MATH 210B WINTER 2005: FINAL

Instructor: Hans Wenzl

1. Consider the integral equation $f = 1 + K(f)$, where $K(f)(x) = 9\int_0^{1/3}\sqrt{1-xy}f(y)dy$.
(a) Compute the exact solution for the equation $f = 1 + K_0(f)$, with $K_0(f)(x) = 9\int_0^{1/3}(1 - xy/2)f(y)dy$.
(b) Estimate $\|K - K_0\|$; you may use that $|\sqrt{1-z} - (1 - z/2)| \leq |z|^2/4$ for $|z| \leq 1/3$.
(c) Find a good estimate for $\|(I - K_0)^{-1}\|$; you may use that the eigenvalues of K_0 are approximately 0, .04 and 3.1.
(d) Derive the equation $f = (I - K_0)^{-1}(1) + (I - K_0)^{-1}(K - K_0)(f)$ for the exact solution of our integral equation. Using the estimates above, show that the difference in L^2-norm between the solution in (a) and the exact solution is quite small. (If you could not solve (b) or (c), you may use the estimates .01 for (b) and 5 for (c)).

Sol. (a) It follows from the equation $f(x) = 1 + 9\int_0^{1/3} f(y)dy - x9\int_0^{1/3} yf(y)/2\, dy$ that $f(x) = a + bx$ for suitable coefficents a and b. Plugging this into the equation, we obtain

$$a + bx = 1 + 9\int_0^{1/3}(1 - xy/2)(ay + b)dy = (1 + 3a + b/2) - (a/4 + b/18)x.$$

Comparing the coefficients of 1 and x, we obtain two linear equations for a and b. Solving them, we obtain the solution $f(x) = \frac{18}{143}x - \frac{76}{143}$. For (b) we use the estimate

$$\|K - K_0\|^2 \leq 9^2\int_0^{1/3}\int_0^{1/3}|\sqrt{1-xy} - (1 - xy/2)|^2 dxdy \leq 9^2\int_0^{1/3}\int_0^{1/3} x^4y^4/16\, dxdy.$$

Calculating the integral, one obtains $\|K - K_0\| \leq 1/(27\cdot 20) \leq .002$. For (c), we use the fact that the norm of a selfadjoint compact Hermitian operator is given by its largest eigenvalue (see homework problem). Hence we get $\|(I - K_0)^{-1}\|$ is equal to the maximum of $(1-\lambda)^{-1}$, with λ an eigenvalue of K_0. For the given (slightly incorrect values), this would be 1. Finally, we obtain from $f = (I - K_0)^{-1}(1) + (I - K_0)^{-1}(K - K_0)(f)$, where the solution of (a) is given by $f_0 = (I - K_0)^{-1}(1)$, that

$$\|f - f_0\| \leq \|(I - K_0)^{-1}(K - K_0)(f)\| \leq \|(I - K_0)^{-1}\|\|(K - K_0)\|\|(f)\| \leq .002\|(f)\|.$$

2. Let H be a self-adjoint compact operator. Then $H - iI$ is invertible. Show that $(H + iI)(H - iI)^{-1}$ is a unitary operator.

Sol. *Method 1* Let $U = (H + iI)(H - iI)^{-1}$. Then, using $H^\dagger = H$ and $(iI)^\dagger = -iI$, we get

$$U^\dagger U = (H - iI)(H + iI)^{-1}(H + iI)(H - iI)^{-1} = I,$$

with the proof for $UU^\dagger = I$ similar. *Method 2*: We know that the space V on which H acts has an orthonormal basis of eigenvectors (v_n) with real eigenvalues λ_n. But then

$$(H + iI)(H - iI)^{-1}v_n = \alpha_n v_n = (\lambda_n + i)(\lambda_n - i)^{-1}v_n.$$

As $|\lambda_n + i| = \sqrt{\lambda_n^2 + 1} = |\lambda_n - i|$, because λ_n is real, we have $|\alpha_n| = 1$. Let $v = \sum \beta_n v_n$ be some vector in V. Then

$$\|Uv\|^2 = \|\sum_n \alpha_n \beta_n v_n\|^2 = \sum_n |\alpha_n \beta_n|^2 = \sum_n |\beta_n|^2 = \|v\|^2;$$

here we used the fact that the (v_n) are an orthonormal basis, and that $|\alpha_n| = 1$ for all n.

3. Let H_n be the n-th Hermite polynomial and let $y_n(x) = H_n(x)e^{-x^2/2}$. The only things you need to know about Hermite polynomials is that $H_0 = 1, H_{n+1} = 2xH_n - H_n'$ and $H_n' = 2nH_{n-1}$.

(a) Show that $\hat{f}'(k) = ik\hat{f}(k)$ for $f(x) = P(x)e^{-x^2/2}$ with $P(x)$ a polynomial, and that $2y_n' = -y_{n+1} + 2ny_{n-1}$.

(b) Show that $\hat{y}_n(k) = (-i)^n H_n(k)e^{-k^2/2}$.

Sol. The first statement of (a) is shown using integration by parts and the fact that $\lim_{x\to\infty} f(x) = 0$. Moreover, we have

$$2(\frac{d}{dx}H_n e^{-x^2/2}) = 2H_n' e^{-x^2/2} - 2xH_n e^{-x^2/2} =$$

$$= (2H_n' - (H_{n+1} + H_n'))e^{-x^2/2} = (2nH_{n-1} - H_{n+1})e^{-x^2/2},$$

where we used $H_n' = 2nH_{n-1}$. This shows the second claim of (a). The proof of (b) goes by induction on n, where $\hat{y}_0(k) = y_0(k)$ was proved in class, and $\hat{y}_1 = -\hat{y}_0'$ (by second statement in (a) for $n = 0$, using $y_{-1} = 0) = -2ik\hat{y}_0$ (by first statement in (a); observe that $H_1(x) = 2x$). To prove the general case, observe that

$$\hat{y}_{n+1} = 2n\hat{y}_{n-1} - 2\hat{y}_n' = 2n(-i)^{n-1}y_{n-1}(k) - 2(-i)^n iky_n(k) =$$

by induction assumption,

$$= (-i)^{n+1}[2ky_n(k) - 2ny_{n-1}(k)] = (-i)^{n+1}(2kH_n(k) - 2nH_{n-1}(k)]e^{-k^2/2} = (-i)^{n+1}y_{n+1}.$$

4. Calculate $\|\delta(x)\|^2$ (Suggestion: Choose easy-to-integrate functions f_n which approximate the delta function $\delta(x)$ if $n \to \infty$).

Sol. The easiest choice for f_n would be $f_n(x) = n\chi_{[-1/2n,1/2n]}(x)$, where $\chi_{[-n/2,n/2]}(x)$ is equal to 1 or 0 depending on whether x is in the interval $[-n/2, n/2]$ or not. Then $\int_{-1}^{1} f_n dx = 1$ for all $n \in \mathbf{N}$ and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \neq 0$. This shows that the f_n's approximate $\delta(x)$ and

$$\|\delta(x)\|^2 = \lim_{n\to\infty} \int_1^1 n^2 \chi^2_{[-1/2n,1/2n]}(x)dx = \lim_{n\to\infty} \int_{-1/2n}^{1/2n} n^2 dx = \lim_{n\to\infty} n = \infty.$$