SUBFIELDS OF NONDEGENERATE TAME SEMIRAMIFIED DIVISION ALGEBRAS

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Abstract. We show in this article that in many cases the subfields of a nondegenerate tame semiramified division algebra of prime power degree over a Henselian valued field are inertial field extensions of the center [Th. 2.5, Th. 2.12 and Prop. 2.16].

Introduction

In their work on abelian crossed product algebras [AS], Amitsur and Saltman defined a condition they called *nondegeneracy* for matrices encoding the multiplicative structure of such algebras. They used nondegenerate generic abelian crossed product algebras to prove the existence of noncyclic p-algebras, thereby settling a question that had been open since Albert's work on p-algebras in the 1930's. Saltman at that time also showed in [S_1] that all Galois maximal subfields of a nondegenerate generic abelian crossed product p-algebra have the same Galois group; he used this to prove the existence of noncrossed product p-algebras. Later, in [S_2, Th. 7.17] he used nondegenerate generic abelian crossed products to give examples of indecomposable division algebras of exponent p and degree p^2, for any odd prime p, over a field containing a primitive p-th root of unity. More recently, McKinnie in [Mc_1, Def. 2.4] defined nondegeneracy for inertially split semiramified divsion algebras over Henselian valued fields in terms of nondegeneracy of certain matrices over the residue field; she used this to study prime-to-p extensions of generic crossed product p-algebras. In [Mc_2] she further proved the indecomposability of nondegenerate inertially split semiramified p-algebras over Henselian fields of characteristic p. Independently of McKinnie's work, the first author defined in [M_2] nondegeneracy in the somewhat more general context of inertially split division algebras over Henselian fields; in the semiramified case considered in [Mc_2] this definition agrees with McKinnie's definition, and also that of Amitsur and Saltman. He proved in particular in [M_2, Th. 3.5] that in all characteristics a nondegenerate inertially split semiramified division algebra of prime power degree over a Henselian valued field is indecomposable.

The various formulations of nondegeneracy will be reviewed at the beginning of §2 below.

In all the work just described, the nondegeneracy condition for the algebras was crucial in obtaining constraints on the possible subfields of the algebras which are normal over the center. Thus, it seems worthwhile to investigate more closely the possible subfields of a nondegenerate division algebra, particularly the normal subfields. We do this here in the Henselian setting, i.e., where E is a field with a Henselian valuation v, and D is a division algebra of prime power degree over E; v extends uniquely to a valuation w on D, and it is assumed that D is inertially split and semiramified over E with respect to w. (The valuation-theoretic terminology used here will be recalled later in this Introduction.) There is then a unique up to isomorphism maximal subfield N of D which is inertial (= unramified) over E, and N is abelian Galois over E. The inertial field extensions of E in D are fully classified up to isomorphism as

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the subfields of N, and they are all abelian Galois over E. The question is thus what other subfields of D may exist. This will be studied in §2 below.

Whenever there is a valuation v on a division algebra D, the filtration of D induced by v yields an associated graded ring GD in which every nonzero homogeneous element is a unit—this is called a graded division ring. When the valuation on the center $Z(D)$ is Henselian, the structure of the graded ring closely mimics that of D. Likewise, algebraic field extensions of a Henselian valued field correspond to graded field extensions of the associated graded field. In §1 we will prove some properties for algebraic extensions of valued and graded fields, which have some interest in their own right and are needed for §2. We show in Th. 1.5 that if (E, v) is a valued field and (M, w) is a normal finite-dimensional valued field extension of (E, v), then the associated graded field GM is a normal graded field extension of GE. We give in Th. 1.9 an extension of Hensel's Lemma for polynomials over a valued field all of whose roots have the same value. We prove also in Prop. 1.12 that if E is a Henselian valued field with residue characteristic $p > 0$ and L is a purely wild (resp., simple purely wild) finite-dimensional graded field extension of GE, then there is a defectless field extension (resp., a defectless simple field extension) K of E such that $GK = L$. Moreover, if $\mathrm{char}(E) = p$, then K can be a purely inseparable field extension of E. We give in Cor. 1.13 a new (and more explicit) proof of [HW_1, Th. 5.2] which for an arbitrary Henselian valued field E establishes a one-to-one correspondence between the set of isomorphism classes of finite-dimensional tame field extensions of E and the set of isomorphism classes of finite-dimensional tame graded field extensions of GE.

In §2 we consider subfields of nondegenerate algebras. Let E be a field with a Henselian valuation v, with residue field $\overline{E}$ and value group Γ_E. Let D be a nondegenerate inertially split semiramified division algebra with center E, of degree p^n for some prime p. We show in Th. 2.5 that if $\mathrm{char}(\overline{E}) = p$ and Γ_D/Γ_E is not cyclic, then any subfield of D normal over E is an abelian Galois inertial extension of E, and that all maximal subfields of D Galois over E have the same Galois group. We prove also in Prop. 2.8 that for E of any residue characteristic and Γ_D/Γ_E noncyclic, D is an elementary abelian crossed product if and only if $\mathrm{Gal}(\overline{D}/\overline{E})$ is elementary abelian. Further, we prove in Prop. 2.10 that if $\mathrm{rk}(\Gamma_D/\Gamma_E) \geq 3$, then all subfields of D abelian Galois over E are inertial over E. We show also that if $\mathrm{rk}(\Gamma_D/\Gamma_E)$ is arbitrary but $\exp(\Gamma_D/\Gamma_E) = p$, then any non-maximal subfield of D is inertial over E. In this case, we show in Th. 2.12 that if Γ_D/Γ_E is noncyclic and K is a maximal subfield of D which is normal over E with Galois group not the quaternion group, then either K is cyclic Galois over E with $[K : E] = p^2$ or K is inertial over E. More results concerning subfields in the case where $\mathrm{char}(\overline{E}) \nmid \deg(D)$ are proved at the end of §2.

We now recall some basic terminology from the theory of valued and graded division algebras which will be used throughout the paper.

Let E be a field, D a finite-dimensional division algebra over E, and Γ a totally ordered abelian group. Let ∞ be an element of a set strictly containing Γ with $\infty \notin \Gamma$; extend the order on Γ to $\Gamma \cup \{\infty\}$ by setting $\gamma < \infty$ for all $\gamma \in \Gamma$, and define $\gamma + \infty = \infty + \infty = \infty$. A map $v\colon D \to \Gamma \cup \{\infty\}$ is called a valuation on D if it satisfies the following conditions (for all $c, d \in D$):

(1) $v(c) = \infty$ if and only if $c = 0$;
(2) $v(cd) = v(c) + v(d)$;
(3) $v(c + d) \geq \min\{v(c), v(d)\}$.

We will say that (D, v) is a valued division algebra over E. The value group of v on D is $\Gamma_D = v(D^*)$, where $D^* = D \setminus \{0\}$, the group of multiplicative units of D. The residue division algebra is

$$\overline{D} \;=\; V_D/M_D \;=\; \{d \in D \mid v(d) \geq 0\} \big/ \{d \in D \mid v(d) > 0\}.$$

Of course, v restricts to a valuation on E; we write $|\Gamma_D : \Gamma_E|$ for the ramification index of D over E, which is the index in Γ_D of its subgroup Γ_E. Also, we identify the residue field $\overline{E}$ with its canonical image in $\overline{D}$, and write $[\overline{D} : \overline{E}]$ for the residue degree of D over E, which is the dimension of $\overline{D}$ as an $\overline{E}$-vector space. For background on valued division algebras, the reader can consult [JW] or [W].

Recall the Fundamental Inequality ([Sch, p. 21, Lemma 18; p. 22, Remark]):

$$[\overline{D} : \overline{E}]\,|\Gamma_D : \Gamma_E| \;\leq\; [D : E] \;<\; \infty.$$

D is said to be *defectless* over E if $[D : E] = [\overline{D} : \overline{E}]\,|\Gamma_D : \Gamma_E|$. We say D *inertial* (or *unramified*) over E if $[D : E] = [\overline{D} : \overline{E}]$ and the center $Z(\overline{D})$ is separable over $\overline{E}$. At the other extreme, D is *totally ramified* over E if $[D : E] = |\Gamma_D : \Gamma_E|$. Now assume $E = Z(D)$. Then, D is *inertially split* if it has a maximal subfield which is inertial over E. Also, D is said to be *semiramified* if it is defectless over E, $\overline{D}$ is a field, and $[\overline{D} : \overline{E}] = |\Gamma_D : \Gamma_E|$. It is called *nicely semiramified* if it is inertially split and it has a *totally ramified of radical type* maximal subfield, i.e., a maximal subfield K totally ramified over E such that $K = E[t_1^{1/n_1}, ..., t_r^{1/n_r}]$, where $t_1, ..., t_r \in E^*$, $\Gamma_K/\Gamma_E = \bigoplus_{i=1}^{r} \langle v(t_i^{1/n_i}) + \Gamma_E \rangle$ and $\operatorname{ord}(v(t_i^{1/n_i}) + \Gamma_E) = n_i$. For $a \in V_D$, we write $\overline{a}$ for the image of a in $\overline{D} = V_D/M_D$. There is a well-defined canonical group homomorphism θ_D mapping Γ_D/Γ_E to the Galois group $\operatorname{Gal}(Z(\overline{D})/\overline{E})$, given by $\theta_D(\gamma + \Gamma_E) \colon \overline{a} \mapsto \overline{dad^{-1}}$ for all $a \in V_D$ with $\overline{a} \in Z(\overline{D})$, where d is an arbitrary element of D^* with $v(d) = \gamma$. By [JW, Prop. 1.7], $Z(\overline{D})$ is normal over $\overline{E}$ and θ_D is surjective. Moreover, if D is inertially split, then so is $HD = HF \otimes_F D$, where HF is the Henselization of F re v; by [JW, Lemma 5.1], applied to HD, and [Mor, Th. 2] it then follows that $Z(\overline{D})$ is Galois over $\overline{E}$ and θ_D is an isomorphism. If the valuation on E is Henselian, we say that D is *tame* if D is defectless over E, $Z(\overline{D})$ is separable over $\overline{E}$, and $\operatorname{char}(\overline{E}) \nmid |\ker(\theta_D)|$. Equivalently (see [HW$_2$, Prop. 4.3]), D is tame iff it is split by the maximal tamely ramified field extension of E.

We will be working with graded division rings and fields as well as valued ones. We recall some of the terminology and basic facts in the graded setting, and the connections between the valued setting and the graded setting.

Let F be an associative ring (with 1) and let Γ be a totally ordered abelian group. We say that F is a *graded ring of type* Γ if there are additive subgroups F_γ ($\gamma \in \Gamma$) of F such that $F = \bigoplus_{\gamma \in \Gamma} F_\gamma$ and $F_\gamma F_\delta \subseteq F_{\gamma+\delta}$, for all γ, $\delta \in \Gamma$. In this case, the set $\Gamma_F = \{\gamma \in \Gamma \mid F_\gamma \neq 0\}$ is called the *support* of F. If F is a graded ring of type Γ and $x \in F_\gamma$ for some $\gamma \in \Gamma_F$, we say that x is a homogeneous element of F; if $x \neq 0$, we say that x has *grade* γ and we write $\operatorname{gr}(x) = \gamma$.

A graded ring F (of type Γ) which is commutative and for which all nonzero homogeneous elements are invertible is called a *graded field*. Note that, because of the total ordering on Γ, in a graded field F every element of the group F^* of multiplicative units must be homogeneous; so F^* coincides with the set of nonzero homogeneous elements of F. Furthermore, the total ordering on Γ implies that F is an integral domain. It is easy to see also that if M is a graded F-module (i.e., $M = \bigoplus_{\gamma \in \Gamma} M_\gamma$ with $F_\gamma M_\delta \subseteq M_{\gamma+\delta}$ for all $\gamma, \delta \in \Gamma$) then M is a free F-module with a homogeneous base, and any two such bases have the same cardinality. We therefore write $\dim_F(M)$ for the rank of M as free F-module.

