Today § 7.1: The Spectral Theorem

Next: Review

Reminders: Please fill out your CAPEs.

MyMathLab Homework #8: Due **TOMORROW** by 11:59pm

FINAL EXAM: This Saturday, 11:30am–2:30pm
GH 242, PETER 108, YORK 2722
Seating/Room Assignment on Triton Ed
Given a collection \( \{ v_1, v_2, \ldots, v_p \} \) of linearly independent vectors in \( \mathbb{R}^n \), the Gram-Schmidt orthogonalization process produces a new collection \( \{ \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_p \} \) of orthonormal vectors, with the same span as the \( v_j \)’s.

\[
\begin{align*}
\hat{u}_1 &= u_1 / \|u_1\| \\
\hat{u}_2 &= u_2 / \|u_2\| \\
\hat{u}_3 &= u_3 / \|u_3\| \\
&\vdots
\end{align*}
\]

The triangular pattern here can be summarized by noting this means

\[
\begin{bmatrix}
v_1 & v_2 & \cdots & v_p \\
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & \cdots & h_{1p} \\
0 & h_{22} & \cdots & h_{2p} \\
0 & 0 & \cdots & h_{pp} \\
\end{bmatrix}
= 
\begin{bmatrix}
\hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_p \\
\end{bmatrix}
\begin{bmatrix}
v_1 & v_2 & \cdots & v_p \\
\end{bmatrix}
\begin{bmatrix}
\hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_p \\
\end{bmatrix}
\begin{bmatrix}
h_{11} & h_{12} & \cdots & h_{1p} \\
0 & h_{22} & \cdots & h_{2p} \\
0 & 0 & \cdots & h_{pp} \\
\end{bmatrix}
\]

These come up in the G-S process.
Example:

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ -2 & -1 \end{bmatrix}$$

Produce an o.n.b. of span\(\{v_1, v_2\}\) \(\subseteq C(\mathbf{A})\)

\(u_1 = v_1\)

\[\|u_1\|^2 = 2^2 + 1^2 + (-2)^2 = 9\] \(\|u_1\| = 3\).

\(v_2 - u_1 = v_2 - v_1\) = 3.

\[\hat{u}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}\]

\(u_2 = v_2 - \frac{v_2 \cdot u_1}{\|u_1\|^2} u_1 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{3}{9} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}\)

\[\|u_2\|^2 = \left(\frac{-2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1\]

\(\hat{u}_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}\)

\[\|\hat{u}_2\|^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1\]

\[
\hat{A} = \hat{Q} \hat{R}
\]

\(Q = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 \end{bmatrix} = \begin{bmatrix} -2/3 & 2/3 \\ 1/3 & 1/3 \\ -2/3 & -2/3 \end{bmatrix}\)

\(Q^T = Q\)

\(\hat{R} = R = \begin{bmatrix} 2/3 & 1/3 & -2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}\)

\(A = QR\)

\(Q^T Q = I\)
Note: $Q^TQ = I$ does not mean $Q^{-1} = Q^T$! (Only if $Q$ is square; in general $QQ^T = \text{orthogonal projection onto Col}(Q)$.)

Why should you care about $QR$-factorization?
Take the square matrix case.

\[ A = QR \rightarrow A_1 = RQ = Q^{-1}AQ \sim A \text{ similar } \because \text{they have the same eigenvalues} \]

\[ \begin{align*}
    Q^T A_1 &= R \\
    Q^{-1} A &= A_1 = Q_1 R_1 \\
    A_2 &= R_1 Q_1 = Q^{-1} A_1 Q_1 \sim A_1 \sim A \\
    \vdots \\
    A, A_1, A_2, A_3, \ldots, A_{100}, \ldots
\end{align*} \]

**Thm:** $A_n \underset{n \to \infty}{\to} (\text{rapidly})$ to an upper-$\Delta$ matrix.
Recall that $A \in \mathbb{M}_{n \times n}$ is diagonalizable if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of $A$. \((\Leftarrow) A = PDP^{-1}\)

What kinds of matrices have an orthonormal basis of eigenvectors?

$$A = PDP^{-1} = PDPT$$

$P$ is a matrix whose columns are orthonormal eigenvectors of $A$; if they are o.n., \(P^{-1} = P^T\)

$$A^T = (PDPT)^T = (P^T)^T D^T P^T = PDPT = A \text{ "symmetric"}$$

Conclusion: if $A$ is orthogonally diagonalizable, then $A = A^T$.

What about the converse? $Au = \lambda u$, $Av = \mu v$ \((\lambda \neq \mu)\)

\[\begin{align*}
(\lambda - \mu)u \cdot v &= \lambda u \cdot u - \mu u \cdot v = (\lambda u) \cdot v - u \cdot (\mu v) = (Au) \cdot v - u \cdot (Av), \\
(Au)^T v - u^T Av &= u^T A^T v - u^T Av = 0.
\end{align*}\]

\[
\begin{align*}
(Au)^T = u^T A^T \quad \therefore \quad u \cdot v = 0 \quad \text{i.e.} \quad u \perp v
\end{align*}\]
Suppose $A$ is symmetric, and happens to be diagonalizable.

\[ \text{The eigenspaces of } A \text{ span all of } \mathbb{R}^n. \]

\[ \Rightarrow \text{we just saw that eigenvectors for distinct eigenvalues are } \perp. \]

So eigenspaces for distinct eigenvalues are orthogonal.

\[ \Rightarrow \text{if an eigenvalue has geometric multiplicity } > 1, \text{ can take any basis of this eigenspace and produce an o.n.b.} \]

\[
\text{Conclusion: There is an o.n.b. of eigenvectors of } A. \]

\[ \text{I.e. } A \text{ is orthogonally diagonalizable.} \]
The Spectral Theorem

Every symmetric matrix is orthogonally diagonalizable.

This gives the spectral decomposition of a symmetric matrix:

\[ (A^T = D D P^T = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} \hat{u}_1^T \\ \hat{u}_2^T \\ \vdots \\ \hat{u}_n^T \end{bmatrix} \]

\[ = \lambda_1 \hat{u}_1 \hat{u}_1^T + \lambda_2 \hat{u}_2 \hat{u}_2^T + \cdots + \lambda_n \hat{u}_n \hat{u}_n^T 
\]

\[ \Rightarrow \lambda_1 \text{Proj} \hat{u}_1 + \lambda_2 \text{Proj} \hat{u}_2 + \cdots + \lambda_n \text{Proj} \hat{u}_n. \]
Why “spectral”? What has any of this got to do with a spectrum?

Heisenberg → “Matrix mechanics”
The vector space that describes the state of the Hydrogen atom is like $\mathbb{P}$: a space of functions.

They are "wave-functions" that represent probability distributions of how likely it is to find the particle near each point in space.

The eigenvectors are an orthonormal basis for these wave functions; they are the states with a well-defined (quantized) energy (the eigenvalues).

You’ve seen pictures of them before...