, then its coordinate vector with respect to the standard basis $E=\{e_1, e_2\}$ is just $[v]_E = \begin{bmatrix} a \\ b \end{bmatrix}$. But what if we use the basis $B=\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$?
Eg. $P_2 = \{ \text{polynomials of degree } \leq 2 \}$

"Standard" basis $B = \{1, x, x^2\}$

Then the polynomial $p = (x-1)^2 = $ ...

**Theorem**: If $B = \{b_1, b_2, \ldots, b_n\}$ is a basis for $V$, then the function $T : V \to \mathbb{R}^n : T(v) = [v]_B$ is a one-to-one linear transformation from $V$ onto $\mathbb{R}^n$.

Such a linear transformation is called an
E.g. \( A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \{ \} = \varnothing \) basis for \( \text{Col}(A) \).
The point of isomorphisms is: they preserve all linear properties.

**Theorem:** Let \( \{ b_1, b_2, \ldots, b_n \} = \mathcal{B} \) be a basis for \( V \).

Then * \( \{ v_1, \ldots, v_k \} \in V \) are linearly independent in \( V \)
iff \( [v_1]_{\mathcal{B}}, \ldots, [v_k]_{\mathcal{B}} \) are linearly independent in \( IR^n \).

* " " \( \text{span} \quad V \)
* " " \( \text{span} \quad IR^n \).

Eg. Show that \( \{ 1, x-1, (x-1)^2 \} \) is a basis for \( \mathbb{P}^2 \).