Let F be a commutative graded ring of type Γ. An algebra A over F is called a *graded algebra* (of type Γ) over F if A is a graded ring of type Γ and $F_\gamma \subseteq A_\gamma$, for all $\gamma \in \Gamma$. If F and A are graded fields, we call A a graded field extension of F. If F is a graded field, then a graded algebra over F in which every nonzero homogeneous element is a unit is called a *graded division algebra* over F. If F is the center of a graded division algebra A, then A is called a *graded central division algebra* over F. We write $[A : F]$ for $\dim_F(A)$. For a graded division algebras A, the support set Γ_A is a subgroup of Γ, and A_0 is a division ring which is an algebra over F_0. Furthermore, it is easy to prove the Fundamental Equality:

$$[A : F] \;=\; [A_0 : F_0]\,|\Gamma_A : \Gamma_F|.$$

Let F be a graded field, let $q(F)$ be its quotient field, and let $q(F)_{\rm alg}$ be an algebraic closure of $q(F)$. Clearly, for any element λ of the divisible hull Δ_F of Γ_F (i.e., $\Delta_F = \Gamma_F \otimes_{\mathbb{Z}} \mathbb{Q}$), there is a unique grading of type Δ_F on the polynomial ring $F[X]$ extending the grading of F and for which X is a homogeneous element with $\mathrm{gr}(X) = \lambda$. We denote $F[X]$ with this grading by $F[X]^{(\lambda)}$. A polynomial $f \in F[X]$ is called λ-*homogenizable* if f is homogeneous in $F[X]^{(\lambda)}$. Let $x \in q(F)_{\rm alg}$ and let $f_{x,q(F)}$ denote its minimal polynomial over $q(F)$. We say that x is *gr-algebraic* over F if $f_{x,q(F)}$ is a homogenizable polynomial of $F[X]$, i.e., a λ-homogenizable polynomial for some λ. It is shown in [HW$_1$, Prop. 2.2] that x is gr-algebraic over F if and only if the ring $F[x]$ is a graded field extension of F and x is homogeneous in $F[x]$. If K is a graded field extension of F, we say that K is *gr-algebraic* over F if every homogeneous element of K is gr-algebraic over F. This holds, in particular, whenever $[K : F] < \infty$, by [HW$_1$, Prop. 2.2]. Let $F_{\text{gr-alg}} = F[\{x \in q(F)_{\rm alg} \mid x \text{ is gr-algebraic over } F\}]$, then as proved in [HW$_1$, Cor. 2.7(c)], $F_{\text{gr-alg}}$ is a gr-algebraic graded field extension of F which contains every other gr-algebraic graded field extension of F in $q(F)_{\rm alg}$. We call $F_{\text{gr-alg}}$ 'the' *graded algebraic closure* of F.

Let K be a graded field extension of a graded field F with $[K : F] < \infty$. In analogy with the valuation terminology, K is said to be *totally ramified* over F if $[K : F] = |\Gamma_K : \Gamma_F|$. We say K is *inertial* over F if $[K : F] = [K_0 : F_0]$ and K_0 is separable over F_0. Also, K is *tame* over F if K_0 is separable over F_0 and Γ_K/Γ_F has no p-torsion if $\mathrm{char}(F) = p \neq 0$. Further, K is *purely wild* over F if $\mathrm{char}(F) = p \neq 0$, K_0 is purely inseparable over F_0, and Γ_K/Γ_F is a p-group. By [HW$_1$, Lemma 3.6] K/F is purely wild if and only if $q(K)/q(F)$ is purely inseparable. If A is a graded central division algebra over F, we say that A is *semiramified* if A_0 is a field and $[A_0 : F_0] = |\Gamma_A : \Gamma_F|$; A is *nicely semiramified* if A has a maximal subfield inertial over F and another which is totally ramified over F.

If F is a graded field and A is a graded division algebra of type Γ which is finite-dimensional over F, we denote by $q(A)$ the algebra of central quotients of A. So, $q(A) \cong A \otimes_F q(F)$, which is a division ring over $q(F)$ with $[q(A) : q(F)] = [A : F] < \infty$. The graded structure on A and the total ordering on Γ induce a canonical valuation v on $q(A)$ as follows (see [B$_3$, §4] or [HW$_2$, §4]): For nonzero $a = \sum_{\gamma\in\Gamma} a_\gamma \in A$ (with each $a_\gamma \in A_\gamma$) set $v(a)$ to be the least γ for which $a_\gamma \neq 0$. Then for nonzero $a \in A$, $b \in F$, define $v(ab^{-1}) = v(a) - v(b)$. It is easy to check that v is well-defined and is a valuation on $q(A)$ with $\Gamma_{q(A)} = \Gamma_A$ and $\overline{q(A)} \cong A_0$. Note that this canonical valuation depends not only on Γ as a group, but also on the choice of ordering on Γ. Let $Hq(F)$ denote the Henselization of $q(F)$ with respect to its canonical valuation (see [EP, §5.2] or [E, §16]), and set $Hq(A) = q(A) \otimes_{q(F)} Hq(F)$. If $F = Z(A)$ (so $q(F) = Z(q(A))$, it is known by Morandi's Henselization theorem [Mor, Th. 2] that $Hq(A)$ is a division algebra over $Hq(F)$. The Henselian valuation on $Hq(F)$ has a unique extension to a valuation on $Hq(A)$, and it is known that $Hq(A)$ is a tame central division algebra over $Hq(F)$ (see [B$_2$, Cor. 4.4] or [HW$_2$, Th. 5.1]).

Going in the other direction, suppose we start with a field E with a valuation v. Then, the filtration of E induced by v yields a canonical graded field GE. Namely, for $\gamma \in \Gamma$ let $E^\gamma = \{x \in E \mid v(x) \geq \gamma\}$ and $E^{>\gamma} = \{x \in E \mid v(x) > \gamma\}$. Obviously, $E^{>\gamma}$ is a subgroup of the additive group E^γ. So, we can define the factor group $GE_\gamma = E^\gamma/E^{>\gamma}$. For $x \in E\backslash\{0\}$, we denote by $\widetilde{x}$ the element $x + E^{>v(x)}$ of $GE_{v(x)}$; for $0 \in E$, set $\widetilde{0} = 0$ in GE. One can easily see that the additive group $GE = \bigoplus_{\gamma\in\Gamma} GE_\gamma$ with the multiplication law defined for homogeneous elements by $\widetilde{x}\widetilde{y} = \widetilde{xy}$, is a graded field. Similarly, if D is a valued division algebra finite dimensional over a field E, then the analogous filtration of D yields a graded division algebra $GD = \bigoplus_{\gamma\in\Gamma} GD_\gamma$ where $GD_\gamma = D^\gamma/D^{>\gamma}$ (see [B$_3$, §4] or [HW$_2$, §4]). Note that $GD_0 = \overline{D}$ and $\Gamma_{GD} = \Gamma_D$. It is easy to see that if F is a graded field and D is a graded central division algebra over F, then D is canonically isomorphic as a graded ring to $Gq(D)$, the associated graded ring of the valued division algebra $q(D)$, via the mapping $x = \sum_{\gamma\in\Gamma} x_\gamma \mapsto \sum_{\gamma\in\Gamma} \widetilde{x_\gamma}$. Likewise, $D \cong GHq(D)$, the associated graded ring of $Hq(D)$.

It is known that graded central division algebras over a graded field F play an analogous role to central division algebras over a Henselian valued field. Indeed, their equivalence classes form a *graded Brauer group* $\mathrm{GBr}(F)$, and there is a group isomorphism $\mathrm{GBr}(F) \to \mathrm{TBr}(Hq(F))$, where $\mathrm{TBr}(Hq(F))$ is the tame part of the Brauer group $\mathrm{Br}(Hq(F))$ [HW_2, Th. 5.1]. Conversely, for any Henselian valued field E, there is a canonical group isomorphism $\mathrm{TBr}(E) \to \mathrm{GBr}(GE)$ [HW_2, Th. 5.3].

1. Graded and valued field extensions

Lemma 1.1. *Let F be a graded field, take any λ in the divisible hull of Γ_F, and let f be a nonzero λ-homogenizable polynomial in $F[X]$. Then,*

(1) *If $h \in F[X]$ and $h \mid f$, then h is λ-homogenizable.*
(2) *For $g \in F[X]$, $f \mid g$ in $F[X]$ if and only if $f \mid g$ in $q(F)[X]$.*
(3) *f is irreducible in $F[X]$ if and only if f is irreducible in $q(F)[X]$. When this occurs, f is a prime element of $F[X]$.*

Thus, unique factorization holds for λ-homogenizable polynomials in $F[X]$.

Proof. (1) This holds because $\Gamma_{F[X]^{(\lambda)}}$ is totally ordered. Therefore, for nonzero h, k in $F[X]$, the lowest [resp. highest] grade homogeneous component of hk is the product of the lowest [resp. highest] grade components of h and k. So, if hk is homogeneous, then h and k must also be homogeneous.

(2) Write $f = \sum_{i=0}^{n} a_i X^i$. Since f is λ-homogenizable, each nonzero a_i is homogeneous in F, so lies in F^*. (2) thus follows by the division algorithm for polynomials, since the leading coefficient of f is a unit.

(3) Since the leading coefficient of f lies in F^*, we may assume that f is monic. Because the integral domain F is integrally closed by [HW_1, Cor. 1.3], if f is irreducible in $F[X]$, then f is irreducible in $q(F)[X]$. Conversely, if f is irreducible in $q(F)[X]$, then $fF[X]$ is a prime ideal of $F[X]$, since (2) shows that $fF[X] = (f\,q(F)[X]) \cap F[X]$. Hence f is a prime element of $F[X]$, so it is irreducible in $F[X]$.

Since nonzero constant λ-homogenizable polynomials are units of $F[X]$, it follows by induction on degree and by (1) and (3) above that every λ-homogenizable polynomial of positive degree is a product of prime λ-homogenizable polynomials. The usual argument gives the uniqueness of such a factorization. □

Let F be a graded field and let L be an algebraic graded field extension of F. Then, we say that L is *normal* over F if every homogenizable irreducible polynomial g of $F[X]$ which has a root in L factors into polynomials of degree one in $L[X]$. When this occurs, each root x of such a g is homogeneous in L, since $X - x$ is homogenizable in $L[X]$ by Lemma 1.1(1). Moreover, the minimal polynomial $f_{x,q(F)}$ of x over $q(F)$ lies in $F[X]$ (as L is integral over F, which is integrally closed), and $f_{x,q(F)}$ is λ-homogenizable, where $\lambda = \mathrm{gr}(x)$. So by Lemma 1.1, $g = af_{x,q(F)}$ for some $a \in F^*$. Thus, L is normal over F if and only if for any $x \in L^*$, $f_{x,q(F)}$ factors into polynomials of degree one in $L[X]$.

Lemma 1.2. *Let L/F be an algebraic graded field extension. Then, L is normal over F if and only if $q(L)$ is a normal field extension of $q(F)$.*

Proof. Suppose that L is normal over F and consider a $q(F)$-monomorphism σ from $q(L)$ into $q(L)_{\mathrm{alg}}$, the algebraic closure of $q(L)$. Let $x \in L^*$ and let $f_{x,q(F)}$ be its minimal polynomial over $q(F)$. Obviously, we have $f_{x,q(F)}(\sigma(x)) = 0$. It follows by the normality of L/F that $\sigma(x) \in L$. So, $\sigma(L) = L$, as L is generated over F by L^*. Since $q(L) = L \cdot q(F)$, it follows that $\sigma(q(L)) = q(L)$. Therefore, every irreducible polynomial in $q(F)[X]$ with a root in $q(L)$ must split over $q(L)$. Hence, $q(L)$ is normal over $q(F)$.

Conversely, suppose that $q(L)$ is a normal field extension of $q(F)$ and let g be a homogenizable irreducible polynomial of $F[X]$ with a root in L. By Lemma 1.1(3) g remains irreducible over $q(F)$, so by the normality g splits over $q(L)$. Clearly, the roots of g are integral over F, hence integral over L, so they all lie in L as L is integrally closed. □

Proposition 1.3. *Let L/F be a finite-dimensional graded field extension. Then, the following are equivalent:*

1. *L/F is tame and normal.*
2. *L is a Galois graded field extension of F.*

Proof. This follows by [HW_1, Th. 3.11(a),(b)] and Lemma 1.2 above. □

Let L/F be a normal finite-dimensional graded field extension. The Galois group of L over F is the group $\mathrm{Gal}(L/F)$ consisting of graded (i.e., grade-preserving) F-isomorphisms of L. Let $G = \mathrm{Gal}(L/F)$ and let L^G denote the set of elments of L invariant under the action of G; so, L^G is a graded subfield of L. It was proved in [B_5, p. 26] that L is tame over L^G. The following proposition gives a more general result.

Proposition 1.4. *Let L/F be a finite-dimensional normal graded field extension with Galois group G. Then, L^G is purely wild over F and L is Galois over L^G. Moreover, if T is the tame closure of F in L, then $L = T \cdot L^G \cong_g T \otimes_F L^G$.*

Proof. Since L/F is normal , by Lemma 1.2 $q(L)/q(F)$ is a normal field extension. By [HW_1, Cor. 2.5(d)], every $\sigma \in \mathrm{Gal}(q(L)/q(F))$ restricts to a graded F-automorphism of L. Furthermore, the map $\mathrm{Gal}(q(L)/q(F)) \to \mathrm{Gal}(L/F) = G$ given by restriction is an isomorphism as $q(L) = L \otimes_F q(F)$. Therefore, we identify $\mathrm{Gal}(q(L)/q(F))$ with G. Recall from field theory (see, e.g., [K, Prop. 3.2, p. 316]) that the normality of $q(L)$ over $q(F)$, implies $q(L)$ is Galois over $q(L)^G$, which is purely inseparable over $q(F)$. Moreover, if S is the separable closure of $q(F)$ in $q(L)$, then $q(L) = S \cdot q(L)^G \cong S \otimes_{q(F)} q(L)^G$. Now, since every x in $q(L)$ is expressible as ab^{-1} with $a \in L$ and $b \in F\backslash\{0\}$, we have $q(L)^G = L^G \cdot q(F) = q(L^G)$. Hence, L is Galois over L^G by [HW_1, Th. 3.11(b)] as $q(L)$ is Galois over $q(L^G)$, and L^G is purely wild over F by [HW_1, Lemma 3.6] as $q(L^G)$ is purely inseparable over $q(F)$.

