Math 102

Midterm Solutions

① a) Do Gaussian elimination on A:

$$\begin{pmatrix} 1 & 2 & 4 & 3 \\ 2 & 4 & 9 & 7 \\ 3 & 6 & 11 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 4 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

↑ ↑ basis for col (A). ↑ ↑ pivot cols.

$$\therefore \boxed{\operatorname{col}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 9 \\ 11 \end{pmatrix} \right\}.}$$

Find Nul (A) by solving homogeneous equation:

$x_1 + 2x_2 + 4x_3 + 3x_4 = 0$

$x_3 + x_4 = 0 \Rightarrow x_4 = -x_3$

$x_1 + 2x_2 + x_3 = 0$

$\Rightarrow x_3 = -x_1 - 2x_2$

$x_4 = x_1 + 2x_2$

$$\therefore \operatorname{Nul}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ -x_1 - 2x_2 \\ x_1 + 2x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}$$

$$\Rightarrow \boxed{\operatorname{Nul}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 2 \end{pmatrix} \right\}.}$$

b)

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 2 & 4 & 9 & 7 & 5 \\ 3 & 6 & 11 & 8 & 5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$

$x_1 + 2x_2 + 4x_3 + 3x_4 = 2$

$x_3 + x_4 = 1$

$\Rightarrow x_4 = 1 - x_3$

$\Rightarrow x_1 = -1 - 2x_2 - x_3$

$\therefore$ solutions to the system are

$$\left\{ \begin{pmatrix} -1 - 2x_2 - x_3 \\ x_2 \\ x_3 \\ 1 - x_3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

c) $\text{Col } A = \{ b \in \mathbb{R}^3 \text{ s.t. } Ax=b \text{ is consistent} \}$

Let $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ and row-reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 2 & 4 & 9 & 7 & b_2 \\ 3 & 6 & 11 & 8 & b_3 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 0 & 0 & 1 & 1 & b_2-2b_1 \\ 0 & 0 & -1 & -1 & b_3-3b_1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 4 & 3 & b_1 \\ 0 & 0 & 1 & 1 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-3b_1+b_2-2b_1 \end{array}\right)$$

So in order for $b \in \text{Col}(A)$, need

$$b_3 + b_2 - 5b_1 = 0$$

Any b for which this equation does <u>not</u> hold is a vector for which $Ax=b$ has no solution.

$\boxed{\text{Key Idea}}$:

$b \in \text{Col}(A)$ if and only if $Ax=b$ has a solution!

(2)

a) A has LI columns $\Rightarrow$ $\boxed{\text{Rank}(A) = 3}$

$\text{Rank}(A) + \dim(\text{Nul } A) = 3$ $\Rightarrow$ $\boxed{\dim(\text{Nul } A) = 0}$

b) $\boxed{\text{False}}$ since $\dim(\mathbb{R}^4) = 4 > \text{Rank}(A)$

(i.e. there are vectors in $\mathbb{R}^4$ that aren't in $\text{Col}(A)$).

c) $\boxed{\text{True}}$ because A has a trivial nullspace (equivalently, because the columns of A are LI).

(3)

a) Apply Gram-Schmidt to $\{a_1, a_2, a_3\}$ and get $w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $w_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$.

b) cols of Q are orthonormal basis for $\text{Col}(A)$. We already found an orthogonal basis in (a); we just need to normalize to get the columns of Q:

$$\|w_1\| = 2 \Rightarrow q_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{pmatrix}$$

$$Q = [q_1 \ q_2 \ q_3], \quad R = Q^T A = \begin{pmatrix} 2 & 4 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

#3

c) We already found an orthogonal basis $\{w_1, w_2\}$ for span$\{a_1, a_2\}$, so we can project b onto w_1 and w_2 and then add them.

$\boxed{\text{Key Idea}}$: You can only project onto an orthogonal basis!

$$\text{proj}_{sp\{a_1,a_2\}} b = \text{proj}_{w_1} b + \text{proj}_{w_2} b$$

$$= \frac{(1\ 6\ 0\ 2\ 0)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{(1\ 1\ 1\ 1)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{(1\ 6\ 0\ 2\ 0)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}}{(1\ -1\ -1\ 1)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}3\\1\\1\\3\end{pmatrix}}.$$

d) $\boxed{\text{True}}$ Nul$(A^T) = (\text{Col } A)^\perp$, so any vector q_4 in Nul(A^T) extends $\{q_1, q_2, q_3\}$ into an orthogonal basis for $\mathbb{R}^4$.

4) $\begin{pmatrix}2\\3\\1\end{pmatrix} = 2\begin{pmatrix}1\\1\\0\end{pmatrix} + 1\begin{pmatrix}0\\1\\1\end{pmatrix}$, so by linearity of A,

$$A\begin{pmatrix}2\\3\\1\end{pmatrix} = 2A\begin{pmatrix}1\\1\\0\end{pmatrix} + A\begin{pmatrix}0\\1\\1\end{pmatrix} = 2\begin{pmatrix}2\\1\\3\end{pmatrix} + \begin{pmatrix}1\\1\\2\end{pmatrix} = \boxed{\begin{pmatrix}5\\3\\8\end{pmatrix}}.$$