.

On the Winner Problem

November 21, 2022

Let $X_1, X_2, \ldots$ be non-negative and independent r.v., $F_i(x) = P(X_i \le x)$, and

$$\nu_i(x) = -\ln F_i(x).$$

If $F_i(x) = 0$, we set $\nu(0) = \infty$.
Clearly, for all i,

$$F_i(x) = \exp\{-\nu_i(x)\}, \tag{1}$$

$$\nu_i(x) \ \text{ is non-increasing, } \ \nu_i(0) = \infty, \ \nu_i(\infty) = 0, \tag{2}$$

and the asymptotic behavior of $\nu_i(x)$ as $x \to \infty$ is equivalent to that of $1 - F_i(x)$.

EXAMPLE

$$F_i(x) = \exp\left\{-\frac{c_i}{x^\alpha}\right\},$$

Let

$$p_{in} = P(X_i = \max\{X_1, \ldots, X_n\}).$$

Assuming $\nu_i(x)$ to be smooth, we have

$$\begin{aligned} p_{in} &= \int_0^\infty \prod_{j=1,\, j\neq i}^n F_j(x) dF_i(x) = -\int_0^\infty \exp\left\{-\sum_{j=1, j\neq=i}^n \nu_j(x)\right\} \exp\{-\nu_i(x)\} d\nu_i(x) \\ &= -\int_0^\infty \exp\left\{-\sum_{j=1}^n \nu_j(x)\right\} d\nu_i(x). \end{aligned}$$

Integrating by parts and taking into account (1)–(2), we have

$$p_{in} = -\int_0^\infty \nu_i(x) \exp\left\{-\sum_{j=1}^n \nu_j(x)\right\} d\sum_{j=1}^n \nu_j(x). \qquad (3)$$

For any non-increasing function $r(x)$, we define its inverse as

$$r^{-1}(y) = \sup\{x : r(x) \geq y\}.$$

Let $x_n(y)$ be the inverse of the function $\sum_{i=1}^n \nu_i(x)$; in other words, a solution (in the above sense) to the equation

$$\sum_{i=1}^n \nu_i(x) = y.$$

Then, from (3) it follows that

$$p_{in} = \int_0^\infty \nu_i(x_n(y)) e^{-y} dy. \qquad (4)$$

Let, for example,

$$\nu_i(x) = c_i r(x). \qquad (5)$$

Then

$$x_n(y) = r^{-1}\left(\frac{y}{\sum_{i=1}^n c_i}\right), \qquad (6)$$

and

$$\nu_i(x_n(y)) = c_i r\left(r^{-1}\left(\frac{y}{\sum_{i=1}^n c_i}\right)\right) = \frac{c_i}{\sum_{i=1}^n c_i} y. \qquad (7)$$

Thus, in this case,

$$p_{in} = \frac{c_i}{\sum_{i=1}^n c_i} \int_0^\infty y e^{-y} dy = \frac{c_i}{\sum_{i=1}^n c_i}.$$

For a general scheme, we can assume, *for example*, the following.

1. Uniformly in i,

$$\frac{\nu_i(x)}{c_i r(x)} \to 1 \quad \text{as} \quad x \to \infty. \tag{8}$$

2.

$$\sum_{i=1}^n c_i \to \infty \quad \text{as} \quad n \to \infty. \tag{9}$$

3. There exists a non-negative function $c(x)$ on $[0,1]$ such that $\int_0^1 c(x)dx = 1$ and if

$$\frac{k_n}{n} \to z \in [0,1],$$

then

$$\frac{\sum_{i=1}^{k_n} c_i}{\sum_{i=1}^n c_i} \to \int_0^z c(x)dx \tag{10}$$

For example, this is true if we consider a triangular array scheme ($c_i = c_{in}$) and there exists a function $g(x)$ on $[0,1]$ such that $c_{in} = g(i/n)$. Then

$$c(x) = \frac{g(x)}{\int_0^1 g(x)dx}.$$

CONJECTURE 1

Under conditions (8)–(9), relation (6) may be replaced by

$$x_n(y) \sim r^{-1}\left(\frac{y}{\sum_{i=1}^n c_i}\right) \quad \text{as} \quad n \to \infty, \tag{11}$$

CONJECTURE 2

Consider probability measures μ_n on $[0,1]$, that assigns to points i/n. $i = 1,...,n$ the probabilities p_{in}, $i = 1,...,n$, respectively.

Then μ_n weakly converge to an absolutely continuous measure μ on $[0,1]$ whose density equals $c(x)$.

CONJECTURE 3

In the case where there may be several ”winners”, we should pick one at random.

EXAMPLE regarding Conjecture 3

Let all $X_i = 0$ or 1 with equal probabilities, i.e., $1/2$.

Then
$$p_{in} = \frac{1}{2} \times 1 + \frac{1}{2} \times \frac{1}{2^{n-1}} = \frac{1}{2} + \frac{1}{2^n}.$$
However, in the case of selecting a winner at random (throwing lots), just by symmetry, for all i
$$p_{in} = \frac{1}{n}.$$

I would like to thank professor Vadim Ponomarenko (SDSU) a bygone conversation (about an application problem) with whom helped me to come to the statement of this problem.