Turning to T, by [HW_1, (3.8)], $q(T) = S$. Because $T \otimes_F L^G$ is a torsion-free F-module, it injects into $(T \otimes_F L^G) \otimes_F q(F) \cong q(T) \otimes_{q(F)} q(L^G) \cong q(L)$. Let $L' = T \cdot L^G$, which is the image of $T \otimes_F L^G$ under its injective mapping to $q(L)$. Then, L' is a graded subfield of L, and the isomorphism $T \otimes_F L^G \to L'$ respects the gradings. Also, $[L : L'] = [q(L) : q(L')] = 1$, as $q(L') = q(T) \cdot q(L^G) = q(L)$. Thus, $L = L' = T \cdot L^G \cong_g T \otimes_F L^G$. □

Theorem 1.5. *Let (E, v) be a valued field and (M, w) a finite-dimensional valued field extension of (E, v). If M is normal over E, then GM is normal over GE.*

Proof. Assume first that w is the unique valuation of M extending v on E. Let $x \in M$ and let $f_{x,E}$ be its minimal polynomial over E. Since M is normal over E, we can write $f_{x,E} = \prod_{i=1}^n (X - x_i)$, where $x_i = \sigma_i(x)$ for some $\sigma_i \in \mathrm{Gal}(M/E)$. Moreover, since w is the unique extension of v to M, $w(x_i) = w(x)$ for $1 \le i \le n$. Therefore, the polynomial $\prod_{i=1}^n (X - \widetilde{x}_i)$ lies in $GE[X]$, by [B_4, Lemma 2.1, Lemma 2.4] (or by Lemma 1.8 below, which shows that $h = f^{(\lambda)} \in GE[X]$). Hence, the minimal polynomial $f_{\widetilde{x},q(GE)}$ of $\widetilde{x}$ over $q(GE)$ splits into polynomials of degree one in $GM[X]$, showing that GM is normal over GE.

Now, without assuming that w is the unique extension of v to M, let $I = M^{\mathrm{Gal}(M/E)}$ and let K be the decomposition field of w over I. Since M is normal over K and w is the unique extension of its restriction $w|_K$ to M, by the first part of the proof GM is normal over GK. So by Lemma 1.2, $q(GM)$ is normal over $q(GK)$. Moreover, since $(K, w|_K)$ is an immediate field extension of $(I, w|_I)$ by [EP, Cor. 5.3.8(0), pp. 134–135], we have $GK = GI$. Note that because I is purely inseparable over E, we have GI is purely wild over GE, so $q(GI)$ is purely inseparable over $q(GE)$. Therefore, $q(GM)$ is normal over $q(GE)$, so again by Lemma 1.2, GM is normal over GE. □

In what follows we will consider polynomials over a valued field (E, v) for which all the roots in an algebraic closure E_{alg} of E have the same value for any valuation that extends v to E_{alg}. The following proposition generalizes [B_4, Lemma 2.1], which gives (1) ⇔ (3) under the additional assumptions that v is Henselian and f is monic.

Proposition 1.6. *Let (E, v) be a valued field, E_{alg} an algebraic closure of E, and let $f = \sum_{i=0}^n a_i X^i \in E[X]$ with $a_0 a_n \neq 0$. Let $\lambda = \frac{1}{n}\big(v(a_0) - v(a_n)\big)$ in the divisible hull of Γ_E. Then, the following statements are equivalent:*

1. *For some extension of v to E_{alg}, all the roots of f in E_{alg} have the same value.*
2. *For every extension of v to E_{alg}, all the roots of f in E_{alg} have value λ.*
3. *$v(a_i) \geq (n-i)\lambda + v(a_n)$ for every i, $0 \leq i \leq n$.*
4. *Let w be an extension of v to E_{alg}, $c \in E_{\mathrm{alg}}$ with $w(c) = \lambda$, and let $h = \frac{1}{a_n c^n} f(cX)$. Then h is a monic polynomial of $V_{\mathrm{alg}}[X]$, where V_{alg} is the valuation ring of w.*

Proof. (2) ⇒ (1) is clear.

(1) ⇒ (3) Let $f = a_n(X - x_1)\dots(X - x_n)$ in $E_{\mathrm{alg}}[X]$, and let s_j be the j-th symmetric polynomial in $x_1, \dots, x_n$ for $1 \leq j \leq n$. Suppose the x_i all have the same value for some extension w of v to E_{alg}. Then $w(x_i) = \lambda$ for all i, as $a_0 = (-1)^n a_n x_1 \dots x_n$. Since s_j is a sum of products of j of the x_i, $w(s_j) \geq j\lambda$. For $0 \leq i \leq n-1$ we have $a_i = (-1)^{n-i} s_{n-i} a_n$; hence, $v(a_i) = w(s_{n-i}) + v(a_n) \geq (n-i)\lambda + v(a_n)$.

(3) ⇒ (2) Let w be an extension of v to E_{alg}, and let x be any root of f in E_{alg}. Since $\sum_{i=0}^n a_i x^i = 0$, in the list of values $w(a_0), w(a_1 x), \dots, w(a_n x^n)$ the least value must occur at least twice. If $w(x) > \lambda$, then (3) yields for $i > 0$,

$$w(a_i x^i) \ > \ v(a_i) + i\lambda \ \geq \ (n-i)\lambda + v(a_n) + i\lambda = n\lambda + v(a_n) \ = \ v(a_0).$$

This is not possible, since then the least value on the list would be $w(a_0)$, occurring only once. Similarly, if $w(x) < \lambda$, then for $i < n$,

$$\begin{aligned} w(a_n x^n) \ &= \ v(a_n) + i\, w(x) + (n-i) w(x) \\ &< \ v(a_n) + i\, w(x) + (n-i)\lambda \ \leq \ v(a_i) + i\, w(x) \ = \ w(a_i x^i). \end{aligned}$$

This is also ruled out, since the least value on the list would be then $w(a_n x^n)$, occurring only once. Therefore, $w(x) = \lambda$ for any root x of f.

(3) ⇔ (4) Clearly, h is a monic polynomial. Write $h = \sum_{i=0}^n b_i X^i$, where $b_i = a_i a_n^{-1} c^{i-n}$. Then, $w(b_i) = v(a_i) - v(a_n) + (i-n)\lambda$. Hence, $w(b_i) \geq 0$ if and only if $v(a_i) \geq (n-i)\lambda + v(a_n)$. □

Definition 1.7. If $f = \sum_{i=0}^n a_i X^i \in E[X]$ satisfies the equivalent conditions of Prop. 1.6, then we call f a *λ-polynomial*, where $\lambda = \frac{1}{n}\big(v(a_0) - v(a_n)\big)$ is the common value of all the roots of f. We then write $\widetilde{f}^{(\lambda)} := \sum_{i=0}^n \overline{a}_i^{(\lambda)} X^i \in GE[X]$, where $\overline{a}_i^{(\lambda)}$ is the class of a_i in $GE_{(n-i)\lambda + v(a_n)}$. Observe that $\overline{a}_i^{(\lambda)} = \widetilde{a}_i$ if $v(a_i) = (n-i)\lambda + v(a_n)$ and $\overline{a}_i^{(\lambda)} = 0$ if $v(a_i) > (n-i)\lambda + v(a_n)$. Thus, $\widetilde{f}^{(\lambda)}$ is a λ-homogenizable polynomial in $GE[X]$ with $\mathrm{gr}(\widetilde{f}^{(\lambda)}) = v(a_0)$ and $\deg(\widetilde{f}^{(\lambda)}) = \deg(f)$.

Lemma 1.8. *Let (E,v) be a valued field, and let $f=\sum_{i=0}^n a_iX^i$ be a λ-polynomial in $E[X]$. Let K be an algebraic field extension of E over which f splits, say $f=a_n(X-x_1)\dots(X-x_n)$ in $K[X]$, and let w be any extension of v to K. Then, $\widetilde{f}^{(\lambda)}=\widetilde{a}_n(X-\widetilde{x}_1)\dots(X-\widetilde{x}_n)$ in $GK[X]$.*

Proof. We have $f=\sum_{i=0}^n a_iX^i = a_n\prod_{i=1}^n(X-x_i)=\sum_{k=0}^n(-1)^k a_n s_k X^{n-k}$, where $s_0=1$ and s_k is the k-th symmetric polynomial in $x_1,\dots,x_n$ for $1\le k\le n$. Likewise, let $g=\widetilde{a}_n\prod_{i=0}^n(X-\widetilde{x}_i)=\sum_{k=0}^n\widetilde{(-1)}^k\widetilde{a}_n t_k X^{n-k}$, where $t_0=\widetilde{1}$ and t_k is the k-th symmetric polynomial in $\widetilde{x}_1,\dots,\widetilde{x}_n$ for $1\le k\le n$. Now, each s_k is a sum of monomials of degree k in the x_i (so of value $k\lambda$). Hence, $w(s_k)\ge k\lambda$. We have $(x_{j_1}\dots x_{j_k})^\sim=\widetilde{x}_{j_1}\dots\widetilde{x}_{j_k}$ for all indices $j_1,\dots,j_k$. When $w(s_k)=k\lambda$, $\widetilde{s}_k$ is the sum of the images of its monomials in GK, i.e, $\widetilde{s}_k=t_k\ne 0$ in $GK_{k\lambda}$. When this occurs, $v(a_{n-k})=v(a_n)+k\lambda$, so $\overline{a}_{n-k}^{(\lambda)}=\widetilde{a}_{n-k}=\widetilde{(-1)}^k\widetilde{a}_n\widetilde{s}_k=\widetilde{(-1)}^k\widetilde{a}_nt_k$, which is the $(n-k)$-th coefficient of g. On the other hand, when $w(s_k)>k\lambda$, the sum of the images of its monomials in GK is 0, i.e., $t_k=0$. When this occurs, $v(a_{n-k})=v(a_n)+v(s_k)>v(a_n)+k\lambda$, so $\overline{a}_{n-k}^{(\lambda)}=0$, which is the $(n-k)$-th coefficient of g, as $t_k=0$. Thus, $f^{(\lambda)}=\sum_{k=0}^n\overline{a}_{n-k}^{(\lambda)}X^{n-k}=g$. □

Let (E,v) be a Henselian valued field. The next theorem generalizes to arbitrary λ-polynomials over E well-known basic properties for 0-polynomials, which are those $f=\sum_{i=1}^n a_iX^i$ in $V_E[X]$ with $v(a_n)=v(a_0)=0$ (cf. [EP, Th. 4.1.3, pp. 87–88]).

Theorem 1.9. *Let (E,v) be a Henselian valued field, $f=\sum_{i=0}^n a_iX^i$ a polynomial of $E[X]$ with $a_0a_n\ne 0$ and let $f'=\sum_{i=0}^n\widetilde{a}_iX^i\in GE[X]$. Then,*

(1) *If f is a λ-polynomial and $f=gh$ in $E[X]$, then g and h are λ-polynomials and $\widetilde{f}^{(\lambda)}=\widetilde{g}^{(\lambda)}\widetilde{h}^{(\lambda)}$ in $GE[X]$. So, if $\widetilde{f}^{(\lambda)}$ is irreducible in $GE[X]$, then f is irreducible in $E[X]$.*

(2) *If f is irreducible in $E[X]$, then f is a λ-polynomial for $\lambda=\frac{1}{n}(v(a_0)-v(a_n))$. Furthermore, $\widetilde{f}^{(\lambda)}=\widetilde{a}_nk^s$ for some irreducible monic homogeneous polynomial k of $GE[X]^{(\lambda)}$ and some positive integer s.*

(3) *If f is a λ-polynomial in $E[X]$ and if $\widetilde{f}^{(\lambda)}=\ell m$ in $GE[X]$ with $\gcd(\ell,m)=1$, then there exist λ-polynomials g, h in $E[X]$ such that $f=gh$, $\widetilde{g}^{(\lambda)}=\ell$, and $\widetilde{h}^{(\lambda)}=m$.*

(4) *If f is a λ-polynomial and $\widetilde{f}^{(\lambda)}$ has a simple root b in GE, then f has a simple root a in E with $\widetilde{a}=b$.*

(5) *Suppose f' is a λ-homogenizable polynomial of $GE[X]$. Then, f is a λ-polynomial and $\widetilde{f}^{(\lambda)}=f'$.*

Proof. (1) If x is any root of g or of h in E_{alg}, then x is also a root of f, so x has value λ. Thus, g and h are λ-polynomials. Let K be any algebraic extension of E over which f splits (so g and h split), and let w be any extension of v to K. In $K[X]$ write $g=b\prod_{i=1}^r(X-x_i)$ and $h=c\prod_{i=r+1}^n(X-x_i)$. Then the leading coefficient of f is bc, and $\widetilde{bc}=\widetilde{b}\widetilde{c}$ in GE. By applying Lemma 1.8 to f, g, and h, we obtain $\widetilde{f}^{(\lambda)}=\widetilde{g}^{(\lambda)}\widetilde{h}^{(\lambda)}$.

(2) Let N be a normal field extension of E that contains the roots $(x_i)_{i=1}^n$ of f and denote the unique extension of the Henselian valuation v to N by w. Since f is irreducible in $E[X]$, for any $1\le i\le n$, there is an E-automorphism σ_i of N such that $\sigma_i(x_1)=x_i$. Because v is Henselian, $w\circ\sigma_i=w$. So, $w(x_i)=w(x_1)$. This shows that f is a λ-polynomial for $\lambda=w(x_1)$. Moreover, as σ_i preserves w, it induces a graded GE-automorphism $\widetilde{\sigma}_i$ on GN for which $\widetilde{\sigma}_i(\widetilde{x}_1)=\widetilde{x}_i$. This automorphism of course extends to a $q(GE)$-automorphism of $q(GN)$. So, the minimal polynomial k of $\widetilde{x}_1$ over $q(GE)$ is also the minimal polynomial of $\widetilde{x}_i$. Since the monic irreducible factors of $\widetilde{f}^{(\lambda)}$ in $q(GE)[X]$ are the minimal polynomials of its roots and Lemma 1.8 shows that the roots of $\widetilde{f}^{(\lambda)}$ are the $\widetilde{x}_i$, we must have $\widetilde{f}^{(\lambda)}=\widetilde{a}_nk^s$ in $q(GE)[X]$. This k lies in $GE[X]$ as noted above (because the graded field GE is integrally closed), and k is λ-homogenizable by Lemma 1.1(1) above.

(3) Note that ℓ and m are λ-homogenizable by Lemma 1.1(1). Without loss of generality, we may assume that f, ℓ, and m are monic polynomials. Write $f = \prod_{i=1}^r p_i^{t_i}$, where p_i are distinct monic irreducible polynomials in $E[X]$. By (1) above the p_i are also λ-polynomials and by (2) each $\widetilde{p}_i^{(\lambda)}$ equals $q_i^{s_i}$ for some monic irreducible λ-homogenizable polynomial q_i in $GE[X]$, so $\widetilde{f}^{(\lambda)} = \prod_{i=1}^r q_i^{s_i t_i}$. Since each q_i is a prime element of $GE[X]$ by Lemma 1.1(3), q_i divides ℓ or m but not both. By the unique factorization for λ-homogenizable polynomials (see Lemma 1.1), the product of the $q_i^{s_i t_i}$ for those q_i dividing ℓ is ℓ, and the product of the remaining $q_i^{s_i t_i}$ is m. Let g be the product of those $p_i^{t_i}$ with q_i dividing ℓ, and let h be the product of the remaining $p_i^{t_i}$. Then, $f = gh$, and $\widetilde{g}^{(\lambda)} = \ell$ and $\widetilde{h}^{(\lambda)} = m$.

(4) Write $\widetilde{f}^{(\lambda)} = (X - b)m$ in $GE[X]$ with $(X - b) \nmid m$. By Lemma 1.1, $X - b$ and m are λ-homogenizable, and since $X - b$ is prime in $GE[X]$, $\gcd(X - b, m) = 1$. So by (3) above, $f = gh$ for λ-polynomials g and h in $E[x]$ with $\widetilde{g}^{(\lambda)} = X - b$ and $\widetilde{h}^{(\lambda)} = m$. Write $g = c_1 X + c_0$; so, $\widetilde{c}_1 = \widetilde{1}$ and $\widetilde{c}_0 = -b$. Then, $a = -c_0 c_1^{-1}$ is a root of g, so of f, and $\widetilde{a} = -\widetilde{c}_0 \widetilde{c}_1^{-1} = b$. Since $\widetilde{a}$ is not a root of $\widetilde{h}^{(\lambda)}$, a cannot be a root of h by Lemma 1.8. So, a is a simple root of f.

(5) Since f' is λ-homogenizable in $GE[X]$, for any i with $a_i \neq 0$, $v(a_i) + i\lambda = v(a_n) + n\lambda$, hence $v(a_i) = (n - i)\lambda + v(a_n)$. In particular, we have $v(a_0) = n\lambda + v(a_n)$, hence $\lambda = \frac{1}{n}(v(a_0) - v(a_n))$. So, by Prop. 1.6(3) f is a λ-polynomial. Moreover, we have $\overline{a}_i^{(\lambda)} = \widetilde{a}_i$ for any i, so $\widetilde{f}^{(\lambda)} = f'$. □

For monic λ-polynomials over a Henselian valued field, Lemma 1.8 and Th. 1.9(2) and (4) were essentially proved by Boulagouaz in [B_4, Lemme 2.4, Th. 2.5, Cor. 2.6].

Corollary 1.10. *Let E be a Henselian valued field, and let $g = \sum_{i=0}^n b_i X^i$ be a λ-homogenizable irreducible polynomial of $GE[X]$, with $b_0 \neq 0$. Choose any $a_i \in E$ with $\widetilde{a}_i = b_i$, $0 \leq i \leq n$, and let $f = \sum_{i=0}^n a_i X^i$. Then, for any root a of f in an algebraic extension K of E such that $\widetilde{a}$ is a root of g in GK, we have $G(E[a]) = (GE)[\widetilde{a}]$ and $[E[a] : E] = [G(E[a]) : GE]$.*

Proof. By Th. 1.9(5) f is a λ-polynomial in $E[X]$, with $\widetilde{f}^{(\lambda)} = g$. So by Th. 1.9(1), f is irreducible in $E[X]$. Now, clearly $(GE)[\widetilde{a}] \subseteq G(E[a])$. But, as f and g are irreducible,

$$[(GE)[\widetilde{a}] : GE] = \deg(g) = \deg(f) = [E[a] : E] \geq [G(E[a]) : GE] \geq [(GE)[\widetilde{a}] : GE].$$

Hence, equality holds throughout, which implies that $G(E[a]) = (GE)[\widetilde{a}]$. □

Remark 1.11. Note that (1) and (5) of Th. 1.9 are true without assuming that v is Henselian. So, the Henselian assumption can be omitted from Cor. 1.10, as well.

Proposition 1.12. *Let E be a Henselian valued field with residue characteristic $p > 0$ and L a purely wild finite-dimensional graded field extension of GE, then there is a defectless field extension K of E such that $GK = L$. If $\operatorname{char}(E) = p$, then K can be chosen to be purely inseparable field extension of E.*

Proof. Assume first that $L = (GE)[c]$ for some $c \in L^*$. Since L is purely wild over GE, $q(L)$ is purely inseparable over $q(GE)$ by [HW_1, Lemma 3.6]. Let $g = X^{p^n} - d$ be the minimal polynomial ove c over $q(GE)$. Then, $d \in q(GE)$ and $d = c^{p^n} \in L$, which is integral over GE as $[L : GE] < \infty$; so, $d \in GE$, as GE is integrally closed. Furthermore, d is homogeneous in L, so also in GE. Now, g is clearly λ-homogenizable for $\lambda = \deg(c)/p^n$, so g is irreducible in $GE[X]$ by Lemma 1.1(3). The evaluation homomorphism $q(GE)[X] \to q(L)$ given by $X \mapsto c$ has kernel $g\, q(GE)[X]$. Since $(g\, q(GE)[X]) \cap GE[X] = g\, GE[X]$ by Lemma 1.1(2), the evaluation at c map yields $GE[X]/(g\, GE[X]) \cong L$, a graded GE-algebra isomorphism when X is given degree λ. Write $d = \widetilde{b}$ for some $b \in E^*$, and let $f = X^{p^n} - b \in E[X]$. Let a be a root of f in some algebraic field extension of E, and let $K = E[a]$. Since $a^{p^n} = b$, we have $\widetilde{a}^{p^n} = \widetilde{b} = d$ in the graded field GK. Since $\widetilde{a}$ is a homogeneous root of g in a graded field extension of GE, we have $GE[X]/(g\, GE[X]) \cong (GE)[\widetilde{a}]$ by the same argument as for the preceding isomorphism. By combining these isomorphisms, we obtain a graded GE-algebra isomorphism $(GE)[\widetilde{a}] \cong L$. So, we may identify L with $(GE)[\widetilde{a}]$. By Cor. 1.10, $GK = (GE)[\widetilde{a}] = L$, and clearly K is purely inseparable over E if $\mathrm{char}(E) = p$.

Now, let L be an arbitrary finite-dimensional purely wild graded field extension of GE. Then, we can write $L = GE[c_1, \ldots, c_r]$ with each $c_i \in L^*$, and the result follows by induction on r. □

As a consequence of Th. 1.9, we have the following Corollary which makes explicit the correspondence between (finite-dimensional) tame valued field extensions over a Henselian valued field and tame graded field extensions. This correspondence was given in [HW$_1$, Th. 5.2] in a somewhat cumbersome and indirect way. Recall that if L is a finite-dimensional extension of a Henselian valued field E, then L is *tame* (or tamely ramified) over E if $\mathrm{char}(\overline{E}) = 0$ or $\mathrm{char}(\overline{E}) = p > 0$, $\overline{L}$ is separable over $\overline{E}$, $p \nmid |\Gamma_L : \Gamma_E|$, and $[\overline{L} : \overline{E}]\,|\Gamma_L : \Gamma_E| = [L : E]$.

Corollary 1.13. [HW$_1$, Th. 5.2] *Let (E, v) be a Henselian valued field. Then, the map $K \mapsto GK$ gives a one-to-one correspondence between the set of E-isomorphism classes of finite-dimensional tame field extensions of E and the set of graded GE-isomorphism classes of finite-dimensional tame graded field extensions of GE. Moreover, K is a Galois tame (finite-dimensional) field extension of E if and only if GK is a Galois (finite-dimensional) graded field extension of GE, in which case* $\mathrm{Gal}(K/E) \cong \mathrm{Gal}(GK/GE)$.

Proof. If K is a tame field extension of E, then obviously GK is a tame graded field extension of GE. Let K' be a tame field extension of E such that $K' \cong K$. Since E is Henselian, the isomorphism respects the valuations on K and K' extending v on E; so, $GK \cong_g GK'$.

Conversely, if L is a tame finite-dimensional graded field extension of GE, then $q(L)$ is separable over $q(GE)$ by [HW$_1$, Prop. 3.5]; so, we can write $L = GE[\widetilde{x}_1, ..., \widetilde{x}_r]$, where $x_i \in E_{\mathrm{alg}}$ with $\widetilde{x}_i$ separable over $q(GE)$. Let g be the minimal polynomial of $\widetilde{x}_1$ over $q(GE)$. Then g is λ-homogenizable in $GE[X]$ where $\lambda = \mathrm{gr}(\widetilde{x}_1)$, g is irreducible in $GE[X]$ by Lemma 1.1(3), and $\widetilde{x}_1$ is a simple root of g. Take any $f = \sum_{i=1}^{n} c_i X^i \in E[X]$ such that $\sum_{i=1}^{n} \widetilde{c}_i X^i = g$. By Th. 1.9(5), f is a λ-polynomial with $\widetilde{f^{(\lambda)}} = g$. So, f is irreducible in $E[X]$ by Th. 1.9(1), and Th. 1.9(4) applied over $E[x_1]$ shows that f has a simple root $a_1 \in E[x_1]$ with $\widetilde{a}_1 = \widetilde{x}_1$. By Cor. 1.10, $G(E[a_1]) = GE[\widetilde{a}_1]$. Moreover, $E[a_1]$ is tame over E, as $G(E[a_1])$ is tame over GE and $[E[a_1] : E] = [G(E[a_1]) : GE]$ by Cor. 1.10. Since $L = G(E[a_1])[\widetilde{x}_2, \ldots, \widetilde{x}_r]$, which is tame over $G\big(E[a_1]\big)$, by induction on r there exist $a_2, \ldots, a_r \in E[a_1][x_2, \ldots, x_r]$ such that each $\widetilde{a}_i = \widetilde{x}_i$ and $E[a_1][a_2, \ldots, a_r]$ is tame over $E[a_1]$ with $G(E[a_1][a_2, \ldots, a_r]) = G(E[a_1])[\widetilde{a}_2, \ldots, \widetilde{a}_r] = L$. Let $K = E[a_1, \ldots, a_r]$. Then $GK = L$ and K is tame over E, as K is tame over $E[a_1]$ and $E[a_1]$ is tame over E. For the uniqueness of K up to isomorphism, suppose K'' is another tame field extension of E with a graded GE-isomorphism $\eta\colon L \to GK''$. Let $b = \eta(\widetilde{a}_1)$, which is a root of the g above in GK''. With the f above, Th. 1.9(4) applied over K'' shows that f has a root a_1'' in K'' with $\widetilde{a}_1'' = b$. Since f is irreducible in $E[X]$ with roots a_1 and a_1'', we have an E-isomorphism $\psi\colon E[a_1] \to E[a_1'']$ with $\psi(a_1) = a_1''$. The induced GE-isomorphism $\widetilde{\psi}\colon G(E[a_1]) \to G(E[a_1''])$ maps $\widetilde{a}_1$ to $\widetilde{a}_1'' = \eta(\widetilde{a}_1)$. So, $\widetilde{\psi} = \eta|_{G(E[a_1])}$, as $G(E[a_1]) = GE[\widetilde{a}_1]$. Since $K = E[a_1][a_2, \ldots, a_r]$, it follows by induction on r that there is an E-isomorphism $K \to K''$ inducing η on the graded fields.

Now, let K be a Galois tame finite-dimensional field extension of E, let $G = \mathrm{Gal}(K/E)$, and let w be the unique extension of v to K. By Prop. 1.3 and Th. 1.5, GK is a Galois graded field extension of GE. Take any $\sigma \in G$. Since $w \circ \sigma = w$, σ induces a graded GE-automorphism $\widetilde{\sigma}\colon GK \to GK$ satisfying $\widetilde{\sigma}(\widetilde{x}) = \widetilde{\sigma(x)}$ for all $x \in E$. Let $\varphi\colon G \to \mathrm{Gal}(GK/GE)$ be the group homomorphism defined by $\varphi(\sigma) = \widetilde{\sigma}$, and let $G^v = \ker(\varphi)$. So,

$$G^v \;=\; \{\sigma \in G \mid \widetilde{\sigma(x)} = \widetilde{x} \text{ for all } x \in K\} \;=\; \{\sigma \in G \mid w(\sigma(x) - x) > w(x) \text{ for all } x \in K \setminus \{0\}\},$$

which shows that G^v is the ramification group for w over E (cf. [E, Th. (20.5)(c)]). But, since K is tame over E the ramification group is trivial, e.g., by the table in [E, p. 171] as K/E is defectless; hence, φ is injective. Since $|\mathrm{Gal}(GK/GE)| = [GK : GE] = [K : E] = |G|$, φ is a group isomorphism.

Let M be a tame finite-dimensional field extension of E such that GM is a Galois graded field extension of GE and consider a Galois tame finite-dimensional field extension N of E containing M. By the above GN is a Galois graded field extension of GE [resp., of GM] and $\mathrm{Gal}(N/E) \cong \mathrm{Gal}(GN/GE)$ [resp., $\mathrm{Gal}(N/M) \cong \mathrm{Gal}(GN/GM)$]. Since GM is a Galois graded field extension of GE, then $\mathrm{Gal}(GN/GM)$ is a normal subgroup of $\mathrm{Gal}(GN/GE)$, therefore $\mathrm{Gal}(N/M)$ is a normal subgroup of $\mathrm{Gal}(N/E)$. Hence, M is a Galois field extension of E. □

2. Subfields of nondegenerate tame semiramified division algebras

For a central simple algebra A over a field E, as usual we set $\deg(A) = \sqrt{[A : E]}$ and $\exp(A) =$ the order of $[A]$ in the Brauer group Br(E).

Before reviewing the notion of nondegeneracy, we recall Tignol's Dec groups. Let N be a finite-dimensional Galois field extension of a field E with abelian Galois group $G = \mathrm{Gal}(N/E)$. Since G is abelian, there is a base $(\sigma_1, \ldots, \sigma_m)$ of G, i.e., $G = \langle\sigma_1\rangle \oplus \ldots \oplus \langle\sigma_m\rangle$. Let r_i be the order of σ_i in G. For each j, let K_j be the fixed field of the subgroup of G generated by all the σ_i for $i \neq j$. So, K_j is a cyclic Galois extension of E with $[K_j : E] = r_j$ and $\mathrm{Gal}(K_j/E) = \langle\sigma_j|_{K_j}\rangle$; also, $N = K_1 \otimes_E \ldots \otimes_E K_m$. The group $\mathrm{Dec}(N/E)$, introduced by Tignol in [T], is the subgroup of the relative Brauer group $\mathrm{Br}(N/E)$ ($= \ker(\mathrm{Br}(E) \to \mathrm{Br}(N))$) generated by all the subgroups $\mathrm{Br}(L/E)$ as L ranges over the fields with $E \subseteq L \subseteq N$ and $\mathrm{Gal}(L/E)$ cyclic. Equivalently, $\mathrm{Dec}(N/E)$ is the subgroup of $\mathrm{Br}(E)$ generated by the $\mathrm{Br}(K_i/E)$ for $1 \leq i \leq m$. Tignol showed in [T, Cor. 1.4] that $\mathrm{Dec}(N/E)$ consists of the Brauer classes of central simple E-algebras T containing N such that $\deg(T) = [N : E]$ and T is a tensor product of cyclic algebras with respect to the K_i, i.e., $T \cong (K_1/E, \sigma_1, c_1) \otimes_E \ldots \otimes_E (K_m/E, \sigma_m, c_m)$. Such algebras T were said by Tignol to decompose according to N, whence the name $\mathrm{Dec}(N/E)$. As the definition makes clear, $\mathrm{Dec}(N/E)$ is intrinsic to N and E, and is independent of the choice of cyclic decomposition of $\mathrm{Gal}(N/E)$.

Now, with E, N, G, and the σ_i as above, let A be a central simple E-algebra containing N as a maximal subfield with $\deg(A) = [N : E]$. For $1 \leq i \leq m$, choose (by Skolem-Noether) $z_i \in A^*$ with $z_i c z_i^{-1} = \sigma_i(c)$ for all $c \in N$. Let $u_{i,j} = z_i z_j z_i^{-1} z_j^{-1} \in C_A(N)^* = N^*$, and let $b_i = z_i^{r_i}$, which also lies in N, as $\sigma_i^{r_i} = \mathrm{id}_N$. Then, $A = \bigoplus_{0 \leq j_1 \leq r_1 - 1} \ldots \bigoplus_{0 \leq j_m \leq r_m - 1} N z_1^{j_1} \ldots z_m^{j_m}$, and the multiplication in A is completely determined by N, G, the u_{ij}, and the b_i, so we write $A = (N/E, G, S, U, \mathbf{b})$, where $U = (u_{i,j})_{1 \leq i,j \leq m}$, $\mathbf{b} = (b_i)_{1 \leq i \leq m}$, and $S = (\sigma_i)_{1 \leq i \leq m}$ is the chosen base of G. Amitsur and Saltman defined in [AS, p. 81] a condition that they called *degeneracy* for the matrix $(u_{i,j})$, which by [BM, Prop. 0.13] is equivalent to: there is a field L, $E \subseteq L \subseteq N$ such that $\mathrm{Gal}(N/L)$ is noncyclic and $[C_A(L)] \in \mathrm{Dec}(N/L)$. When there is such an L, we say that N is *degenerate in* A, or (when N is understood) A is *degenerate*. When there is no such L, we say that N is *nondegenerate* in A. Note that the characterization in [BM] makes it clear that degeneracy is intrinsic to N and A, independent of the presentation of G and of the choice of the z_i. (However, degeneracy is not intrinsic to A. Indeed, McKinnie has recently given in [Mc_3] an example of a central division algebra A over a field E with maximal subfields N and N' each abelian Galois over A such that N is degenerate in A but N' is not.) Clearly, if $\mathrm{Gal}(N/E)$ is cyclic, then N is nondegenerate in A. Also, one can check that if $[N : E]$ has more than one distinct prime factor, then N is nondegenerate in A if and only if each primary component of N is nondegenerate in the corresponding primary component of A. Therefore, our focus will be on nondegenerate algebras of prime power degree with $\mathrm{Gal}(N/E)$ noncyclic. The first examples of nondegenerate algebras (with $\mathrm{Gal}(N/E)$ noncyclic) given in [AS, Remark, p. 82] satisfied $\exp(A) = \deg(A)$, (for which the nondegeneracy holds trivially, see [AS, Lemma 1.7]). Subsequently, Saltman gave in [S_2, Cor. 12.15][1] an example of a nondegenerate algebra A with $\deg(A) = p^2$ and $\exp(A) = p$ for any odd prime p over a field E containing a primitive p-th root of unity. Recently, McKinnie has given in [Mc_2] more examples of nondegenerate division algebras of odd prime exponent, from which further examples can be built as well. See Remarks 2.2 below.

Now let F be a graded field, and let B be an inertially split graded F-central division algebra, i.e. B has a maximal graded subfield which is inertial over F. Then, as defined in [M_2, Remark 2.13], B is said to be *degenerate* if it has a graded subfield L inertial over F such that $C_B(L)$ is nicely semiramified with $\Gamma_{C_B(L)}/\Gamma_L$ noncyclic.

Our focus is on the case when the F-central graded division algebra B is semiramified, i.e., $[B_0 : F_0] = |\Gamma_B : \Gamma_F| = \deg(B)$. Recall from [$\mathrm{HW}_2$, Prop. 2.6] that for every F-central graded division algebra D, we have $Z(D_0)$ is Galois over F_0, and conjugation by nonzero homogeneous elements induces an epimorphism $\Gamma_D/\Gamma_F \to \mathrm{Gal}(Z(D_0)/F_0)$. Thus, for our semiramified B, we have B_0 is abelian Galois over F_0, and by comparing group orders, $\mathrm{Gal}(B_0/F_0) \cong \Gamma_B/\Gamma_F$. Thus, the graded maximal subfield B_0F of B is inertial over F, so B is inertially split. Furthermore, the graded subfields L of B which are inertial over F are the graded subfields of B_0F containing F, and they are in one-to-one correspondence with the subfields of B_0 containing F_0. ($L \leftrightarrow L_0$; note that $L = L_0F$.)

Remark 2.1. For a semiramified graded division algebra B over F there are several notions of degeneracy related to B and $q(B)$, and we want to point out that they are actually the same. Indeed, the following conditions are equivalent (for B semiramified):

(1) B is degenerate, as defined above.
(2) The graded field B_0F is degenerate in B, i.e., there is an inertial graded field extension L of F in B such that $\Gamma_{C_B(L)}/\Gamma_L$ ($\cong \mathrm{Gal}(B_0F/L) \cong \mathrm{Gal}(B_0/L_0)$) is noncyclic and $C_B(L)$ is isomorphic to a tensor product of cyclic graded algebras over L with respect to cyclic Galois graded field extensions of L within B_0F.
(3) The maximal subfield $q(B_0F)$ is degenerate in the $q(F)$-central division algebra $q(B)$.
(4) Let $\sigma_1, \ldots, \sigma_m$ be any base of $\mathrm{Gal}(B_0/F_0)$, choose any $y_i \in B^*$ with $y_i c y_i^{-1} = \sigma_i(c)$ for all $c \in B_0$, and let $u_{i,j} = y_i y_j y_i^{-1} y_j^{-1} \in B_0^*$. Then, the collection $(u_{i,j})_{1 \le i,j \le m}$ satisfies the Amitsur-Saltman degeneracy condition as elements of the abelian Galois field extension B_0 of F_0.

Here, (1) ⇒ (2) is clear and (2) ⇒ (1) holds because $C_L(B)$ is semiramified by [M_2, Prop. 1.3], so every cyclic subalgebra of $C_L(B)$ determined by an inertial cyclic graded field extension of L is nicely semiramified, by [M_2, Prop. 1.3]. (1) ⇔ (3) is given in [M_2, Prop. 2.15], and (1) ⇔ (4) in [M_2, Prop. 2.17].

The generic abelian crossed products of Amitsur and Saltman are associated to semiramified graded division algebras. Specifically, let $A = (N/E, G, S, U, \mathbf{b})$ be an abelian crossed product over a field E, as described above. From the data associated to A, Amitsur and Saltman defined in [AS, p. 83] a generic abelian crossed product $A' = \mathcal{K}(N/E, G, S, U)$ which is a division algebra of the same degree as A whose center $Z = Z(A')$ is purely transcendental over E; A' has a maximal subfield $N' = N \otimes_E Z$ which is abelian Galois over Z with $\mathrm{Gal}(N'/Z) \cong \mathrm{Gal}(N/E) = G$, and A'^* contains elements $y_1, \ldots, y_m$ such that y_i induces σ_i on N' by conjugation, and $y_i y_j y_i^{-1} y_j^{-1} = u_{i,j}$ for all i, j. This A' depends up to isomorphism on the choice of base $S = (\sigma_1, \ldots \sigma_m)$ of G and on $U = (u_{i,j})$, but not on $\mathbf{b} = (b_1, \ldots, b_m)$.

[1] The referee points out that the proof of [S_2, Cor. 12.15] uses [S_2, Th. 12.14], which is known to be true though the proof given there has an error.

Also, it follows from [T, Prop. 2.3] that N' is degenerate in A' iff N is degenerate in A. A' is definable as the ring of quotients of an iterated twisted polynomial ring, and it was shown in [BM, Th. 1.1] that A' is therefore also $q(B)$ for a graded division ring B, which is an iterated twisted Laurent polynomial ring. Let $F = Z(B)$, a graded field. It was shown further in [BM, Th. 1.1] that $q(F) = Z$, $F_0 = E$, and B is semiramified over F with $B_0 = N$ and $\Gamma_B = \mathbb{Z}^m$. Moreover, by [M_2, Prop. 2.15], N' is degenerate in A' iff B is degenerate. We will see in Cor. 2.7 below how results on subfields of nondegenerate algebras over Henselian fields yield another proof of one of Saltman's key results about maximal subfields of generic abelian crossed product p-algebras.

