MATH 110
EXAM #1

Please answer the following questions. Because this test is open book and open note, you *will not get credit* for answers unless you demonstrate how you arrived at them. In short, please show all work.

PROBLEM 1.

Please find the **specific** solution to the transport equation:

$$\frac{1}{2+\sin(x)}\partial_x u \ + \ (y+1)\partial_y u \ = \ 0 \ ,$$

with the initial data at $y = 0$:

$$u(x,0) \ = \ \cos(x) - 2x \ .$$

Problem 2.

For this problem, let $u(x,t)$ be the specific solution to the wave equation:

$$\begin{aligned} u_{tt} &= u_{xx} \ , \qquad\qquad \text{on} \quad -\infty < x < \infty \\ u(x,0) &= 0 \ , \\ u_t(x,0) &= e^{-x^2} \ . \end{aligned}$$

Please answer the following:

a) Compute the energy at time $t = 100$ for this solution $E(100)$. Please explain your answer carefully.

b) Is it true that the solution $u(x,t)$ is always *strictly* positive for $0 < t$? That is, is it true that $0 < u(x,t)$ for all $0 < t$.

Problem 3.

Please answer the following:

a) Suppose that one has two solutions $u(x,t)$ and $v(x,t)$ to the heat equation with Dirichlet boundary conditions on the interval $[0,\pi]$:

$$\begin{aligned} u_t &= u_{xx} \ , \\ u(0,t) &= u(\pi,t) \ = \ 0 \ , \end{aligned}$$

and:

$$\begin{aligned} v_t &= v_{xx} \ , \\ v(0,t) &= v(\pi,t) \ = \ 0 \ . \end{aligned}$$

Suppose that at the initial time $t = 0$ one has the inequality:

$$v(x,0) \ \leqslant \ u(x,0) \ .$$

Show that for *all* positive times $0 < t$ one has that:

$$v(x,t) \ \leqslant \ u(x,t) \ .$$

Please explain carefully your answer. (Hint: Consider the function $w = u - v$.)

b) Notice that the function:

$$u(x,t) \ = \ e^{-t}\sin(x) \ ,$$

solves the above heat equation with zero Dirichlet boundary conditions on the interval $[0,\pi]$. Show that if $v(x,t)$ is *any* solution to the (same) heat equation boundary value problem:

$$\begin{aligned} v_t &= v_{xx} \ , \\ v(0,t) &= v(\pi,t) \ = \ 0 \ , \end{aligned}$$

with the additional property that at $t = 0$ (on $[0,\pi]$):

$$-\sin(x) \ \leqslant \ v(x,t) \ \leqslant \ \sin(x) \ ,$$

then one always has:

$$|v(x,t)| \ \leqslant \ e^{-t} \ .$$

(Hint: Recall that if $-u \leqslant v \leqslant u$, with $0 \leqslant u$, then one also has $|v| \leqslant u$.)