Today: § 6.4: Gram-Schmidt Orthogonalization
Next: § 7.1: Spectral Theorem

Reminders:
Please fill out CAPEs
MyMathLab Homework #8 Due next Tuesday
Final Exam: Friday, March 24 11:30a-2:30p
CENTR 101 & 217A
How do we find $\text{Proj}_V$ for a given subspace $V$?

We already saw that if $V = \text{span}\{y\}$ (1-dimensional),

$$\text{Proj}_V(y) = \frac{y \cdot u}{y \cdot y} u.$$

(What happens if you scale $u$?)

**Theorem:** Let $\{u_1, u_2, \ldots, u_p\}$ be an orthogonal basis for $V$. Then

$$\text{Proj}_V(y) = \frac{y \cdot u_1}{\|u_1\|^2} u_1 + \frac{y \cdot u_2}{\|u_2\|^2} u_2 + \cdots + \frac{y \cdot u_p}{\|u_p\|^2} u_p.$$
This allows us to compute the matrix of $\text{Proj}_V$. Start with an orthonormal basis for $V$; then

$$\text{Proj}_V(y) = (y \cdot u_1)u_1 + (y \cdot u_2)u_2 + \ldots + (y \cdot u_p)u_p$$

**Theorem:** Let $V \subseteq \mathbb{R}^n$ be a subspace, and fix an orthonormal basis $\{u_1, \ldots, u_p\}$ for $V$. Let $U = [u_1 \ u_2 \ldots \ u_p]$. 
Eg. Compute the matrix of the orthogonal projection in $\mathbb{R}^3$ onto the subspace $\text{span}\{[1, -2, 2]^T\}$. 
What is an orthogonal projection, really?

**Theorem:** \( \text{Proj}_V(y) \) is the point in \( V \) that is closest to \( y \).

I.e. it is the best approximation of \( y \) in \( V \).

E.g. \( \{u_1 = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \ u_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \} \) is an orthogonal basis for \( V \).

Find the closest point in \( V \) to \( y = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \).
What if you aren't given an orthogonal basis?
Can you even be sure that one exists?

Eg. \( V = \text{Nul} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 3 \end{bmatrix} \)
The Gram–Schmidt Orthogonalization Process

Given a basis \( B = \{v_1, v_2, \ldots, v_p\} \) of a subspace \( V \), we can find a new basis \( \mathcal{O} = \{u_1, u_2, \ldots, u_p\} \) with the properties:

1) \( \mathcal{O} \) is orthogonal

2) \( \text{span}\{v_1, v_2, \ldots, v_k\} = \text{span}\{u_1, u_2, \ldots, u_k\} \) for \( 1 \leq k \leq p \).
Eg. Find an orthogonal basis for $\mathbb{R}^2$ that includes the vector \[
\begin{bmatrix}
1 \\
2 \\
\end{bmatrix}
\]
Gram-Schmidt takes a linearly independent set \([v_1, v_2, \ldots, v_n]\) in \(\mathbb{R}^m\) and produces an orthogonal set \([u_1, u_2, \ldots, u_n]\) with the same span. What does it tell us about the matrices \[
abla_1 \quad v_2 \quad \cdots \quad v_n \] \(\text{vs.}\) \[
abla_1 \quad u_2 \quad \cdots \quad u_n \)?
QR decomposition

Using Gram-Schmidt, any matrix $A \in \mathbb{M}_{m \times n}$ with linearly independent columns can be decomposed

$$A = QR$$

Eg. Find the QR decomposition of

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
How MATLAB actually computes eigenvalues: