Getting CDF from PDF

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Given a probability density function (pdf, or just density function), \( p(x) \), we have the following properties:

1. \( \int_{-\infty}^{\infty} p(x) \, dx = 1 \)
2. \( p(x) \geq 0 \) always

Now, given a cumulative distribution function (cdf), \( P(x) \), we have the properties:

1. \( P(x) \to 1 \) as \( x \to \infty \) (as we move to the right \( P(x) \) must go to 1)
2. \( P(x) \to 0 \) as \( x \to -\infty \) (as we move to the left \( P(x) \) must go to 0)
3. \( P(x) \) is never decreasing
4. \( P(x) \) does NOT need to be smooth, but IS continuous

A way to remember this is that \( P(x) \) must ‘start’ at 0 and ‘end’ at 1. The relationship between \( P(x) \) and \( p(x) \) is that \( P(x) = \int_{-\infty}^{x} p(t) \, dt \), which tells us 1) that \( P(x) \) is an anti-derivative of \( p(x) \) and 2) that the derivative of \( P(x) \) is \( p(x) \); that is, \( \frac{d}{dx} P(x) = p(x) \).

A question that came up in section was, ”given the graph of \( p(x) \), how do we find the graph \( P(x) \)?” Here are step by step examples of how to solve this problem:

**Example 1: Question 9 section 8.7**

We are given that the pdf is

\[
p(x) = \begin{cases} 
4x & 0 \leq x \leq 1/2 \\
-4x + 4 & 1/2 \leq x \leq 1 
\end{cases}
\]

and we want to find the equation/graph of \( P(x) \). We have that \( P(x) \) is an anti-derivative of \( p(x) \), so integrating the terms of \( p(x) \) we get:

\[
P(x) = \begin{cases} 
2x^2 + A & 0 \leq x \leq 1/2 \\
-2x^2 + 4x + C & 1/2 \leq x \leq 1 
\end{cases}
\]

where \( A \) and \( C \) are constants of integration. Now, we use the properties of \( P(x) \) to find those constants. Since \( P(x) \) must go to 0 as we move to the left and we have assumed that \( P(x) \) ’starts’ at 0, so we must have \( P(0) = 0 = 2(0)^2 + A \). So \( A = 0 \). Also, as we move to the right \( P(x) \) must
go to 1, and we have assumed that $P(x)$ 'ends’ at 1, so we need $P(1) = 1 = -2(1)^2 + 4(1) + C$, so $C = -1$. So, if we were to write out what $P(x)$ was, we have

$$P(x) = \begin{cases} 
2x^2 & 0 \leq x \leq 1/2 \\
-2x^2 + 4x - 1 & 1/2 \leq x \leq 1 
\end{cases}$$

Another way we could have found the constant $C$ would be by using the fact that $P(x)$ must be continuous. This means that the graph of $P(x)$ must touch when $x = 1/2$ (the two pieces of the piecewise graph must touch). So, looking at the equation for $P(x)$ just to the left of $1/2$ we have that $P(1/2) = 2(1/2)^2 = 1/2$. Using the equation of $P(x)$ just to the right of $1/2$ we get that $P(1/2) = -2(1/2)^2 + 4(1/2) + C = -1/2 + 2 + C = 1.5 + C$. Setting the value of $P(1/2)$ just to the left equal to the value of $P(1/2)$ just to the right, we get: $1.5 + C = 1/2$. So, we again get that $C = -1$. Notice that both ways of solving for $C$ gives us the same answer!

**Example 2**

Suppose now we are given that $p(t) = 5e^{-5t}$, where $t$ is time. Now, we assume that the graph of $p(t)$ starts at $t = 0$, or another way to view it is that $p(t) = 0$ for any $t < 0$. So, we can think of $p(t)$ starting at 0 also.

The way we solved this during section is by using the equation $P(t) = \int_0^t 5e^{-5x}dx$ where here the lower bound on the integration is 0, where $p(t)$ starts. Solving this integral and plugging in the bounds we have $P(t) = -e^{-5t} + 1$.

Another way to find $P(t)$ we can first find the general anti-derivative of $p(t)$. We get that $\int p(t)dt = -e^{-5t} + C$ for some constant $C$. So, $P(t) = -e^{-5t} + C$. To find out what $C$ is we know that when $P(x)$ starts it must be 0, so, since $P(t)$ starts at $t = 0$ we have $0 = P(0) = -e^{-5(0)} + C = -1 + C$, so $C = 1$.

Another way to solve for $C$ is to know that as we go to the right $P(t)$ must go to 1. But here, $P(t)$ doesn’t ”end”, but goes off to $\infty$. So, what we want to say is that $\lim_{b \to \infty} -e^{-5b} + C = 1$. Since $\lim_{b \to \infty} -e^{-5b} = 0$, we are just left with $C = 1$. Again, we get the same answer! So, $P(t) = -e^{-5t} + 1$. 

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