Math 210C Spring 2005
Fourth Homework Assignment

1. This problem deals with solving the three-dimensional wave equation. The approach is similar to the one in problem 3 of the last homework assignment. Although the approach here is not exactly the same, you may also wish to consult the treatment in Byron and Fuller, chapter 7.8).

(a) Show that as distributions we have $\delta(x) = \frac{1}{2\pi} \int \cos xy dy$ (*Hint*: Consider the formula for the inverse Fourier transform).

(b) Find the inverse Fourier transform of $\frac{\sin |\xi| t}{|\xi|}$. (*Hint* : Use spherical coordinates. Moreover, you can assume that the inverse Fourier transform is a function only depending on the length $|\mathbf{x}|$ of the vector $\mathbf{x}$. Hence it suffices to calculate the inverse Fourier transform for $\mathbf{x}$ being a multiple of the z-unit vector.)

(c) Find the solution of the initial value problem

$$u_{tt} - \Delta u = 0, \quad u(0, \mathbf{x}) = f(\mathbf{x}), \quad u_t(0, \mathbf{x}) = g(\mathbf{x}).$$

(d) Find the fundamental solution $E(t, \mathbf{x})$ such that $E_{tt} - \Delta E = \delta^3(\mathbf{x})\delta(t)$. (*Hint*: Consider $E(t, \mathbf{x}) = H(t)\mathcal{F}^{-1}(\frac{\sin |\xi| t}{|\xi|})$, where $\mathcal{F}$ is the Fourier transform, and $H(t)$ is the Heaviside function.)

2. Find the fundamental solution of the one-dimensional diffusion equation with drift, given by

$$E_t - E_{xx} + vE_x = \delta(t, x),$$

where v is a real constant. (*Hint*: Let $E = F(t, x)exp(\frac{vx}{2} - \frac{v^2 t}{4})$.)

3. Consider the function $f(x)$ in $L^2(\mathbf{R})$ which is x when $-1 < x < 1$ and 0 otherwise.

(a) Compute the coefficients a_{kj} in the expansion $f(x) = \sum_{k,j} a_{kj}\psi_{kj}$, where ψ_{kj} are the Haar basis functions.

(b) Compute the Fourier coefficients in the expansion $f(x) = \sum_n c_n e^{2\pi i n x}$. Compare how quickly the Haar coefficients a_{kj} and the Fourier coefficients up to c_{2^k} go to zero.

4. Suppose that ϕ, ψ are the scaling function and wavelet for a multiresolution analysis, that both are real, continuous and have compact support, and that $\int \phi^2(x)\ dx = \int \phi(x)\ dx = 1$, and that $\int \psi(x)\ dx = \int x\psi(x)\ dx = 0$.

(a) Show that these conditions imply the following for the coefficients in the scaling relation $\phi(x) = \sum_j \phi(2x - j)$:

$$\sum_j a_j^2 = \sum_j a_j = 2; \quad \sum_j (-1)^j a_j = \sum_j (-1)^j j a_j = 0.$$

(b) Verify that one solution to these equations is

$$a_0 = \frac{1 + \sqrt{3}}{4}, \quad a_1 = \frac{3 + \sqrt{3}}{4}, \quad a_2 = \frac{3 - \sqrt{3}}{4}, \quad a_3 = \frac{1 - \sqrt{3}}{4},$$

with all other a_j vanishing.