Remarks 2.2. (i) Let p be an odd prime number, and let G be a noncyclic finite abelian p-group of order p^n, $n \geq 2$. McKinnie gave in [Mc_2, Cor. 3.2.11] an example of central division algebras A over any suitable field E of any characteristic with maximal subfield N nondegenerate in A with $\text{Gal}(N/E) \cong G$ and $\exp(A) = p$. This yields nondegenerate algebras of higher exponent, as follows: Say $A = (N/E, G, S, U, \mathbf{b})$, as above. Let $A' = \mathcal{K}(N/E, G, S, U)$ be the associated generic abelian crossed product. Let $E' = Z(A')$, which is purely transcendental over E, and let $N' = N \otimes_E E'$, which is a maximal subfield which is nondegenerate in A' with $\text{Gal}(N'/E') \cong \text{Gal}(N/E) \cong G$. But, $\exp(A') = \text{lcm}\big(\exp(A), \exp(G)\big)$ by [T, Prop 2.3]. Thus, simply by choosing G to have exponent p^r for $1 \leq r \leq n-1$, we obtain nondegenerate abelian crossed products of exponent p^r and degree p^n (cf. [Mc_2, Ex. 3.3.1]).

(ii) From any nondegenerate generic abelian crossed product $A' = \mathcal{K}(N/E, G, S, U)$ of degree p^n and exponent p^r, one can obtain a nondegenerate inertially split semiramified division algebra A'' over a Henselian valued field with $\deg(A'') = p^n$ and $\exp(A'') = p^r$. For A'', nondegeneracy is defined to mean nondegeneracy in A'' of the (unique up to isomorphism) maximal subfield of A'' which is inertial over $Z(A'')$. One such A'' is what McKinnie calls the "power series generic abelian crossed product" [Mc_1, Def. 3.6], in which the iterated twisted polynomials in A' are replaced by iterated twisted Laurent series. Another way to produce such an A'' is to view A' as $q(B)$ for B an inertially split graded division algebra, and let $A'' = A' \otimes_{Z(A')} HZ(A')$, where $HZ(A')$ is the Henselization of $Z(A')$ with respect to a valuation on $Z(A')$ induced by the grading on $Z(B)$. (See the proof of Cor. 2.7 below.)

(iii) If one wants nondegenerate abelian crossed product algebras with specified exponent exceeding $\exp(G)$, these are obtainable by a slight adaptation of McKinnie's examples as follows: Her A is obtained as $\mathcal{A} \otimes_{\mathcal{E}} E$, where $\mathcal{A}$ is a division algebra with center $\mathcal{E}$ and with a nondegenerate maximal subfield $\mathcal{N}$ such that $\text{Gal}(\mathcal{N}/\mathcal{E}) \cong G$ and $\exp(\mathcal{A}) = \deg(\mathcal{A}) = |G|$; also, E is the function field $E = \mathcal{E}(Y)$, where Y is the Brauer-Severi variety of $\mathcal{A}^{\otimes p}$. Passage from $\mathcal{E}$ to E generically splits $\mathcal{A}^{\otimes p}$, so generically reduces the exponent of $\mathcal{A}$ to p (while assuring that $\mathcal{A} \otimes_{\mathcal{E}} E$ is a division ring.) For any r with $1 \leq r < n$, let $\mathcal{E}' = \mathcal{E}(Z)$, where Z is the Brauer-Severi variety of $\mathcal{A}^{\otimes p^r}$, and let $\mathcal{A}' = \mathcal{A} \otimes_{\mathcal{E}} \mathcal{E}'$ and $\mathcal{N}' = \mathcal{N} \otimes_{\mathcal{E}} \mathcal{E}'$. Then, by Amitsur's theorem [GS, Th. 5.4.1, p. 125], $\ker\big(\text{Br}(\mathcal{E}) \to \text{Br}(\mathcal{E}')\big)$ is the cyclic group generated by $\big[\mathcal{A}^{\otimes p^r}\big]$; so, $\exp(\mathcal{A}') = p^r$. Also, $\mathcal{A}'$ is a division algebra (of degree p^n) by the Schofield-van den Bergh index reduction formula [SB, Th. 1.3] for function fields of Brauer-Severi varieties. Clearly, $\mathcal{N}'$ is a maximal subfield of $\mathcal{A}'$ which is Galois over $\mathcal{E}'$ with $\text{Gal}(\mathcal{N}'/\mathcal{E}') \cong \text{Gal}(\mathcal{N}/\mathcal{E}) \cong G$. Furthermore, $\mathcal{N}'$ is nondegenerate in $\mathcal{A}'$. To see this, let $\mathcal{E}'' = \mathcal{E}' \cdot E$, the free composite of $\mathcal{E}'$ and E over $\mathcal{E}$; so, $\mathcal{E}''$ is the function field over E of the Brauer-Severi variety of $A^{\otimes p^r}$. Since $A^{\otimes p^r}$ is split, $\mathcal{E}''$ is purely transcendental over E, by [GS, Th. 5.1.3, p. 115]. Therefore, since N is nondegenerate in A (as McKinnie proved), $N \otimes_E \mathcal{E}''$ is nondegenerate in $A \otimes_E \mathcal{E}''$, which follows from by [T, Prop. 2.3]. Then, as $\mathcal{A}' \otimes_{\mathcal{E}'} \mathcal{E}'' \cong A \otimes_E \mathcal{E}''$ and $\mathcal{N}' \otimes_{\mathcal{E}'} \mathcal{E}'' \cong N \otimes_E \mathcal{E}''$, $\mathcal{N}'$ must be nondegenerate in $\mathcal{A}'$.

Throughout the rest of this section, we will have the following standing hypotheses:

Hypotheses 2.3. Let E be a field with Henselian valuation v, and let D be a division algebra with center E and $[D : E] = p^n$ for some prime number p and some $n \in \mathbb{N}$. We assume further that D is inertially split semiramified with respect to the unique extension of v to a valuation of D. There is a distinguished maximal subfield N of D, namely, an inertial maximal subfield. This N is unique up to isomorphism in D, and, since $\overline{N} = \overline{D}$ is abelian Galois over $\overline{E}$ (see Prop. 2.4(1) below), N is abelian Galois over E, with $\mathrm{Gal}(N/E) \cong \mathrm{Gal}(\overline{D}/\overline{E})$. We assume further that D is nondegenerate, by which is meant that N is nondegenerate in D.

Note that by [M_2, Prop. 2.11, Prop. 2.2], the nondegeneracy of D in 2.3 is equivalent to: for every subfield K of D containing E, Γ_K/Γ_E is cyclic. Also, the nondegeneracy of D is equivalent to the nondegeneracy of its associated graded division ring GD, by [M_2, Lemma 2.14].

D and N exist satisfying hypotheses 2.3, as we noted in Remark 2.2(ii). The goal of this section is to obtain information about subfields of D (containing E). Of course, the inertial subfields are known: their isomorphism classes are in one-to-one correspondence with the subfields of $\overline{D}$ containing $\overline{E}$. The interest, therefore, is with the noninertial subfields. We first recall some known properties of D and its subfields which will be used repeatedly below.

Proposition 2.4. *Assume hypotheses 2.3.*

(1) *$\overline{D}$ is abelian Galois over $\overline{E}$ with* $\mathrm{Gal}(\overline{D}/\overline{E}) \cong \Gamma_D/\Gamma_E$.
(2) *If Γ_D/Γ_E is noncyclic, then D has no (non-trivial) subfield totally ramified over E.*
(3) *If K is a subfield of D containing E such that* $\mathrm{Gal}(\overline{D}/\overline{K})$ *is noncyclic, then K is inertial over E.*
(4) *Let M be a subfield of D with M inertial over E, and let $C = C_D(M)$. Then C is inertially split, semiramified, and nondegenerate, with $\overline{C} = \overline{D}$.*

Proof. (1) This holds by [JW, Lemma 5.1], as D is inertially split with $\overline{D}$ a field. (2) holds by [M_2, Prop. 3.2], and (3) by [M_2, Prop. 3.3]. (4) C is inertially split since D is, and by [JW, Th. 3.1(b)] (or by embedding M in an inertial lift of $\overline{D}$ over E in D), $\overline{C} = \overline{D}$. Since C is inertially split with $\overline{C}$ a field, by [M_2, Prop. 1.3(1)] C is semiramified; the nondegeneracy of C is immediate from the nondegeneracy of D. □

Theorem 2.5. *Assume hypotheses 2.3, and assume further that* $\mathrm{char}(\overline{E}) = p$ *and Γ_D/Γ_E is noncyclic. Let K be a subfield of D with K normal over E. Then, K is Galois and inertial over E. So, there is a subgroup H of* $\mathrm{Gal}(\overline{D}/\overline{E})$ *such that* $\mathrm{Gal}(K/E) \cong \mathrm{Gal}(\overline{D}/\overline{E})/H$. *In particular, if K is a maximal subfield of D which is Galois over E, then* $\mathrm{Gal}(K/E) \cong \mathrm{Gal}(\overline{D}/\overline{E})$.

Proof. Let K be a subfield of D which is normal over E. Then by Th. 1.5, GK is a normal graded field extension of GE. Let $L = GK^{\mathrm{Gal}(GK/GE)}$. By Prop. 1.4, L is a purely wild graded field extension of GE. Therefore, by [M_2, Lemma 3.1], $L = GE$. Hence, by Prop. 1.4 again, GK is a Galois graded field extension of GE. In particular, by [HW_1, Th. 3.11] GK is a tame graded field extension of GE; since also $[GK : GE]$ is a power of p, we must have $\Gamma_{GK} = \Gamma_{GE}$. Hence, GK is inertial over GE. Therefore, K is an inertial valued field extension of E, and by Cor. 1.13, K is a Galois field extension of E. So, $\mathrm{Gal}(K/E) \cong \mathrm{Gal}(\overline{K}/\overline{E})$, which is a homomorphic image of $\mathrm{Gal}(\overline{D}/\overline{E})$. The rest is obvious. □

Remark. McKinnie has proved in [Mc_2, Th. 1.2.1] that if D is a semiramified division p-algebra over a Henselian valued field E with $\overline{D}$ separable over $\overline{E}$ (which is equivalent to saying D is an inertially split semiramified division algebra over E) and with $\overline{D}/\overline{E}$ not strongly degenerate (see [Mc_1, Def. 1.5]), and if K is a Galois subfield of D, then $\mathrm{Gal}(K/E)$ has the form $\mathrm{Gal}(\overline{D}/\overline{E})/H$, where H is a subgroup of $\mathrm{Gal}(\overline{D}/\overline{E})$. Since nondegeneracy of D implies non-strong degeneracy of $\overline{D}/\overline{E}$ (see [Mc_1, p. 815]), this yields Th. 2.5 above for p-algebras. McKinnie also deduced from her result (see [Mc$_2$, Remark 2.1.5]) Saltman's key theorem on Galois groups in nondegenerate generic abelian crossed product p-algebras, which is reproved here as Cor. 2.7 below. The work given here was done independently of McKinnie's work.

Corollary 2.6. *Assume hypotheses 2.3, and suppose* $\operatorname{char}(\overline{E}) = p$. *Then D is a cyclic algebra if and only if Γ_D/Γ_E is cyclic.*

Proof. Recall from Prop. 2.4(1) that $\Gamma_D/\Gamma_E \cong \operatorname{Gal}(\overline{D}/\overline{E})$. Thus, if D is a cyclic algebra, then Th. 2.5 shows that $\operatorname{Gal}(\overline{D}/\overline{E})$ is a cyclic group, so Γ_D/Γ_E is also cyclic. Conversely, if Γ_D/Γ_E is cyclic, then 'the' inertial lift of $\overline{D}$ over E in D (see [JW, Th. 2.9]) is a cyclic maximal subfield of D. □

Corollary 2.7. [S$_1$, Th. 3.2] *Let $\mathcal{E}$ be a field with* $\operatorname{char}(\mathcal{E}) = p \neq 0$, *let $\mathcal{N}$ be a noncylic abelian Galois field extension of $\mathcal{E}$ with $[\mathcal{N} : \mathcal{E}] = p^n$, $n \geq 2$, and let $G = \operatorname{Gal}(\mathcal{N}/\mathcal{E})$. Let $\mathcal{A} = \mathcal{K}(\mathcal{N}/\mathcal{E}, G, S, U)$ be any associated generic abelian crossed product with $U = (u_{i,j})$ nondegenerate; let $Z = Z(\mathcal{A})$. For any subfield L of $\mathcal{A}$ with L Galois over Z, there is a subgroup H of G such that $\operatorname{Gal}(L/Z) \cong G/H$. In particular, if L is a Galois maximal subfield of $\mathcal{A}$, then $\operatorname{Gal}(L/Z) \cong G$.*

