Math 210C Spring 2005
Third Homework Assignment

1. Consider the Laplacian PDE $\Delta u = u_{xx} + u_{yy} = 0$. This can be rewritten as a system $v_y - w_x = 0$ and $v_x + w_y = 0$, where $v = u_x$ and $w = u_y$. Show that this system has no characteristics.

2. Consider the PDE $u_{xx} + u_{yy} = 0$ on the half-plane $x > 0$, $-\infty < y < \infty$ with the boundary conditions $u(0, y) = \epsilon \sin \alpha y$ and $u_x(0, y) = 0$.

(a) Show that $u(x, t) = \epsilon \sin \alpha y \cosh \alpha x$ is a solution of this boundary value problem.

(b) Show that the solutions of this boundary value problem do not depend continuously on the boundary values. More precisely, show that for any $\epsilon > 0$ and for fixed (x, y) with $x > 0$ one can find a solution u for which $|u(0, y)| \leq \epsilon$ for all $x > 0$ and $|\frac{\partial u}{\partial x}(0, y)| \leq \epsilon$ for all $y \in \mathbf{R}$, but for which $u(x, y)$ can be arbitrarily large.
Remark: This shows that even though the boundary conditions tend to 0, the solution of the PDE need not go to 0. Such problems are called ill-posed problems.

3. (a) Use Fourier transforms to find the solution of the wave equation

$$u_{tt} - \Delta u = 0, \quad u(0, x) = f(x), \quad u_t(0, x) = g(x).$$

See page 7 on the hand-out on Fourier transform (see webpage how to download it).

(b) In the setting of problem 3, prove the energy identity

$$\int |\partial_t u(t, x)|^2 + \sum_{i=1}^{n} |\partial_i u(t, x)|^2 dx = \int |g|^2 + \sum_{i=1}^{n} |\partial_i f|^2 dx.$$

Here ∂_i means partial derivative with respect to x_i. (Again, look at page 7 of hand-out).

4. Find the solution of the initial value problem for the Schrödinger equation

$$u_t + i u_{xx} = F, \quad u(0, x) = g(x),$$

where $t > 0$, $-\infty < x < \infty$ and F and g are continuous functions.