Today: § 4.6 Rank & § 4.4 Coordinates

Next: § 4.7 Change of Basis

Homework:
My MathLab Homework #4: Due TODAY by 11:59p.
My MathLab Homework # : Due Tuesday
MATLAB Assignment #4: Due February 24
The **dimension** of a vector space $V$ is the number of vectors in any basis. (Makes sense, since all bases have the same number of vectors.

Eg. Given a matrix $A$, we can find a basis for

$$\text{Col}(A) = \text{span}\{\text{the columns of } A\}$$

$\rightarrow$ Basis : $\{\text{the pivotal columns in } A\}$

$\therefore \dim(\text{Col}(A)) = \# \text{ pivotal columns in } A$

$=: \text{rank}(A)$. 
What about the nullspace?

$$A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$
\[
\text{Eg.} \quad A = \begin{bmatrix}
-3 & 6 & -1 & 1 & -7 \\
1 & -2 & 2 & 3 & -1 \\
2 & -4 & 5 & 8 & -4 \\
\end{bmatrix} \quad \xrightarrow{\text{RREF}} \quad \begin{bmatrix}
1 & -2 & 0 & -1 & 3 \\
0 & 0 & 1 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
**Theorem:** The non-pivotal columns of $A$ form a basis for $\text{Nul}(A)$.

\[\therefore \text{nullity}(A) = \text{dim}(\text{Nul}(A)) = \# \text{non-pivotal columns}\]

One more important subspace:

**Definition:** The row space of $A$, $\text{Row}(A)$, is the span of the rows of $A$. 
Example: \( A = \begin{bmatrix} 1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & 1 & 4 \\ 1 & 2 & 3 & 0 & 6 \end{bmatrix} \)

\( \text{Row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 6 \end{bmatrix} \right\} \)

**Theorem:** \( \text{Row}(A) = \text{Row}(\text{RREF}(A)) \)

**Theorem:** \( \text{dim}(\text{Row}(A)) = \text{rank}(A) + \text{nullity}(A) = \)
4.4 What's so great about a basis, anyway?

**Theorem:** If $B = \{b_1, \ldots, b_n\}$ is a basis for $V$, then each vector $v \in V$ has a unique expansion

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n$$

for some unique scalars $x_1, \ldots, x_n \in \mathbb{R}$.
This means, if $V$ has a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ (so $\dim(V) = n$) then we can identify $V$ with $\mathbb{R}^n$ by identifying each vector $v$ with its $\mathcal{B}$-coordinate vector

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n \mapsto [v]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$ 

Eg. If $v = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, then its coordinate vector with respect to the standard basis $\mathcal{E} = \{e_1, e_2\}$ is just $[v]_{\mathcal{E}} = [a].$ But what if we use the basis $\mathcal{B} = \{[0], [1]\}$?
Eg. $P_2 = \{ \text{polynomials of degree } \leq 2 \}$

"Standard" basis $B_3 = \{ 1, x, x^2 \}$.

Then the polynomial $p = (x-1)^2$

**Theorem:** If $B = \{ b_1, b_2, \ldots, b_n \}$ is a basis for $V$, then the function $T : V \rightarrow \mathbb{R}^n : T(v) = [v]_B$ is a one-to-one linear transformation from $V$ onto $\mathbb{R}^n$.

Such a linear transformation is called an
E.g. Is \( \{ (x-1)^2, (x+1)^2, x^2-1 \} \) a basis for \( \mathbb{P}_2 \)?