Proof. As recalled preceding Remark 2.2 above, by [BM, Th. 1.1], $\mathcal{A} = q(B)$ for some semiramified graded division algebra B with $B_0 = \mathcal{N}$, $Z(B)_0 = \mathcal{E}$, and $q(Z(B)) = Z$. Let HZ be the Henselization of Z with respect to the canonical valuation on Z determined by the grading on $Z(B)$ with some chosen total ordering of $\Gamma_{Z(B)}$; let $H\mathcal{A} = \mathcal{A} \otimes_Z HZ$. Because the valuation on Z extends to $\mathcal{A} = q(B)$, by Morandi's Henselization Theorem [Mor, Th. 2] $H\mathcal{A}$ is a division ring with valuation extending the valuation on HZ, and $\overline{H\mathcal{A}} \cong \overline{\mathcal{A}} \cong B_0 \cong \mathcal{N}$ and $\Gamma_{H\mathcal{A}} = \Gamma_{\mathcal{A}}$; also, $\overline{HZ} \cong \overline{Z} \cong Z(B)_0 \cong \mathcal{E}$. So, $H\mathcal{A}$ is inertially split with $\operatorname{Gal}(\overline{H\mathcal{A}}/\overline{HZ}) \cong \operatorname{Gal}(\mathcal{N}/\mathcal{E}) = G$ and associated graded ring $GH\mathcal{A} \cong G\mathcal{A} \cong B$. Since $U = (u_{i,j})$ is nondegenerate in $\mathcal{N}$ for G with base S, by [M$_2$, Prop. 2.17] B is a nondegenerate graded division algebra, so by [M$_2$, Lemma 2.14] $H\mathcal{A}$ is nondegenerate. Let L be a subfield of $\mathcal{A}$ with L Galois over Z. Then $L \otimes_Z HZ$ is a subfield of $H\mathcal{A}$ Galois over HZ with $\operatorname{Gal}((L \otimes_Z HZ)/HZ) \cong \operatorname{Gal}(L/Z)$. So, by Th. 2.5, there is a subgroup H of G such that $\operatorname{Gal}(L/Z) \cong G/H$. In particular, if L is a Galois maximal subfield of $\mathcal{A}$, then $\operatorname{Gal}(L/Z) \cong G$. □

Proposition 2.8. *Assume hypotheses 2.3. Assume further that Γ_D/Γ_E is noncylic, and let K be a subfield of D which is elementary abelian Galois over E. Then, K is inertial over E. Therefore, D is an elementary abelian crossed product if and only if $\operatorname{Gal}(\overline{D}/\overline{E})$ is elementary abelian.*

Proof. Since K is an elementary abelian field extension of E, we can write $K = K_1 \otimes_E K_2 \otimes_E ... \otimes_E K_r$, where each K_i is a cyclic field extension of E with $[K_i : E] = p$. By Prop. 2.4(2), D contains no proper totally ramified field extensions of E. Hence, each K_i is inertial over E; so, K is also inertial over E. If K is in addition a maximal subfield of D, then $\overline{D}$ $(= \overline{K})$ is elementary abelian over $\overline{E}$. Conversely, suppose that $\operatorname{Gal}(\overline{D}/\overline{E})$ is elementary abelian and let M be the inertial lift of $\overline{D}$ over E in D. Then, M is a maximal subfield of D which is Galois over E with $\operatorname{Gal}(M/E) \cong \operatorname{Gal}(\overline{M}/\overline{E})$. □

Remark 2.9. Prop. 2.8 is not true if Γ_D/Γ_E is cyclic. Indeed, let k be a field containing a primitive p-th root of unity, with a cyclic Galois extension L with $[L : k] = p^2$. Let N be the field with $k \subsetneq N \subsetneq L$. Let E be the Laurent series field $k((X))$, and let D be the cyclic algebra $\big(L((X))/E, \sigma, X\big)$, where σ is any generator of $\operatorname{Gal}(L((X))/E)$. Then, D is a division algebra of degree p^2 with center E. Moreover, with respect to the Henselian X-adic valuation on E, D is tame and semiramified, with $\overline{D} = L$. This D is trivially nondegenerate since $\operatorname{Gal}(\overline{D}/\overline{E})$ is cyclic. We have $\Gamma_D/\Gamma_E \cong \operatorname{Gal}(\overline{D}/\overline{E}) \cong \mathbb{Z}/p^2\mathbb{Z}$. But, take t in D with $tat^{-1} = \sigma(a)$ for all $a \in L((X))$ and $t^{p^2} = X$. Then, D also contains the maximal subfield $N((X))[t^p]$ which is elementary abelian Galois over E.

For a finite abelian p-group P, $\mathrm{rk}(P)$ denotes the number of summands in a cyclic decomposition of P; so $\mathrm{rk}(P) = \dim_{\mathbb{Z}/p\mathbb{Z}}(P/pP)$.

Proposition 2.10. *Assume hypotheses 2.3. Suppose* $\mathrm{rk}(\Gamma_D/\Gamma_E) \geq 3$, *and let K be a subfield of D which is abelian Galois over E. Then, K is inertial over E.*

Proof. Write $K = K_1 \otimes_E K_2 \otimes_E \ldots \otimes_E K_r$, where each K_i is a cyclic Galois field extension of E. So, $\overline{K_i}$ is cyclic over $\overline{E}$. Therefore, $\overline{D}$ cannot be cyclic over $\overline{K_i}$ (since $\mathrm{rk}(\mathrm{Gal}(\overline{D}/\overline{E})) \geq 3$). So, by Prop. 2.4(3) each K_i is inertial over E. Hence, K is inertial over E. □

Lemma 2.11. *Assume hypotheses 2.3, and let K be any subfield of D containing E with K not inertial over E. Then,* $\mathrm{Gal}(\overline{D}/\overline{K})$ *is cyclic and Γ_K/Γ_E is isomorphic to a subgroup of* $\mathrm{Gal}(\overline{D}/\overline{K})$. *In particular, if* $\exp(\Gamma_D/\Gamma_E) = p$, *then K is a maximal subfield of D.*

Proof. Let M be the maximal unramified extension of E in K, and let $C = C_D(M)$. By Prop. 2.4(4) above, C is a nondegenerate inertially split semiramified division algebra with $\overline{C} = \overline{D}$. Moreover K is a subfield of C which is a non-trivial totally ramified extension of the center M of C. So, by Prop. 2.4(1) and (2) applied to C as an M-algebra, $\mathrm{Gal}(\overline{C}/\overline{M})$ is cyclic. Then, $\mathrm{Gal}(\overline{D}/\overline{K})$ is cyclic, as $\overline{D} = \overline{C}$ and $\overline{K} = \overline{M}$. The canonical isomorphism $\Gamma_D/\Gamma_E \cong \mathrm{Gal}(\overline{D}/\overline{E})$ of Prop. 2.4(1) is induced by conjugation, so it injects Γ_K/Γ_E into $\mathrm{Gal}(\overline{D}/\overline{K})$. If $p = \exp(\Gamma_D/\Gamma_E) = \exp(\mathrm{Gal}(\overline{D}/\overline{E}))$, then the cyclic group $\mathrm{Gal}(\overline{C}/\overline{M})$ has exponent and hence order p. Then, $p = |\mathrm{Gal}(\overline{C}/\overline{M})| = \deg(C)$, as C is semiramified. So the proper extension K of M is a maximal subfield of C, and hence K is also a maximal subfield of D. □

Let H be a finite nonabelian group. Recall that H is said to be a *quaternion group* if H has order 8 and is generated by two elements a and b satisfying the conditions $a^4 = b^4 = 1$, $a^2 = b^2$ and $ba = a^{-1}b$. If K/E is a normal [resp., Galois] field extension with a quaternion Galois group, we say that K is a quaternion normal [resp., Galois] field extension of E. A finite group H is said to be *Hamiltonian* if H is nonabelian and every subgroup of H is normal. Recall that a Hamiltonian group is the direct product of a quaternion group with an abelian group of odd order and an abelian group of exponent two [H, Th. 12.5.4].

Theorem 2.12. *Assume hypotheses 2.3. Suppose* $\exp(\Gamma_D/\Gamma_E) = p$, *and let K be a subfield of D containing E. Then,*

1. *If K is not inertial over E, then K is a maximal subfield D.*
2. *If Γ_D/Γ_E is noncyclic and K is a non-quaternion normal maximal subfield of D, then either K is cyclic Galois over E with $[K : E] = p^2$ or K is inertial and elementary abelian Galois over E.*

Proof. (1) This follows by Lemma 2.11.

(2) Let $G = \mathrm{Gal}(K/E)$ and let $I = K^G$. Then, I is purely inseparable over E, so $\overline{I}$ is purely inseparable over $\overline{E}$. Since $\overline{D}$ is separable over $\overline{E}$, we have $\overline{I} = \overline{E}$; hence I is totally ramified over E. Since D is nondegenerate, Prop. 2.4(2) shows that $I = E$. Therefore, K is Galois over E. If L is any proper subfield of K, then by part (1) of this proof L is inertial over E, hence L is an abelian Galois field extension of E. Therefore, any subgroup of G is a normal subgroup. So, G is Hamiltonian or abelian. If K is inertial over E, then K is Galois over E since $\overline{K}$ is Galois over $\overline{E}$, and $\mathrm{Gal}(K/E) \cong \mathrm{Gal}(\overline{K}/\overline{E})$; this group is elementary abelian since $\mathrm{Gal}(\overline{D}/\overline{E}) \cong \Gamma_D/\Gamma_E$, which is elementary abelian. Suppose now K is not inertial over E, and let M be the maximal unramified extension of E in K. We have seen that every proper subfield of K is inertial over E, so lies in M. Therefore, $\mathrm{Gal}(K/M)$ is the unique

minimal proper subgroup of G. So, G admits no nontrivial direct product decompositions, and since G is assumed non-quaternion, it cannot be Hamiltonian. Hence, G is abelian, and since it has a unique minimal proper subgroup it must be cyclic. The exponent assumption then implies that the cyclic groups Γ_K/Γ_E and $\mathrm{Gal}(\overline{K}/\overline{E})$ have order at most p. So $[K:E]$ is at most p^2. The fact that K is a maximal subfield of D and Γ_D/Γ_E is noncyclic imply that $[K:E]=p^2$. □

Remarks 2.13. (i) Let D be a division algebra over E satisfying hypotheses 2.3 with $\Gamma_D/\Gamma_E \cong (\mathbb{Z}/p\mathbb{Z})^n$ where $n \geq 2$. The results above yield a remarkably complete classification of all the subfields of D containing E, as follows: There is a maximal subfield L of D with L inertial over E. Then, $\overline{L} = \overline{D}$ and L is determined up to isomorphism by $\overline{L}$, as the valuation on E is Henselian. (L is the inertial lift of $\overline{D}$ over E.) This L is abelian Galois over E with $\mathrm{Gal}(L/E) \cong \mathrm{Gal}(\overline{D}/\overline{E}) \cong \Gamma_D/\Gamma_E \cong (\mathbb{Z}/p\mathbb{Z})^n$. The subfields N of L containing E are classified as usual by the subgroups of $\mathrm{Gal}(L/E)$. Moreover, every subfield of D inertial over E is isomorphic to such an N. Now, take any subfield M of L with $[M:E]=p^{n-1}$, and let $C = C_D(M)$. Then, L is a maximal subfield of C, so C is a cyclic algebra, $C = (L/M, \sigma, t)$, where $t \in M^*$, and for the value $v(t)$, we have $\frac{1}{p}v(t) + \Gamma_E$ corresponds to the image $\overline{\sigma}$ of σ in $\mathrm{Gal}(\overline{D}/\overline{E})$ under the isomorphism $\mathrm{Gal}(\overline{D}/\overline{E}) \cong \Gamma_D/\Gamma_E$. Let K be a subfield of C with $K \supsetneqq M$. Then, K is a maximal subfield of C (so of D). Such a K is inertial over M iff $K \cong L$. Assume now that K is ramified over M. Then, K must be totally ramified over M with $\Gamma_K = \Gamma_C = \langle \frac{1}{p}v(t)\rangle + \Gamma_E$. See (ii) below for more about these K. Lemma 2.11 shows that the only proper subfields of K containing E are the subfields of M, which is the maximal unramified extension of E in K. (Therefore, the M used in obtaining K is unique up to isomorphism.) Moreover, if $n \geq 3$ (and $n \geq 4$ if $p=2$), then K is not normal over E by Th. 2.12(2). Note also that *every* subfield N of D containing E which is not inertial over E is isomorphic to such a K. For, by Lemma 2.11 N is a maximal subfield of D with Γ_N/Γ_E a cyclic subgroup of $(\mathbb{Z}/p\mathbb{Z})^n$, hence of order p. Therefore, the maximal unramified extension of E in N has degree p^{n-1} over E.

