Today: §5.1-5.2: Eigenvalues; Characteristic Polynomial

Next: §5.3: Diagonalization

Reminders:
MATLAB Homework #5: Due next Friday (March 10)
MyMathLab Homework #7: Due Monday, March 13
MATLAB QUIZ: March 14 (or 13)
Given an \( n \times n \) matrix \( A \), an eigenvector is a vector \( \mathbf{v} \in \mathbb{R}^n \) with the property that
\[
A \mathbf{v} = \lambda \mathbf{v} \quad (\therefore A(2\mathbf{v}) = 2A\mathbf{v} = 2\lambda \mathbf{v})
\]
for some scalar \( \lambda \in \mathbb{R} \), called the eigenvalue.

- The set of eigenvectors of \( A \) for a given eigenvalue \( \lambda \) is equal to \( \text{Nul}(A - \lambda I) \). So it is a subspace of \( \mathbb{R}^n \), called the eigenspace for \( \lambda \).

- Typically hard to find the eigenvalues of a matrix; once known, finding the eigenspace is routine.

- If \( A \) is triangular, its eigenvalues are the diagonal entries.
Theorem: Eigenvectors with different eigenvalues are linearly independent.

Pf. (Case: 2 eigenvalues \( \lambda, \mu \) (\( \lambda \neq \mu \))

\[ A\mathbf{v} = \lambda \mathbf{v} \quad \text{and} \quad A\mathbf{w} = \mu \mathbf{w} \]

Suppose that \( \mathbf{v}, \mathbf{w} \) are linearly dependent.

They are parallel, so \( \mathbf{w} = c\mathbf{v} \) for some scalar \( c \).

Also \( \mu \mathbf{w} = A\mathbf{w} = A(c\mathbf{v}) = cA\mathbf{v} = c\lambda \mathbf{v} \)

\[ 0 = \mu \mathbf{w} - \mu \mathbf{w} = c\lambda \mathbf{v} - c\mu \mathbf{v} \]

Contradiction: \( c(\lambda - \mu)\mathbf{v} < 0 \neq 0 \)
Corollary

If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues, then there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$. Let's write any vector in the $\mathbb{R}^n$ basis:

$$v = x_1 A v_1 + x_2 A v_2 + \ldots + x_n A v_n$$

Consider the eigenvectors of $A$ consisting of vectors $v_1, v_2, \ldots, v_n$ such that $v_i = \lambda_i v_i$. The eigenvalues, then, are a basis of $\mathbb{R}^n$. If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues, then there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$. Let's write any vector in the $\mathbb{R}^n$ basis:

$$v = x_1 A v_1 + x_2 A v_2 + \ldots + x_n A v_n$$

Corollary: If $A$ is an $n \times n$ matrix with $n$ distinct
Definition: \( n \times n \) matrices \( A \) and \( B \) are called similar if there is an invertible \( n \times n \) matrix \( P \) with the property

\[
A = PBP^{-1} \quad (\therefore AP = PB \quad \therefore P^{-1}AP = B \quad \text{where} \quad Q = P^{-1})
\]

- We just saw that, if \( A \) has all distinct eigenvalues, then it is similar to a diagonal matrix. That's important; more on that next time.

- Similarity is not the same as row equivalent.

Eg. \( A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{row equivalent, but not similar.} \)

\[
(Pz \begin{bmatrix} a & b \\ c & d \end{bmatrix}; \quad \text{want} \quad AP = PB = P \quad \text{\therefore} \quad \begin{bmatrix} \quad \color{red}{a+c} & \color{red}{b+d} \\ \quad \color{red}{c} & \quad \color{red}{d} \end{bmatrix} = \begin{bmatrix} \quad \color{red}{a} & \quad \color{red}{b} \\ \quad \color{red}{c} & \quad \color{red}{d} \end{bmatrix} \quad \text{\therefore c = d = 0.} \quad \text{P = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \quad \color{red}{\text{not invertible.}}}
\]
Theorem: If $A$ and $B$ are similar, then they have the same eigenvalues; and the eigenspaces for $A$ have the same dimensions as the eigenspaces for $B$.

Pf. Let $\lambda$ be an eigenvalue of $A$, and let $E_\lambda$ be its eigenspace: $E_\lambda = \text{Null}(A-\lambda I)$.

Let $B = PAP^{-1}$.

$A\mathbf{v} = \lambda \mathbf{v}$

$P^{-1}BP\mathbf{v} = \lambda \mathbf{v}$

$BP\mathbf{v} = P(\lambda \mathbf{v}) = \lambda (P \mathbf{v})$.

So $P \mathbf{v}$ is an eigenvector of $B$ with eigenvalue $\lambda$. 
Consider the matrices $A$ and $B$ and their eigenvalues.

For $A$, its eigenvalues are $1$ and $2$.

For $B$, its eigenvalues are $1$.

The eigenspace of $A$ with eigenvalue $1$ has dimension $1$.

The eigenspace of $A$ with eigenvalue $2$ has dimension $1$.

The eigenspace of $B$ with eigenvalue $1$ has dimension $1$.

The nullspace of $A - I$ is the nullspace of $B - I$.

$\dim(\text{null}(A - I)) = \dim(\text{null}(B - I))$

$\dim(\text{null}(I - A)) = \dim(\text{null}(I - B))$

$\dim(\text{null}(I - B)) = 1$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
How can we actually find the eigenvalues?

\( \lambda \) is an eigenvalue of \( A \) iff 
\[ \text{Nul} (A - \lambda I) \neq \{0\} \]
iff \( A - \lambda I \) is not invertible
iff \( \det (A - \lambda I) = 0 \).

\[
A - \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}
\]
\[
\det (A - \lambda I) = (a - \lambda)(d - \lambda) - bc = \lambda^2 - (a + d)\lambda + ad - bc
\]

**Definition:** The characteristic polynomial of a square matrix \( A \) is 
\[
P_A(\lambda) = \det (A - \lambda I)
\]

**Theorem:** If \( A \) is \( n \times n \), \( P_A \) is a degree \( n \) polynomial, 
whose roots are the eigenvalues of \( A \).
Eg. \(A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}\)  
\[p_A(\lambda) = \det \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 9\]

Solve \(p_A(\lambda) = 0\)  
\((5-\lambda)^2 - 9 = 0\)
\[(5-\lambda)^2 = 9\]
\[5-\lambda = \pm 3\]  \(\Rightarrow\)  \(\lambda = 2, 8\).

\[A - 2I = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\]  
Note: \(\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}\)  
\(\Rightarrow\)  \(A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}\)

Eg. \(A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}\)  
\(\text{NEXT TIME}\)
Eg. \[ A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix} \]
Theorem: If $A$ and $B$ are similar, they have the same characteristic polynomial. Therefore, they have the same eigenvalues with the same (algebraic) multiplicities.

NEXT TIME