(ii) When $\mathrm{char}(\overline{E}) \neq p$, one can characterize more fully the fields K ramified over M in the cyclic algebras $C = (L/M, \sigma, t)$ appearing in (i). Indeed, these are the fields of the form $M(\sqrt[p]{ut})$, where $u \in M^*$ with $v(u)=0$ and and $\overline{u} \in \mathrm{im}(N_{\overline{L}/\overline{M}})$. For, since a noninertial K in C is tame and totally ramified over M, K is a radical extension of M by [Sch, p. 64, Th. 3]. Because $\Gamma_K = \Gamma_C = \langle\gamma\rangle + \Gamma_M$ where $\gamma = \frac{1}{p}v(t)$, we must have $K = M(\sqrt[p]{ut})$ for some $u \in M^*$ with $v(u)=0$. Such a K lies in C iff K splits C, as $\deg(C) = [K:M] = p$. But, as $t \equiv u^{-1} \pmod{K^{*p}}$, we have $[C \otimes_M K] = [(L \otimes_M K)/K, \sigma \otimes \mathrm{id}, u^{-1})]$ in $\mathrm{Br}(K)$. The latter cyclic algebra is inertial over K since $L \otimes_M K$ is inertial over K and u^{-1} is a valuation unit. Therefore, as K is Henselian, this algebra is split iff the residue algebra $(\overline{L}/\overline{M}, \overline{\sigma}, \overline{u}^{-1})$ is split, which holds iff $\overline{u}$ is a norm from $\overline{L}^*$. Furthermore, two such fields, $K = M(\sqrt[p]{ut})$ and $K' = M(\sqrt[p]{u't})$, are M-isomorphic iff $\overline{u/u'} \in \overline{M}^{*p}$. For, $K' \cong K$ iff K contains a p-th root of $u't$, iff $u/u' \in K^{*p}$, iff $\overline{u/u'} \in \overline{K}^{*p} = \overline{M}^{*p}$, as K is Henselian.

(iii) Note that examples of D and E as in (i) are obtainable as follows: Let $\mathcal{E}$ be a field with a Galois extension $\mathcal{N}$ with $\mathrm{Gal}(\mathcal{N}/\mathcal{E}) \cong (\mathbb{Z}/p\mathbb{Z})^n$, and let $\mathcal{D}$ be a division algebra central over $\mathcal{E}$ and containing $\mathcal{N}$ as a maximal subfield, such that, further, $\exp(\mathcal{D}) = \deg(D)$. As pointed out in [AS, Remark, p. 82], $\mathcal{E}$ could be taken to be any global field; then such $\mathcal{N}$ and $\mathcal{D}$ exist, and the exponent condition holds for any division algebra over a global field. So, $\mathcal{D}$ is an abelian crossed product for $\mathcal{N}/\mathcal{E}$, and we can write $\mathcal{D} = A(\mathcal{N}/\mathcal{E}, G, S, U, \mathbf{b})$ in the notation at the beginning of this section. Moreover, the condition $\exp(\mathcal{D}) = \deg(\mathcal{D})$ assures that $U = \{u_{i,j}\}$ is nondegenerate in $\mathcal{N}$, by [AS, Lemma 1.7]. Let $\mathcal{D}' = \mathcal{K}(\mathcal{N}/\mathcal{E}, G, S, U)$ be the associated generic abelian crossed product, whose center $Z = Z(\mathcal{D}')$ is a field purely transcendental over $\mathcal{E}$. As recalled earlier, $\mathcal{D}' = q(B)$ for a twisted Laurent polynomial ring B, which has the structure of a graded division ring. Any choice of total ordering on the grade group Γ_B determines canonically a valuation v on $\mathcal{D}'$. If we let $E = HZ$, the Henselization of Z with respect to v, and $D = \mathcal{D}' \otimes_Z E$, then D/E satisfies hypotheses 2.3 with $\overline{D} \cong \mathcal{N}$ and $\overline{E} \cong \mathcal{E}$; so, $\mathrm{Gal}(\overline{D}/\overline{E}) \cong (\mathbb{Z}/p\mathbb{Z})^n$, and $\exp(D) = \mathrm{lcm}(\exp(\mathcal{D}), \exp(\mathbb{Z}/p\mathbb{Z})^n) = p^n = \deg(D)$. This exponent information assures the nondegeneracy of D, which also follows from the nondegeneracy of $\{u_{i,j}\}$ in $\mathcal{N}$. Another approach would be to take D to be McKinnie's power series generic abelian crossed product determined from $\mathcal{D} = A(\mathcal{N}/\mathcal{E}, G, S, U, \mathbf{b})$.

Lemma 2.14. *Let K be a tame finite-dimensional Galois extension of the Henselian field E, and let L be a field with $E \subseteq L \subseteq K$, such that $\overline{L} = \overline{K}$. Then, L is Galois over E.*

Proof. Let M be the maximal unramified extension of E in L. Because $\overline{L} = \overline{K}$, M is the maximal unramified extension of E in K. Since K is tame and totally ramified over M, K is a radical extension of M, by [Sch, Th. 3, p. 64]. Because K is also Galois over M, it is a Kummer extension of M. Let e be the exponent of $\mathrm{Gal}(K/M)$. The subgroups of the Kummer group $\mathcal{K} = \{a \in K^* \mid a^e \in M^*\}/M^*$ are in canonical one-to-one correspondence with the fields N with $M \subseteq N \subseteq K$. The valuation induces a map $\mathcal{K} \to \Gamma_K/\Gamma_M$, which must be injective, since a nontrivial kernel would yield a nontrivial unramified extension of M in K. Therefore, each such N, which is totally ramified over M, is determined by Γ_N (as Γ_N/Γ_M coincides with the image of the Kummer set for N in Γ_K/Γ_M). However, for any $\sigma \in \mathrm{Gal}(K/E)$, σ must preserve the unique valuation on K extending the Henselian valuation on E. Hence, $\Gamma_{\sigma(L)} = \Gamma_L$. Because $\sigma(M) = M$ from the uniqueness of the maximal unramified extension, L and $\sigma(L)$ each contain M. Since they have the same value groups, we must have $\sigma(L) = L$. As K is Galois over E, it follows that L is normal, hence also Galois, over E. □

Definition 2.15. Let G be an abelian group and H a non-trivial cyclic subgroup of G. We say that H is *maximally cyclic* in G if there is no cyclic subgroup H' of G such that $H \subsetneqq H'$.

Assuming hypotheses 2.3, suppose K is a normal field extension of E in D. From the nondegeneracy of D over E we have seen in Prop. 2.4(3) that if $\mathrm{Gal}(\overline{D}/\overline{K})$ is noncyclic, then K is inertial over E. In the next proposition we will study the case where $\mathrm{Gal}(\overline{D}/\overline{K})$ is maximally cyclic in $\mathrm{Gal}(\overline{D}/\overline{E})$.

Proposition 2.16. *Assume hypotheses 2.3. Let K be a subfield of D which is normal and tame over E such that $\mathrm{Gal}(\overline{D}/\overline{K})$ is cyclic. Suppose $\mathrm{Gal}(\overline{D}/\overline{K})$ is maximally cyclic in $\mathrm{Gal}(\overline{D}/\overline{E})$. Then, K is Galois over E. Furthermore,*

1. *If $\deg(D)$ is odd, then K is an abelian field extension of E.*
2. *If Γ_D/Γ_E is noncyclic and $\deg(D)$ is a power of 2, then K is either a quaternion or an abelian field extension of E.*

Therefore, if $\mathrm{rk}(\Gamma_D/\Gamma_E) \geq 3$ and K is not a quaternion field extension of E, then K is inertial over E.

Proof. Let $G = \mathrm{Gal}(K/E)$. By the same argument as in the proof of Th. 2.12(2), K is Galois over E. Let L be a field with $E \subseteq L \subseteq K$. If $\overline{L} = \overline{K}$, then by Lemma 2.14 L is Galois over E. On the other hand, if $\overline{L} \subsetneqq \overline{K}$ then $\mathrm{Gal}(\overline{D}/\overline{L})$ is not cyclic by the maximal cyclicity assumption. Therefore, by Prop. 2.4(3), L is inertial, and hence Galois, over E. Since L is Galois over E in all cases, G is Hamiltonian or abelian. In case (1), since $|G|$ is odd, G must be abelian.

(2) Suppose G is Hamiltonian. Since G is nonabelian, K is not inertial over E. Suppose we have a tensor decomposition $K = L_1 \otimes_E L_2$ with fields L_i containing E. Since K is not inertial over E, one of the L_i, say L_1, is not inertial over E. Then, $\overline{L_1} = \overline{K}$, as we saw above. Let M_i be the maximal unramified extension of E in L_i. Then, $\overline{M_2} = \overline{L_2} \subseteq \overline{K} = \overline{L_1} = \overline{M_1}$. Therefore, M_2 is a subfield of M_1. Since $M_1 \otimes_E M_2$ is a field (a subfield of $L_1 \otimes_E L_2$), we must have $M_2 = E$. Therefore, L_2 is totally

ramified over E. Hence, $L_2 = E$, by Prop. 2.4(2). This shows that K admits no nontrivial tensor product decompositions over E, and hence that G admits no nontrivial direct product decompositions. Therefore, the Hamiltonian group G must be quaternionic, so K is quaternionic over E.

The rest of the proposition follows by Prop. 2.10. □

Corollary 2.17. *Assume hypotheses 2.3. Suppose also that* $\deg(D)$ *is odd and* $\operatorname{char}(\overline{E}) \nmid \deg(D)$*. Let K be a maximal subfield of D such that K is Galois over E. If* $|\Gamma_K : \Gamma_E| = \exp(\Gamma_D/\Gamma_E)$*, then* $\operatorname{Gal}(K/E)$ *is abelian and* $\operatorname{rk}(\Gamma_D/\Gamma_E) \le 2$*.*

Proof. Assuming $D \ne E$, the assumption on Γ_K assures that K is not inertial over E, and the hypothesis on $\operatorname{char}(\overline{E})$ implies that K is tame over E. Therefore, by Lemma 2.11, $\operatorname{Gal}(\overline{D}/\overline{K})$ is cyclic. Let M be the maximal unramified extension of E in K, and let $C = C_D(M)$. By Prop. 2.4(4), C is inertially split and semiramified with $\overline{C} = \overline{D}$. Now, K is a maximal subfield of C, since it is maximal in D, and K is totally ramified over M. Since

$$|\Gamma_K : \Gamma_M| \;=\; [K : M] \;=\; \operatorname{ind}(C) \;=\; |\Gamma_C : \Gamma_M|,$$

we have $\Gamma_K = \Gamma_C$. As $\overline{M} = \overline{K}$ and $\Gamma_M = \Gamma_E$, Prop. 2.4(1) applied to C shows that

$$\operatorname{Gal}(\overline{D}/\overline{K}) \;=\; \operatorname{Gal}(\overline{C}/\overline{M}) \;\cong\; \Gamma_C/\Gamma_M \;=\; \Gamma_K/\Gamma_E.$$

Therefore,

$$\big|\operatorname{Gal}(\overline{D}/\overline{K})\big| \;=\; |\Gamma_K : \Gamma_E| \;=\; \exp(\Gamma_D/\Gamma_E) \;=\; \exp\big(\operatorname{Gal}(\overline{D}/\overline{E})\big);$$

so, $\operatorname{Gal}(\overline{D}/\overline{K})$ is maximally cyclic in $\operatorname{Gal}(\overline{D}/\overline{E})$. Hence, by (1) and the last assertion of Prop. 2.16 (which applies as K is tame over E), K is abelian Galois over E and $\operatorname{rk}(\Gamma_D/\Gamma_E) \le 2$ (as K is not inertial over E). □

Remark 2.18. Suppose $\operatorname{char}(\overline{E}) \ne p$. Let x be an element of D^* such that $\operatorname{ord}(v(x) + \Gamma_E) = \exp(\Gamma_D/\Gamma_E)$, and let L be a maximal subfield of D containing x. Then by Cor. 2.17, L cannot be Galois over E if $\operatorname{rk}(\Gamma_D/\Gamma_E) \ge 3$.

Proposition 2.19. *Assume hypotheses 2.3, and suppose further that* $\operatorname{char}(\overline{E}) \ne p$ *and* $\operatorname{rk}(\Gamma_D/\Gamma_E) \ge 3$*. Let K be a subfield of D which is Galois but not inertial over E. Then,* $[K : E] \ge p \deg(D) \exp(\Gamma_D/\Gamma_E)^{-1}$*.*

Proof. We may assume that K is minimal in D with the property that K is Galois but not inertial over E. Let M be the maximal unramified extension of E in K. Then, K is Galois and totally ramified over M, and tame over E (as $\operatorname{char}(\overline{E}) \ne p$). Let L be a field with $M \subseteq L \subsetneqq K$. Then, $\overline{L} = \overline{K}$ since $\overline{M} = \overline{K}$, so L is Galois over E by Lemma 2.14. Hence, L is inertial over E by the minimality of K, i.e., $L = M$. Thus, M is a maximal proper subfield of K. Since K is Galois over M, this implies that $[K : M] = p$. So, $|\Gamma_K : \Gamma_E| = |\Gamma_K : \Gamma_M| = [K : M] = p$. Now, by Prop. 2.4(3), $\operatorname{Gal}(\overline{D}/\overline{K})$ is a cyclic group. Hence, $[\overline{D} : \overline{K}] \le \exp(\operatorname{Gal}(\overline{D}/\overline{E})) = \exp(\Gamma_D/\Gamma_E)$ (see Prop. 2.4(1)). Therefore,

$$[K : E] \;=\; [\overline{K} : \overline{E}]\,|\Gamma_K : \Gamma_E| \;=\; [\overline{D} : \overline{E}]\,[\overline{D} : \overline{K}]^{-1} p \;\ge\; p \deg(D) \exp(\Gamma_D/\Gamma_E)^{-1}.$$

□

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